Mario Paz · Young Hoon Kim

Structural Dynamics

Theory and Computation

Sixth Edition

Springer

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Mario Paz
J.B. Speed School of Engineering, Civil and Environmental Engineering
University of Louisville
Louisville, KY, USA

Young Hoon Kim
J.B. Speed School of Engineering, Civil and Environmental Engineering
University of Louisville
Louisville, KY, USA

ISBN 978-3-319-94742-6 ISBN 978-3-319-94743-3 (eBook)
https://doi.org/10.1007/978-3-319-94743-3

Library of Congress Control Number: 2018949618

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Preface to the Sixth Edition

The basic structure of the five previous editions is still maintained in this Sixth Edition. After the release of the Fifth Edition in 2004, academic and industrial environments have been changed, although the fundamentals have not changed over 15 years. When the author started to teach structural dynamics since 2011, the most challenging part as an instructor has been to present how students can solve and simulate the structural dynamics using the computer program. There is a limited information available to show how we can solve structural dynamics in finite element method–based commercial software. When understanding the background of undergraduate and graduate students who are first exposed to structural dynamics, the fundamentals are mainly considered as core content. The author believes that a line-by-line computer language is a helpful learning and teaching tool for its application of fundamentals. This is the major motivation of the revision of this textbook.

This revised textbook intends to provide enhanced learning materials for students to learn structural dynamics, ranging from basics to advanced topics, including their application. When a line-by-line programming language is included with solved problems, students can learn course materials easily and visualize the solved problems using a program. Among several programming languages, MATLAB® has been adopted by many academic institutions across several disciplines. Many educators and students in the USA and many international institutions can readily access MATLAB®, which has an appropriate programming language to solve and simulate problems in the textbook. It effectively allows matrix manipulations and plotting of data. Therefore, multi-degree-of-freedom problems can be solved in conjunction with the finite element method using MATLAB®. As of 2018, SAP2000 presented in the Fifth Edition is currently outdated, at least regarding user interface procedure. The revision author Young Hoon Kim still believes that SAP2000 includes routines for the analysis and design of structures with linear or nonlinear behavior subjected to static or dynamics loads. However, in this edition exclusion of SAP2000 is necessary to minimize the learner's confusion to link between contents and solving with the aid of computer programming language. The author still believes that SAP2000 can be one of the best tools to solve structural analysis and structural dynamics in complex systems. Practical engineers who are eager to use commercial software can learn from many other textbooks available in the market. Generously, authors

offer the alternative option to navigate other textbooks related to finite element methods. In short, the Sixth Edition mainly targets readers such as senior or master students in structural engineering and earthquake engineering in civil engineering.

This revised edition includes 34 solved examples in Chaps. 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, and 22 with basics: inputs and outputs. The solved problems enhance learners' understanding, as well as effective teaching resources: line-by-line programing language. Additional figures printed out from MATLAB® codes illustrate time-variant structural behavior and dynamic characteristics (e.g., time versus displacement and spectral chart). This textbook updates basics of earthquake design with current design codes (ASCE 7-16 and IBC 2018). Finally, the Sixth Edition uses (1) basic MATLAB® codes for structural dynamics: more than 30 examples in most chapters covering basics and advanced topics, (2) contents to educate undergraduate students and Master of Science/Engineering students who are first exposed to structural dynamics. Graduate and undergraduate students can easily use a contemporary computer program (MATLAB) that is widely used in the USA and other countries.

- Printed code language helps students to understand the application of structural dynamics.
- Graduate students are able to apply the fundamentals to real design problems using current version and practices.
- Enhanced illustrations will enhance the readability of expected readers.

I also like to take this opportunity to thank my colleagues in my home department at Speed School of Engineering at the University of Louisville, KY, especially, Dr. J.P. Mohsen, who continuously encouraged me to revise this book in a timely manner. He recommended the Fifth Edition which also originated from the Third Edition, which I first read in my undergraduate structural dynamics study in South Korea. I also wish to recognize and thank my current PhD student, Jice Zeng. He did diligent work redrawing most figures from the previous edition of this textbook. Especially, I want to thank Dr. Yeesock Kim at California Baptist University. He introduced the application of programming language in my first course of structural dynamics at the University of Louisville which enabled me to proficiently apply structural dynamics using MATLAB®. Finally, I was able to transform my course materials into part of the revision of this version. In addition, I wish to thank Paul Drougas and Flanagan Caroline who patiently waited for my final manuscript.

I am very grateful to serve as the coauthor for original author Mario Paz for enabling my contribution in reviewing and editing this volume, especially those sections which used the computer programs. Finally, I thank my wife, Hye-Jin Baek. Without her support, this revision would be incomplete. Also, my two sons, Edward and William, always provide me all the energy.

For from him and through him and to him are all things. To him be glory forever. Amen. (Romans 11:36)

Louisville, KY, USA Young Hoon Kim
May, 2018

Preface to the First Edition

Natural phenomena and human activities impose forces of time-dependent variability on structures as simple as a concrete beam or a steel pile, or as complex as a multistory building or a nuclear power plant constructed from different materials. Analysis and design of such structures subjected to dynamic loads involve consideration of time-dependent inertial forces. The resistance to displacement exhibited by a structure may include forces which are functions of the displacement and the velocity. As a consequence; the governing equations of motion of the dynamic system are generally nonlinear partial differential equations which are extremely difficult to solve in mathematical terms. Nevertheless, recent developments in the field of structural dynamics enable such analysis and design to be accomplished in a practical and efficient manner. This work is facilitated through the use of simplifying assumptions and mathematical models, and of matrix methods and modem computational techniques.

In the process of teaching courses on the subject of structural dynamics, the author came to the realization that there was a definite need for a text which would be suitable for the advanced undergraduate or the beginning graduate engineering student being introduced to this subject. The author is familiar with the existence of several excellent texts of an advanced nature but generally these texts are, in his view, beyond the expected comprehension of the student. Consequently, it was his principal aim in writing this book to incorporate modern methods of analysis and techniques adaptable to computer programming in a manner as clear and easy as the subject permits. He felt that computer programs should be included in the book in order to assist the student in the application of modern methods associated with computer usage. In addition, the author hopes that this text will serve the practicing engineer for purposes of self-study and as a reference source.

In writing this text, the author also had in mind the use of the book as a possible source for research topics in structural dynamics for students working toward an advanced degree in engineering who are required to write a thesis. At Speed Scientific School, University of Louisville, most engineering students complete a fifth year of study with a thesis requirement leading to a Master in Engineering degree. The author's experience as a thesis advisor leads him to believe that this book may well serve the students in

their search and selection of topics in subjects currently under investigation in structural dynamics.

Should the text fulfill the expectations of the author in some measure, particularly the elucidation of this subject, he will then feel rewarded for his efforts in the preparation and development of the material in this book.

Louisville, KY, USA Mario Paz
December, 1979

Contents

Part I Structures Modeled as a Single-Degree-of-Freedom System

1 Undamped Single Degree-of-Freedom System 3
1.1 Degrees of Freedom 3
1.2 Undamped System 5
1.3 Springs in Parallel or in Series 6
1.4 Newton's Law of Motion 7
1.5 Free Body Diagram 8
1.6 D'Alembert's Principle 9
1.7 Solution of the Differential Equation of Motion 10
1.8 Frequency and Period 12
1.9 Amplitude of Motion 14
1.10 Response of SDF Using MATLAB Program 18
1.11 Summary 20
1.12 Problems 21

2 Damped Single Degree-of-Freedom System 29
2.1 Viscous Damping 29
2.2 Equation of Motion 29
2.3 Critically Damped System 30
2.4 Overdamped System 31
2.5 Underdamped System 32
2.6 Logarithmic Decrement 33
2.7 Response of SDF Using MATLAB Program 39
2.8 Summary 40
2.9 Problems 41

3 Response of One-Degree-of-Freedom System to Harmonic Loading 45
3.1 Harmonic Excitation: Undamped System 45
3.2 Harmonic Excitation: Damped System 47
3.3 Evaluation of Damping at Resonance 54
3.4 Bandwidth Method (Half-Power) to Evaluate Damping . 55
3.5 Energy Dissipated by Viscous Damping 57
3.6 Equivalent Viscous Damping 58

3.7 Response to Support Motion 60
3.7.1 Absolute Motion 60
3.7.2 Relative Motion 65
3.8 Force Transmitted to the Foundation 69
3.9 Seismic Instruments 71
3.10 Response of One-Degree-of-Freedom System to Harmonic Loading Using MATLAB 73
3.11 Summary 76
3.12 Analytical Problem 78
3.13 Problems 79

4 Response to General Dynamic Loading 85
4.1 Duhamel's Integral – Undamped System 85
4.1.1 Constant Force 86
4.1.2 Rectangular Load 88
4.1.3 Triangular Load 90
4.2 Duhamel's Integral-Damped System 95
4.3 Response by Direct Integration 95
4.4 Solution of the Equation of Motion 97
4.5 Summary 104
4.6 Analytical Problems 105
4.7 Problems 108

5 Response Spectra 115
5.1 Construction of Response Spectrum 115
5.2 Response Spectrum for Support Excitation 118
5.3 Tripartite Response Spectra 119
5.4 Response Spectra for Elastic Design 123
5.5 Influence of Local Soil Conditions 126
5.6 Response Spectra for Inelastic Systems 128
5.7 Response Spectra for Inelastic Design 131
5.8 Seismic Response Spectra Using MATLAB 135
5.9 Summary 139
5.10 Problems 140

6 Nonlinear Structural Response 143
6.1 Nonlinear Single-Degree-of-Freedom Model 143
6.2 Integration of the Nonlinear Equation of Motion 145
6.3 Constant Acceleration Method 146
6.4 Linear Acceleration Step-by-Step Method 148
6.5 The Newmark: β Method 150
6.6 Elastoplastic Behavior 156
6.7 Algorithm for Step-by-Step Solution for Elastoplastic Single-Degree-of-Freedom System 158
6.8 Response for Elastoplastic Behavior Using MATLAB . . 163
6.9 Summary 166
6.10 Problems 167

Part II Structures Modeled as Shear Buildings

7 Free Vibration of a Shear Building 173
7.1 Stiffness Equations for the Shear Building 173
7.2 Natural Frequencies and Normal Modes 176
7.3 Orthogonality Property of the Normal Modes 181
7.4 Rayleigh's Quotient 186
7.5 Summary 187
7.6 Problems 188

8 Forced Motion of Shear Buildings 193
8.1 Modal Superposition Method 193
8.2 Response of a Shear Building to Base Motion 199
8.3 Response by Modal Superposition Using MATLAB . . . 205
8.4 Harmonic Force Excitation 207
8.5 Harmonic Response: MATLAB Program 211
8.6 Combining Maximum Values of Modal Response 214
8.7 Summary 214
8.8 Problems 215

9 Reduction of Dynamic Matrices 219
9.1 Static Condensation 219
9.2 Static Condensation Applied to Dynamic Problems 222
9.3 Dynamic Condensation 233
9.4 Modified Dynamic Condensation 241
9.5 Summary 245
9.6 Problems 246

Part III Framed Structures Modeled as Discrete Multi-Degree-of-Freedom Systems

10 Dynamic Analysis of Beams 251
10.1 Shape Functions for a Beam Segment 251
10.2 System Stiffness Matrix 256
10.3 Inertial Properties-Lumped Mass 259
10.4 Inertial Properties—Consistent Mass 260
10.5 Damping Properties 264
10.6 External Loads 265
10.7 Geometric Stiffness 266
10.8 Equations of Motion 270
10.9 Element Forces At Nodal Coordinates 276
10.10 Program 13—Modeling Structures as Beams 278
10.11 Summary 284
10.12 Problems 285

11 Dynamic Analysis of Plane Frames 291
11.1 Element Stiffness Matrix for Axial Effects 291
11.2 Element Mass Matrix for Axial Effects 293
11.3 Coordinate Transformation 297
11.4 Modeling Structures as Plane Frames Using MATLAB . 304

11.5 Dynamic Analysis of Plane Frames Using MATLAB . . . 307
11.6 Summary . . . 312
11.7 Problems . . . 312

12 Dynamic Analysis of Grid Frames . . . 317
12.1 Local and Global Coordinate Systems . . . 317
12.2 Torsional Effects . . . 318
12.3 Stiffness Matrix for a Grid Element . . . 320
12.4 Consistent Mass Matrix for a Grid Element . . . 320
12.5 Lumped Mass Matrix for a Grid Element . . . 321
12.6 Transformation of Coordinates . . . 321
12.7 Modeling Structures as Grid Frames Using MATLAB . . . 327
12.8 Summary . . . 330
12.9 Problems . . . 331

13 Dynamic Analysis of Three-Dimensional Frames . . . 335
13.1 Element Stiffness Matrix . . . 335
13.2 Element Mass Matrix . . . 337
13.3 Element Damping Matrix . . . 337
13.4 Transformation of Coordinates . . . 338
13.5 Differential Equation of Motion . . . 342
13.6 Dynamic Response . . . 342
13.7 Modeling Structures as Space Frames Using MATLAB . . . 342
13.8 Summary . . . 347

14 Dynamic Analysis of Trusses . . . 349
14.1 Stiffness and Mass Matrices for the Plane Truss . . . 349
14.2 Transformation of Coordinates . . . 351
14.3 Stiffness and Mass Matrices for Space Trusses . . . 361
14.4 Equation of Motion for Space Trusses . . . 363
14.5 Modeling Structures as Space Trusses Using MATLAB . . . 364
14.6 Summary . . . 368
14.7 Problems . . . 368

15 Dynamic Analysis of Structures Using the Finite Element Method . . . 371
15.1 Plane Elasticity Problems . . . 372
15.1.1 Triangular Plate Element for Plane Elasticity Problems . . . 373
15.2 Plate Bending . . . 378
15.2.1 Rectangular Element for Plate Bending . . . 379
15.3 Summary . . . 384
15.4 Problems . . . 385

16 Time History Response of Multi-Degree-of-Freedom Systems . . . 389
16.1 Incremental Equations of Motion . . . 389
16.2 The Wilson-θ Method . . . 391

16.3 Algorithm for Step-by-Step Solution of a Linear System Using the Wilson-θ Method . . . 393
16.3.1 Initialization . . . 393
16.3.2 For Each Time Step . . . 393
16.4 Response by Step Integration Using MATLAB . . . 397
16.5 The Newmark Beta Method . . . 401
16.6 Elastoplastic Behavior of Framed-Structures . . . 402
16.7 Member Stiffness Matrix . . . 402
16.8 Member Mass Matrix . . . 405
16.9 Rotation of Plastic Hinges . . . 407
16.10 Calculation of Member Ductility Ratio . . . 408
16.11 Summary . . . 409
16.12 Problems . . . 409

Part IV Structures Modeled with Distributed Properties

17 Dynamic Analysis of Systems with Distributed Properties . . . 415
17.1 Flexural Vibration of Uniform Beams . . . 415
17.2 Solution of the Equation of Motion in Free Vibration . . . 417
17.3 Natural Frequencies and Mode Shapes for Uniform Beams . . . 418
17.3.1 Both Ends Simply Supported . . . 418
17.3.2 Both Ends Free (Free Beam) . . . 421
17.3.3 Both Ends Fixed . . . 422
17.3.4 One End Fixed and the Other End Free (Cantilever Beam) . . . 424
17.3.5 One End Fixed and the Other Simply Supported . . . 425
17.4 Orthogonality Condition Between Normal Modes . . . 426
17.5 Forced Vibration of Beams . . . 427
17.6 Dynamic Stresses in Beams . . . 432
17.7 Summary . . . 433
17.8 Problems . . . 434

18 Discretization of Continuous Systems . . . 437
18.1 Dynamic Matrix for Flexural Effects . . . 437
18.2 Dynamic Matrix for Axial Effects . . . 439
18.3 Dynamic Matrix for Torsional Effects . . . 441
18.4 Beam Flexure Including Axial-Force Effect . . . 443
18.5 Power Series Expansion of the Dynamic Matrix for Flexural Effects . . . 445
18.6 Power Series Expansion of the Dynamic Matrix for Axial and for Torsional Effects . . . 447
18.7 Power Series Expansion of the Dynamic Matrix Including the Effects of Axial Forces . . . 447
18.8 Summary . . . 448

Part V Special Topics: Fourier Analysis, Evaluation of Absolute Damping, Generalized Coordinates

19 Fourier Analysis and Response in the Frequency Domain . . . 453
19.1 Fourier Analysis . . . 453
19.2 Response to a Loading Represented by Fourier Series . . 454
19.3 Fourier Coefficients for Piecewise Linear Functions . . . 456
19.4 Exponential Form of Fourier Series . . . 457
19.5 Discrete Fourier Analysis . . . 458
19.6 Fast Fourier Transform . . . 461
19.7 Response in the Frequency Domain Using MATLAB . . 463
19.8 Summary . . . 469
19.9 Problems . . . 470

20 Evaluation of Absolute Damping from Modal Damping Ratios . . . 477
20.1 Equations for Damped Shear Building . . . 477
20.2 Uncoupled Damped Equations . . . 478
20.3 Conditions for Damping Uncoupling . . . 479
20.4 Summary . . . 487
20.5 Problems . . . 487

21 Generalized Coordinates and Rayleigh's Method . . . 491
21.1 Principle of Virtual Work . . . 491
21.2 Generalized Single-Degree-of-Freedom System–Rigid Body . . . 492
21.3 Generalized Single-Degree-of-Freedom System–Distributed Elasticity . . . 495
21.4 Shear Forces and Bending Moments . . . 500
21.5 Generalized Equation of Motion for a Multistory Building . . . 504
21.6 Shape Function . . . 506
21.7 Rayleigh's Method . . . 510
21.8 Improved Rayleigh's Method . . . 517
21.9 Shear Walls . . . 520
21.10 Summary . . . 523
21.11 Problems . . . 524

Part VI Random Vibration

22 Random Vibration . . . 531
22.1 Statistical Description of Random Functions . . . 532
22.2 Probability Density Function . . . 534
22.3 The Normal Distribution . . . 536
22.4 The Rayleigh Distribution . . . 537
22.5 Correlation . . . 538
22.6 The Fourier Transform . . . 542
22.7 Spectral Analysis . . . 543
22.8 Spectral Density Function . . . 547

22.9 Narrow-Band and Wide-Band Random Processes 550
22.10 Response to Random Excitation: Single-Degree-of-Freedom System 553
22.11 Response to Random Excitation: Multiple-Degree-of-Freedom System 559
22.11.1 Relationship Between Complex Frequency Response and Unit Impulse Response 559
22.11.2 Response to Random Excitation: Two-Degree-of-Freedom System 561
22.11.3 Response to Random Excitation: *N* Degree of Freedom System 565
22.12 Summary 567
22.13 Problems 568

Part VII Earthquake Engineering

23 Dynamic Method 575
23.1 Modal Seismic Response of Building 575
23.1.1 Modal Equation and Participation Factor 575
23.1.2 Modal Shear Force 576
23.1.3 Effective Modal Weight 578
23.1.4 Modal Lateral Forces 579
23.1.5 Modal Displacements 579
23.1.6 Modal Drift 579
23.1.7 Modal Overturning Moment 580
23.1.8 Modal Torsional Moment 580
23.2 Total Design Values 580
23.3 Scaling of Results 582
23.4 Summary 588
23.5 Problems 588

24 IBC-2018 and ASCE 7-16 589
24.1 Response Spectral Acceleration: S_S, S_1 589
24.2 Soil Modified Response Spectral Acceleration: S_{MS}, S_{M1} 590
24.3 Design Response Spectral Acceleration: S_{DS}, S_{D1} 590
24.4 Site Class Definition: A, B . . .F 591
24.5 Risk Category and Seismic Importance Factor (I_e) 592
24.6 Seismic Design Category (A, B, C, D, E and F) 592
24.7 Design Response Spectral Curve: S_a Vs. T 593
24.8 Determination of the Fundamental Period 595
24.9 Minimum Lateral Force Procedure [ASCE 7-16: Section 1.4.3] 596
24.10 Simplified Lateral Force Analysis Procedure [ASCE 7-16: Section 12.14.8 and IBC 2018 Section 1613.2.5.2] 596
24.10.1 Seismic Base Shear 597
24.10.2 Response Modification Factor *R* 597
24.10.3 Vertical Distribution of Lateral Forces 598

24.11 Equivalent Lateral Force Procedure [ASCE 7-16: Section 12.8] 598
24.11.1 Distribution of Lateral Forces [ASCE 7-16: Section 12.8.3] 600
24.11.2 Overturning Moments [ASCE 7-16: Section 12.8.5] 601
24.11.3 Horizontal Torsional Moment 601
24.11.4 *P*-Delta Effect (P–Δ) [ASCE 7-16: Section 12.8.7] 601
24.11.5 Story Drift [ASCE 7-16: Sect. 12.8.6] 602
24.12 Redundancy [ASCE 7-16: Section 12.3.4] 603
24.13 Earthquake Load Effect [ASCE 7-16 Section 12.4.2] . . . 604
24.14 Building Irregularities [ASCE 7-16 Section 12.3.2.1] . . . 604
24.15 Summary 609

Appendices 611

Selected Bibliography 625

Index 629

Part I

Structures Modeled as a Single-Degree-of-Freedom System

1 Undamped Single Degree-of-Freedom System

The analysis and design of structures to resist the effect produced by time dependent forces or motions requires conceptual idealizations and simplifying assumptions through which the physical system is represented by an idealized system known as the analytical or mathematical model. These idealizations or simplifying assumptions may be classified in the following three groups:

1. Material assumptions. These assumptions or simplifications include material properties such as homogeneity or isotrophy and material behaviors such as linearity or elasticity.
2. Loading assumptions. Some common loading assumptions are to consider concentrated forces to be applied at a geometric point, to assume forces suddenly applied, or to assume external forces to be constant or periodic.
3. Geometric Assumptions. A general assumption for beams, frames and trusses is to consider these structures to be formed by unidirectional elements. Another common assumption is to assume that some structures such as plates are two-dimensional systems with relatively small thicknesses. Of greater importance is to assume that continuous structures may be analyzed as discrete systems by specifying locations (nodes) and directions for displacements (nodal coordinates) in the structures as described in the following section.

1.1 Degrees of Freedom

In structural dynamics the number of independent coordinates necessary to specify the configuration or position of a system at any time is referred to as the number of degrees of freedom. In general, a continuous structure has an infinite number of degrees of freedom. Nevertheless, the process of idealization or selection of an appropriate mathematical model permits the reduction to a discrete number of degrees of freedom. Figure 1.1 shows some examples of structures that may be represented for dynamic analysis as one-degree-of-freedom-systems, that is, structures modeled as systems with a single displacement coordinate.

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M. Paz, Y. H. Kim, *Structural Dynamics*, https://doi.org/10.1007/978-3-319-94743-3_1

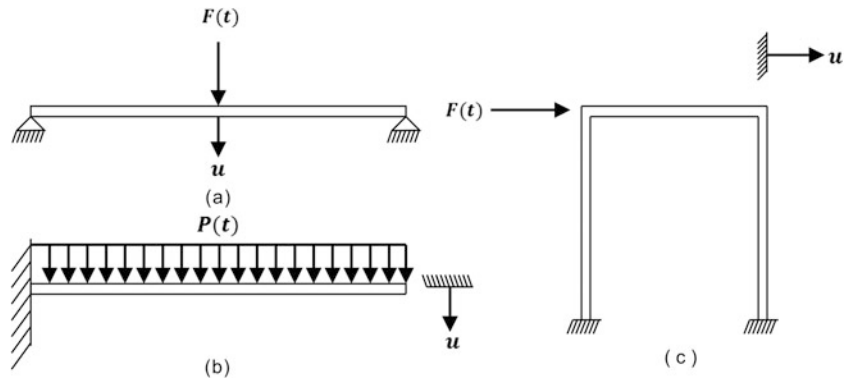

Fig. 1.1 Examples of structures modeled as one-degree-of-freedom systems

These one-degree-of-freedom systems may be conveniently described by the analytical model shown in Fig. 1.2 which has the following elements:

1. A mass element *m* representing the mass and inertial characteristic of the structure.
2. A spring element *k* representing the elastic restoring force and potential energy storage of the structure.
3. A damping element *c* representing the frictional characteristics and energy dissipation of the structure.
4. An excitation force *F(t)* representing the external forces acting on the structural system.

The force *F(t)* is written this way to indicate that it is a function of time. In adopting the analytical model shown in Fig. 1.2, it is assumed that each element in the system represents a single property; that is, the mass *m* represents only the property of inertia and not elasticity or energy dissipation, whereas the spring *k* represents exclusively elasticity and not inertia or energy dissipation. Finally, the damper *c* only dissipates energy. The reader certainly realizes that such "pure" elements do not exist in our physical world and that analytical models are only conceptual idealizations of real structures. As such, analytical models may provide complete and accurate knowledge of the behavior of the model itself, but only limited or approximate information on the behavior of the real physical system. Nevertheless, from a practical point of view, the information acquired from the analysis of the analytical model may very well be sufficient for an adequate understanding of the dynamic behavior of the physical system, including design and safety requirements.

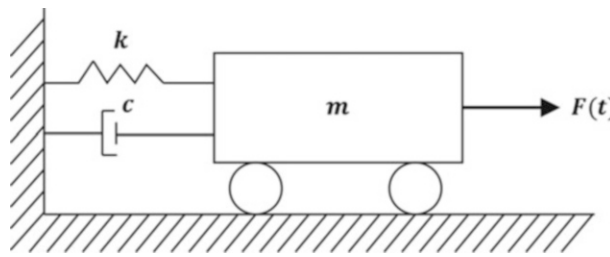

Fig. 1.2 Analytical model for one-degree-of-freedom systems

1.2 Undamped System

We start the study of structural dynamics with the analysis of a fundamental and simple system, the one-degree-of-freedom system in which we disregard or "neglect" frictional forces or damping. In addition, we consider the system, during its motion or vibration, to be free from external actions or forces. Under these conditions, the system is said to be in free vibration and it is in motion governed only by the influence of the so-called initial conditions, that is, the given displacement and velocity at time $t = 0$ when the study of the system is initiated. This undamped, one-degree-of-freedom system is often referred to as the simple undamped oscillator. It is usually represented as shown in Fig. 1.3a or Fig. 1.3b or any other similar arrangement. These two figures represent analytical models that are dynamically equivalent. It is only a matter of preference to adopt one or the other. In these models the mass m is restrained by the spring k and is limited to rectilinear motion along one coordinate axis, designated in these figures by the letter u.

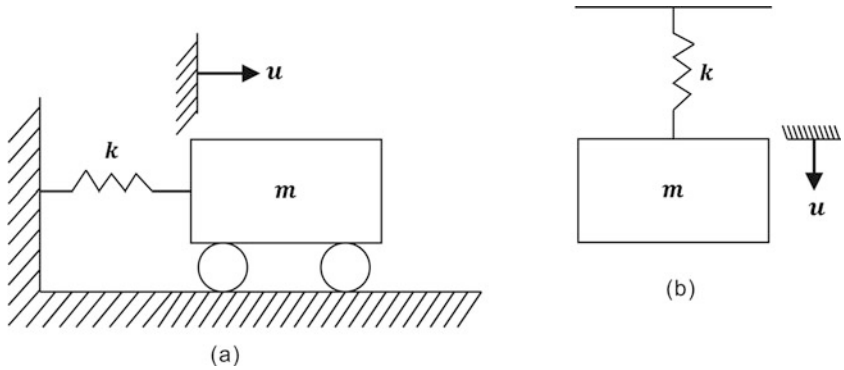

Fig. 1.3 Alternate representations of analytical models for one-degree-of-freedom systems

The mechanical characteristic of a spring is described by the relationship between the magnitude of the force F_s applied to its free end and the resulting end displacement u as shown graphically in Fig. 1.4 for three different springs.

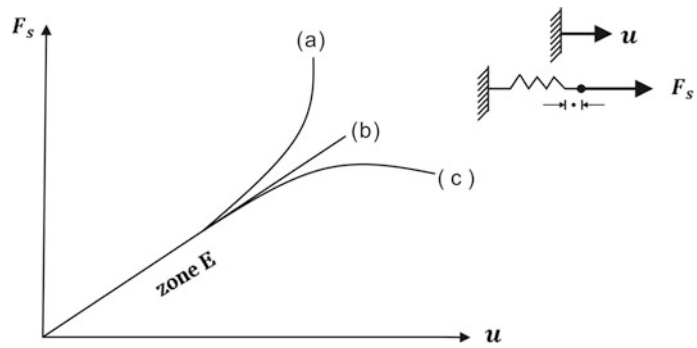

Fig. 1.4 Force-displacement relationship: (**a**) Hard spring, (**b**) Linear spring, (**c**) Soft spring

The curve labeled (a) in Fig. 1.4 represents the behavior of a hard spring in which the force required to produce a given displacement becomes increasingly greater as the spring is deformed. The second spring (b) is designated a linear spring because the deformation is directly proportional to the

force and the graphical representation of its characteristic is a straight line. The constant of proportionality between the force and displacement [slope of the line (b)] of a linear spring is referred to as the stiffness or the spring constant, usually designated by the letter k. Consequently, we may write the relationship between force and displacement for a linear spring as

$$F_s = ku \tag{1.1}$$

A spring with characteristics shown by curve (c) in Fig. 1.4 is known as a soft spring. For such a spring the incremental force required to produce additional deformation decreases as the spring deformation increases. Undoubtedly, the reader is aware from his or her previous exposure to analytical modeling of physical systems that the linear spring is the simplest type to manage mathematically. It should not come as a surprise to learn that most of the technical literature on structural dynamics deals with models using linear springs. In other words, either because the elastic characteristics of the structural system are, in fact, essentially linear, or simply because of analytical expediency, it is usually assumed that the force-deformation properties of the system are linear. In support of this practice, it should be noted that in many cases the displacements produced in the structure by the action of external forces or disturbances are small in magnitude (Zone E in Fig. 1.4), thus rendering the linear approximation close to the actual structural behavior.

1.3 Springs in Parallel or in Series

Sometimes it is necessary to determine the equivalent spring constant for a system in which two or more springs are arranged in parallel as shown in Fig. 1.5a or in series as in Fig. 1.5b.

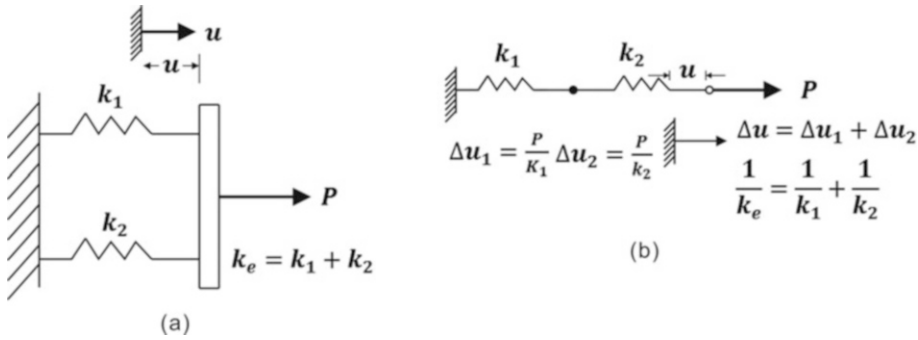

Fig. 1.5 Combination of springs: (**a**) Springs in parallel (**b**) Springs in series

For two springs in parallel the total force required to produce a relative displacement of their ends of one unit is equal to the sum of their spring constants. This total force is by definition the equivalent spring constant k_e and is given by

$$k_e = k_1 + k_2 \tag{1.2}$$

In general for n springs in parallel

$$k_e = \sum_{i=1}^{n} k_i \tag{1.3}$$

For two springs assembled in series as shown in Fig. 1.5b, the force P produces the relative displacements in the springs

$$\Delta u_1 = \frac{P}{k_1}$$

and

$$\Delta u_2 = \frac{P}{k_2}$$

Then, the total displacement u of the free end of the spring assembly is equal to $u = \Delta u_1 + \Delta u_2$, or substituting Δu_1 and Δu_2

$$u = \frac{P}{k_1} + \frac{P}{k_2} \tag{1.4}$$

Consequently, the force k_e necessary to produce one unit displacement (equivalent spring constant) is given by

$$k_e = \frac{P}{u}$$

Substituting u from this last relation into Eq. (1.4), we may conveniently express the reciprocal value of the equivalent spring constant as

$$\frac{1}{k_e} = \frac{1}{k_1} + \frac{1}{k_2} \tag{1.5}$$

In general for n springs in series the equivalent spring constant may be obtained from

$$\frac{1}{k_e} = \sum_{i=1}^{n} \frac{1}{k_i} \tag{1.6}$$

1.4 Newton's Law of Motion

We continue with the study of the simple oscillator depicted in Fig. 1.3. The objective is to describe its motion, that is, to predict the displacement or velocity of the mass m at any tine t, for a given set of initial conditions at time $t = 0$. The analytical relation between the displacement u, and time t, is given by Newton's Second Law of Motion, which in modern notation may be expressed as

$$\boldsymbol{F} = ma \tag{1.7}$$

Where $\boldsymbol{F}$ is the resultant force acting on a particle of mass m and a its resultant acceleration. The reader should recognize that Eq. (1.7) is a vector relation and as such it can be written in equivalent form in terms of its components along the coordinate axes x, y, and z, namely,

$$\sum F_x = ma_x \tag{1.8a}$$

$$\sum F_y = ma_y \tag{1.8b}$$

$$\sum F_z = ma_z \tag{1.8c}$$

The acceleration is defined as the second derivative of the position vector with respect to time; it follows that Eqs. (1.8) are indeed differential equations. The reader should also be reminded that these equations as stated by Newton are directly applicable only to bodies idealized as particles, that is, bodies assumed to possess mass but no volume. However, as is proved in elementary mechanics,

Newton's Law of Motion is also directly applicable to bodies of finite dimensions undergoing translatory motion.

For plane motion of a rigid body that is symmetric with respect to the reference plane of motion (x-y plane), Newton's Law of Motion yields the following equations:

$$\sum F_x = m(a_G)_x \tag{1.9a}$$

$$\sum F_y = m(a_G)_y \tag{1.9b}$$

$$\sum M_G = I_G\alpha \tag{1.9c}$$

In the above equations $(a_G)_x$ and $(a_G)_y$ are the acceleration components, along the x and y axes, of the center of mass G of the body; α is the angular acceleration; I_G is the mass moment of inertia of the body with respect to an axis through G, the center of mass; and $\sum M_G$ is the sum with respect to an axis through G, perpendicular to the x-y plane of the moments of all the forces acting on the body. Equations (1.9) are certainly also applicable to the motion of a rigid body in pure rotation about a fixed axis, alternatively, for this particular type of plane motion, Eq. (1.9c) may be replaced by

$$\sum M_0 = I_0\alpha \tag{1.9d}$$

in which the mass moment of inertia I_0 and the moment of the forces M_0 are determined with respect to the fixed axis of rotation. The general motion of a rigid body is described by two vector equations, one expressing the relation between the forces and the acceleration of the mass center, and another relating the moments of the forces and the angular motion of the body. This last equation expressed in its scalar components is rather complicated, but seldom needed in structural dynamics.

1.5 Free Body Diagram

At this point, it is advisable to follow a method conducive to an organized and systematic analysis in the solution of dynamics problems. The first and probably the most important practice to follow in any dynamic analysis is to draw a free body diagram of the system, prior to writing a mathematical description of the system.

The free body diagram (FBD), as the reader may recall, is a sketch of the body isolated from all other bodies, in which all the forces external to the body are shown. For the case at hand, Fig. 1.6b depicts the FBD of the mass m of the oscillator, displaced in the positive direction with reference to coordinate u and acted upon by the spring force $F_s = ku$ (assuming a linear spring). The weight of the body mg and the normal reaction N of the supporting surface are also shown for completeness, though these forces, acting in the vertical direction, do not enter into the equation of motion written for the u direction. The application of Newton's Law of Motion gives

$$-ku = m\ddot{u} \tag{1.10}$$

where the spring force acting in the negative direction has a minus sign, and where the acceleration has been indicated by $\ddot{u}$. In this notation, double overdots denote the second derivative with respect to time and obviously a single overdot denotes the first derivative with respect to time, that is, the velocity.

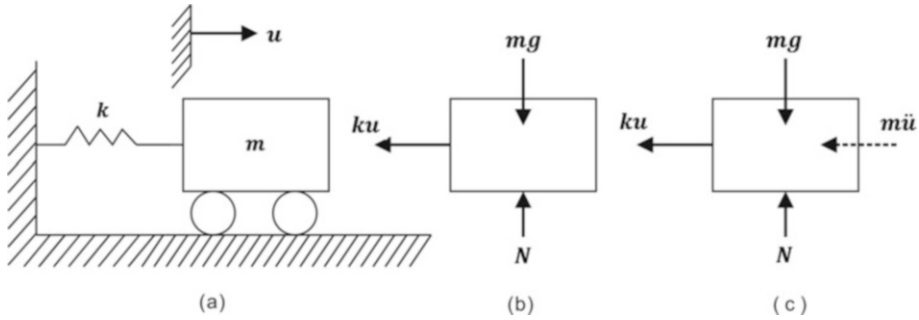

Fig. 1.6 Alternate free body diagrams: (**a**) Single degree-of-freedom system. (**b**) Showing only external forces, (**c**) Showing external and inertial forces

1.6 D'Alembert's Principle

An alternative approach to obtain Eq. (1.10) is to make use of D'Alembert's Principle which states that a system may be set in a state of dynamic equilibrium by adding to the external forces a fictitious force that is commonly known as the inertial force.

Figure 1.6c shows the FBD with inclusion of the inertial force $m\ddot{u}$. This force is equal to the mass multiplied by the acceleration, and should always be directed negatively with respect to the corresponding coordinate. The application of D'Alembert's Principle allows us to use equations of equilibrium in obtaining the equation of motion. For example, in Fig. 1.6c, the summation of forces in the u direction gives directly

$$m\ddot{u} + ku = 0 \tag{1.11}$$

which obviously is equivalent to Eq. (1.10).

The use of D'Alembert's Principle in this case appears to be trivial. This will not be the case for a more complex problem, in which the application of D'Alembert's Principle, in conjunction with the Principle of Virtual Work, constitutes a powerful tool of analysis. As will be explained later, the Principle of Virtual Work is directly applicable to any system in equilibrium. If follows then that this principle may also be applied to the solution of dynamic problems, provided that D'Alembert's Principle is used to establish the dynamic equilibrium of the system.

Illustrative Example 1.1

Show that the same differential equation is obtained for a body vibrating along a horizontal axis or for the same body moving vertically, as shown in Fig. 1.7a, b.

Solution:

The FBDs for these two representations of the simple oscillator are shown in Fig. 1.7c, e, in which the inertial forces have been included. Equating to zero in Fig. 1.7c the sum of the forces along the direction u, we obtain

$$m\ddot{u} + ku = 0 \tag{a}$$

When the body in Fig. 1.7d is in the static equilibrium position, the spring is stretched u_0 units and exerts a force $ku_0 = W$ upward on the body, where W is the weight of the body. When the body is displaced a distance u downward from this position of equilibrium the magnitude of the spring force

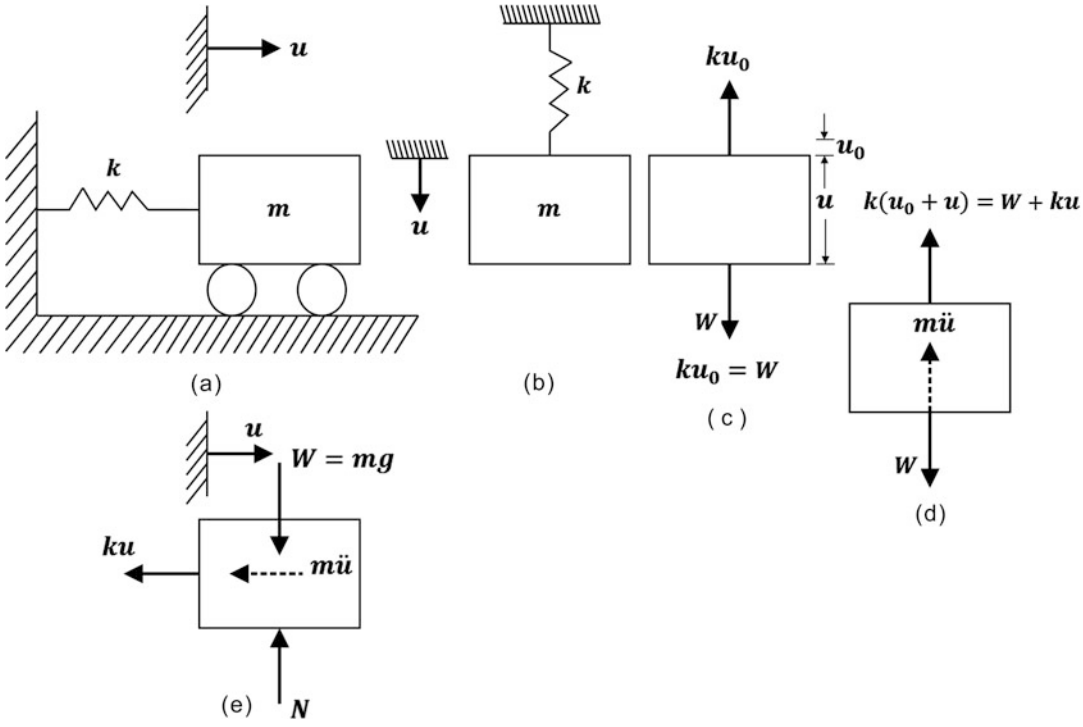

Fig. 1.7 Two representations of the simple oscillator and corresponding free body diagrams. (**a**) Idealized single degree of freedom system, (**b**) Alternative idealized single degree of freedom system, (**c**) dynamic equilibrium with inertial force, (**d**) static displacement due to gravity load, (**e**) dynamic equilibrium with inertial force of alternative model

is given by $F_s = k(u_0 + u)$ or $F_s = W + ku$ since $ku_0 = W$. Using this result and applying it to the body in Fig. 1.7e, we obtain from Newton's Second Law of Motion

$$-(W + ku) + W - m\ddot{u} = 0 \tag{b}$$

or

$$m\ddot{u} + ku = 0$$

which is identical to Eq. (a).

1.7 Solution of the Differential Equation of Motion

The next step toward our objective is to find the solution of the differential Eq. (1.11). We should again adopt a systematic approach and proceed first to classify this differential equation. Since the dependent variable u and second derivative $\ddot{u}$ appear in the first degree in Eq. (1.11), this equation is classified as linear and of second order. The fact that the coefficients of u and of $\ddot{u}$ (k and m, respectively) are constants and that the second member (right-hand side) of the equation is zero further classifies this equation as homogenous with constant coefficients. We should recall, probably with a certain degree of satisfaction, that a general procedure exists for the solution of linear

differential equations (homogenous or non-homogenous) of any order. For this simple, second-order differential equation we may proceed directly by assuming a trial solution given by

$$u = A\cos\omega t \tag{1.12}$$

or by

$$u = B\sin\omega t \tag{1.13}$$

where A and B are constants depending on the initiation of the motion while ω is a quantity denoting a physical characteristic of the system as it will be shown next. The substitution of Eq. (1.12) into Eq. (1.11) gives

$$\left(-m\omega^2 + k\right)A\cos\omega t = 0 \tag{1.14}$$

If this equation is to be satisfied at any time, the factor in parentheses must be equal to zero, or

$$\omega^2 = \frac{k}{m} \tag{1.15}$$

The reader should verify that Eq. (1.13) is also a solution of the differential Eq. (1.11), with ω also satisfying Eq. (1.15).

The positive root of Eq. (1.15),

$$\omega = \sqrt{\frac{k}{m}} \tag{1.16a}$$

is known as the natural frequency of the system for reasons that will soon be apparent.

The quantity ω in Eq. (1.16a) may be expressed in terms of the static displacement resulting from the weight $W = mg$ applied to the spring. The substitution into Eq. (1.16a) of $m = W/g$ results in

$$\omega = \sqrt{\frac{kg}{W}} \tag{1.16b}$$

Hence

$$\omega = \sqrt{\frac{g}{u_{st}}} \tag{1.16c}$$

where $u_{st} = W/k$ is the static displacement of the spring due to the weight W.

Since either Eq. (1.12) or Eq. (1.13) is a solution of Eq. (1.11), and since this differential equation is linear, the superposition of these two solutions, indicated by Eq. (1.17) below, is also a solution. Furthermore, Eq. (1.17), having two constants of integration, A and B, is, in fact, the general solution for this linear second-order differential equation.

$$u = A\cos\omega t + B\sin\omega t \tag{1.17}$$

The expression for velocity, $\dot{u}$, is simply found by differentiating Eq. (1.17) with respect to time, that is,

$$\dot{u} = -A\omega\sin\omega t + B\omega\cos\omega t \tag{1.18}$$

Next, we should determine the constants of integration A and B. These constants are determined from known values for the motion of the system which almost invariably are the displacement u_0 and

the velocity v_0 at the initiation of the motion, that is, at time $t = 0$. These two conditions are referred to as initial conditions, and the problem of solving the differential equation for the initial conditions is called an initial value problem.

After substituting, for $t = 0$, $u = u_0$, and $\dot{u} = v_0$ into Eqs. (1.17) and (1.18) we find that

$$u_0 = A \tag{1.19a}$$

$$v_0 = B\omega \tag{1.19b}$$

Finally, the substitution of A and B from Eq. (1.19) into Eq. (1.17) gives

$$u = u_0 \cos \omega t + \frac{v_0}{\omega} \sin \omega t \tag{1.20}$$

which is the expression of the displacement u of the simple oscillator as a function of the time variable t. Thus, we have accomplished our objective of describing the motion of the simple undamped oscillator modeling structures with a single degree of freedom.

1.8 Frequency and Period

An examination of Eq. (1.20) shows that the motion described by this equation is *harmonic* and therefore periodic, that is, it can be expressed by a sine or cosine function of the same frequency ω. The period may easily be found since the functions sine and cosine both have a period of 2π. The period T of the motion is determined from

$$\omega T = 2\pi$$

or

$$T = \frac{2\pi}{\omega} \tag{1.21}$$

The period is usually expressed in seconds per cycle or simply in seconds, with the tacit understanding that it is "per cycle". The reciprocal value of the period is the natural frequency f. From Eq. (1.21)

$$f = \frac{1}{T} = \frac{\omega}{2\pi} \tag{1.22}$$

The natural frequency f is usually expressed in hertz or cycles per second (cps). Because the quantity ω differs from the natural frequency f only by the constant factor 2π, ω also is sometimes referred to as the natural frequency. To distinguish between these two expressions for natural frequency, ω may be called the circular or angular natural frequency. Most often, the distinction is understood form the context or from the units. The natural frequency f is measured in cps as indicated, while the circular frequency ω should be given in radians per second (rad/sec).

Illustrative Example 1.2

Determine the natural frequency of the beam-spring system shown in Fig. 1.8 consisting of a weight of $W = 50.0$ lb attached to a horizontal cantilever beam through the coil spring k_2. The cantilever beam has a thickness $h = ¼$ in, a width $b = 1$ in. modulus of elasticity $E = 30 \times 10^6$ psi, and length $L = 12.5$ in. The coil spring has a stiffness $k_2 = 100$ (lb/in).

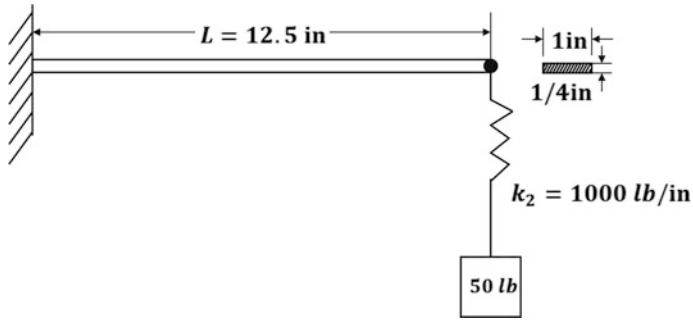

Fig. 1.8 System for Illustrative Example 1.2

Solution:
The deflection Δ at the free end of a uniform cantilever beam acted upon by a static force P at the free end is given by

$$\Delta = \frac{PL^3}{3EI}$$

The corresponding spring constant k_1 is then

$$k_1 = \frac{P}{\Delta} = \frac{3EI}{L^3}$$

where the cross-section moment of inertia $I = \frac{1}{12}bh^3$ (*for* a rectangular section). Now, the cantilever and the coil spring of this system are connected as springs in series. Consequently, the equivalent spring constant as given from Eq. (1.5) is

$$\frac{1}{k_e} = \frac{1}{k_1} + \frac{1}{k_2} \quad \text{(repeated)} \tag{1.5}$$

Substituting corresponding numerical values, we obtain

$$I = \frac{1}{12} \times 1 \times \left(\frac{1}{4}\right)^3 = \frac{1}{768}(\text{in})^4$$

$$k_1 = \frac{3 \times 30 \times 10^6}{(12.5)^3 \times 768} = 60 \text{ lb/in}$$

and

$$\frac{1}{k_e} = \frac{1}{60} + \frac{1}{100}$$

$$k_e = 37.5 \text{ lb/in}$$

The natural frequency for this system is then given by Eq. (1.16a) as

$$\omega = \sqrt{k_e/m} \quad (\text{m} = \text{W/g and g} = 386\,\text{in/sec}^2)$$

$$\omega = \sqrt{37.5 \times 386/50.0}$$

$$\omega = 17.01\,\text{rad/sec}$$

or using Eq. (1.22)

$$f = 2.71\,\text{cps} \tag{Ans}$$

1.9 Amplitude of Motion

Let us now examine in more detail Eq. (1.20), the solution describing the free vibratory motion of the undamped oscillator. A simple trigonometric transformation may show us that we can rewrite this equation in the equivalent forms, namely

$$u = C \sin(\omega t + \alpha) \tag{1.23}$$

or

$$u = C \cos(\omega t - \beta) \tag{1.24}$$

where

$$C = \sqrt{u_0^2 + (v_0/\omega)^2} \tag{1.25}$$

$$\tan \alpha = \frac{u_0}{v_0/\omega} \tag{1.26}$$

and

$$\tan \beta = \frac{v_0/\omega}{u_0} \tag{1.27}$$

The simplest way to obtain Eq. (1.23) or Eq. (1.24) is to multiply and divide Eq. (1.20) by the factor C defined in Eq. (1.25) and to define α (or β) by Eq. (1.26) [or Eq. (1.27)]. Thus

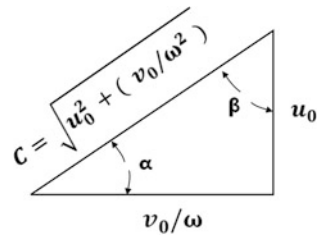

Fig. 1.9 Definition of angle α or angle β

$$u = C\left(\frac{u_0}{C}\cos \omega t + \frac{v_0/\omega}{C}\sin \omega t\right) \tag{1.28}$$

With the assistance of Fig. 1.9, we recognize that

$$\sin \alpha = \frac{u_0}{C} \tag{1.29}$$

and

$$\cos \alpha = \frac{v_0/\omega}{C} \tag{1.30}$$

The substitution of Eqs. (1.29) and (1.30) into Eq. (1.28) gives

$$u = C(\sin \alpha \cos \omega t + \cos \alpha \sin \omega t) \tag{1.31}$$

The expression within the parentheses of Eq. (1.31) is identical to $sin(\omega t + \alpha)$, which yields Eq. (1.23). Similarly, the reader should verify without difficulty, the form of solution given by Eq. (1.24).

The value of C in Eq. (1.23) (or Eq. (1.24)) is referred to as the amplitude of motion and the angle α (or β) as the phase angle. The solution for the motion of the simple oscillator is shown graphically in Fig. 1.10.

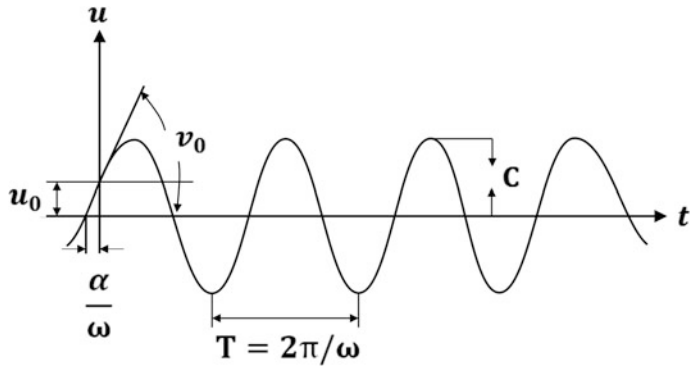

Fig. 1.10 Undamped free-vibration response

Illustrative Example 1.3

Consider the steel frame shown in Fig. 1.11a having a rigid horizontal member to which a horizontal dynamic force is applied. As part of the overall structural design it is required to determine the natural frequency of this structure. Two assumptions are made:

1. The masses of the columns are neglected.
2. The horizontal members are sufficiently rigid to prevent rotation at the tops of the columns.

These assumptions are not mandatory for the solution of the problem, but they serve to simplify the analysis. Under these conditions, the frame may be modeled by the spring-mass system shown in Fig. 1.11b.

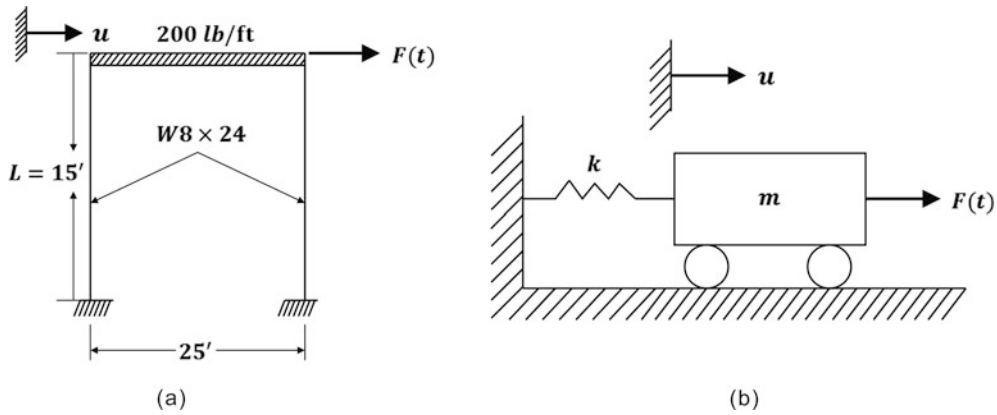

Fig. 1.11 One-degree-of-freedom frame and corresponding analytical model for Illustrative Example 1.3

Solution:
The parameters of this model may be computed as follows:

$$W = 200 \times 25 = 5000\,\text{lb}$$
$$I = 82.5\,\text{in}^4$$
$$E = 30 \times 10^6\,\text{psi}$$
$$k = \frac{12E(2I)}{L^3} = \frac{12 \times 30 \times 10^6 \times 165}{(15 \times 12)^3}$$
$$k = 10,185\,\text{lb/in} \qquad \text{(Ans)}$$

Note: A unit displacement of the top of a fixed column requires a force equal to $12EI/L^3$.
Therefore, the natural frequency from Eqs. (1.16b) and (1.22) is

$$f = \frac{1}{2\pi}\sqrt{\frac{kg}{W}} = \frac{1}{2\pi}\sqrt{\frac{10,185 \times 386}{5000}}$$
$$f = 4.46\,\text{cps} \qquad \text{(Ans)}$$

Illustrative Example 1.4
The elevated water tower tank with a capacity for 5000 gallons of water shown in Fig. 1.12a has a natural period in lateral vibration of 1.0 sec when empty. When the tank is full of water, its period lengthens to 2.2 sec. Determine the lateral stiffness k of the tower and the weight W of the tank. Neglect the mass of the supporting columns (one gallon of water weighs approximately 8.34 lb).

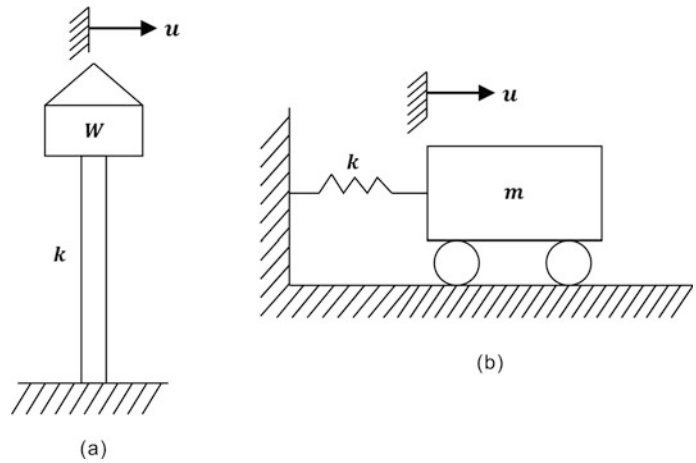

Fig. 1.12 (**a**) Water tower tank of Illustrative Example 1.4. (**b**) Analytical model

Solution:
In its lateral motion, the water tower is modeled by the simple oscillator shown in Fig. 1.12b in which k is the lateral stiffness of the tower and m is the vibrating mass of the tank.

(a) Natural frequency ω_E (tank empty):

$$\omega_E = \frac{2\pi}{T_E} = \frac{2\pi}{1.0} = \sqrt{\frac{kg}{W}} \tag{a}$$

(b) Natural frequency ω_F (tank full of water)

Weight of water W_w:

$$W_w = 5000 \times 8.34 = 41,700\,\text{lb}$$
$$\omega_F = \frac{2\pi}{T_F} = \frac{2\pi}{1.0} = \sqrt{\frac{kg}{W + 41,700}} \tag{b}$$

Squaring Eqs. (a) and (b) and dividing correspondingly the left and right sides of these equations, results in

$$\frac{(2.2)^2}{(1.0)^2} = \frac{W + 41,700}{W}$$

and solving for W

$$W = 10,860\,\text{lb} \tag{Ans}$$

Substituting in Eq. (a), $W = 10{,}860$ lb and $g = 386$ in/sec^2, yields

$$\frac{2\pi}{1.00} = \sqrt{\frac{k386}{10,860}}$$

and

$$k = 1110\,\text{lb/in} \tag{Ans}$$

Illustrative Example 1.5

The steel frame shown in Fig. 1.13a is fixed at the base and has a rigid top W that weighs 1000 lb. Experimentally, it has been found that its natural period in lateral vibration is equal to 1/10 of a second. It is required to shorten or lengthen its period by 20% by adding weight or strengthening the columns. Determine needed additional weight or additional stiffness (neglect the weight of the columns).

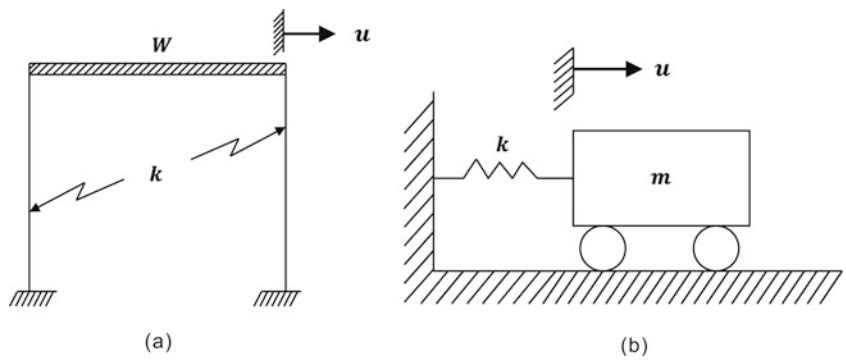

Fig. 1.13 (**a**) Frame of Illustrative Example 1.5. (**b**) Analytical model

Solution:
The frame is modeled by the spring-mass system shown in Fig. 1.13b. Its stiffness is calculated from

$$\omega = \frac{2\pi}{T} = \sqrt{\frac{kg}{W}}$$

as

$$\frac{2\pi}{0.1} = \sqrt{\frac{kg}{1000}} \quad \left(g = 386 \text{ in}/\sec^2\right)$$

or

$$k = 10,228 \text{ lb/in}$$

(a) Lengthen the period to $T_L = 1.2 \times 0.10 = 0.12$ sec by adding weight ΔW:

$$\omega = \frac{2\pi}{0.12} = \sqrt{\frac{10,228 \times 386}{1000 + \Delta W}}$$

Solve for ΔW:

$$\Delta W = 440 \text{ lb} \qquad \text{(Ans)}$$

(b) Shorten the period to $T_s = 0.8 \times 0.1 = 0.08$ sec by strengthening columns in Δk:

$$\omega = \frac{2\pi}{0.08} = \sqrt{\frac{(10,228 + \Delta K)(386)}{1000}}$$

Solve for Δk:

$$\Delta k = 5753 \text{ lb/in} \qquad \text{(Ans)}$$

1.10 Response of SDF Using MATLAB Program

Plot the displacement as a function of time, u(t) ranging from 0 to 5 sec.
Given:

- Initial conditions: $u_0 = 1$ in. and $\dot{u}_0 = 0.2$ in./ sec
- Natural period: $T = 0.5$ sec (Fig. 1.14).

```
clear all
clc

%%%%-GIVEN VALUES-%%%%
%%% Set Initial Conditions

u0=1;                                          %Initial Displacement
v0=2;                                          %Initial Velocity

%%%Define period and frequency

T=0.5;                                         %Natural Period
omega=2*pi/T;                                  %Natural Frequency

%%%%-ESTIMATION-%%%%
%%% Generate time stamp equally between 0 to 5 sec with a total of 500 %%% data

t=linspace(0,5,500);

%%%Calculate the displacement response

A = u0;                                                % Eq. 19a
B = v0/omega;                                          % Eq. 19b

u= A*cos(omega*t)+B*sin(omega*t);                      % Eq. 20

C= sqrt(A^2+B^2);                                      %Maximum amplitude (Eq. 1.25)
udot = -A*omega*sin(omega*t)+B*omega*cos(omega*t);     %Velocity


%%%Plot the response curve

plot(t, u);
title ('Response');
xlabel ('Time (sec)');                                 %Label the x-axis of the plot
ylabel ('Displacement (in.)');                         %Label the y-axis of the plot
```

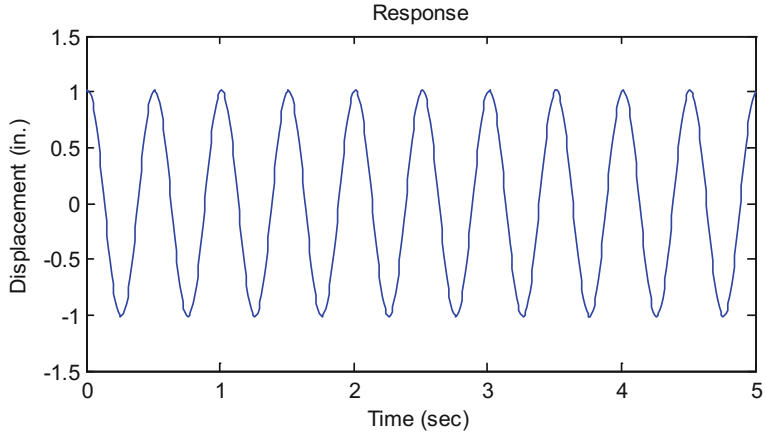

Fig. 1.14 Response of SDF using MATLAB

1.11 Summary

Several basic concepts were introduced in this chapter:

1. The analytical or mathematical model of a structure is an idealized representation for its analysis.
2. The number of degrees of freedom of a structural system is equal to the number of independent coordinates necessary to describe its position.
3. The free body diagram (FBD) for dynamic equilibrium (to allow application of D'Alembert's Principle) is a diagram of the system isolated from all other bodies, showing all the external forces on the system, including the inertial force.
4. The stiffness or spring constant of a linear system is the force necessary to produce a unit displacement.
5. The differential equation of the undamped simple oscillator in free motion is

$$m\ddot{u} + ku = 0$$

and its general solution is

$$u = A\cos\omega t + B\sin\omega t$$

where A and B are constants of integration determined from initial conditions of the displacement u_0 and of the velocity v_0:

$$A = u_0$$

$$B = v_0/\omega$$

$\omega = \sqrt{k/m}$ is the natural frequency in rad/ sec

$f = \dfrac{\omega}{2\pi}$ is the natural frequency in cps

$T = \dfrac{1}{f}$ is the natural period in seconds

6. The equation of motion may be written in the alternate forms:

$$u = C\sin(\omega t + \alpha)$$

or

$$u = C\cos(\omega t - \beta)$$

where

$$C = \sqrt{u_0^2 + (v_0/\omega)^2}$$

and

$$\tan\alpha = \frac{u_0}{v_0/\omega}$$

$$\tan\beta = \frac{v_0/\omega}{u_0}$$

1.12 Problems

Problem 1.1
Determine the natural period for the system in Fig. P1.1. Assume that the beam and springs supporting the weight W are massless.

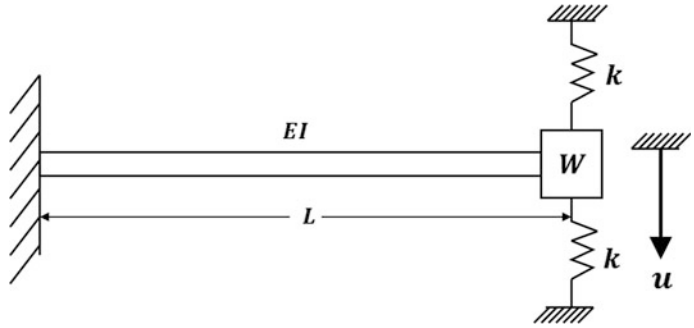

Fig. P1.1

Problem 1.2
The following numerical values are given in Problem 1.1: $L = 100$ in. $EI = 10^8$ (lb.in^2), $W = 3000$ lb, $k = 2000$ lb/in. If the weight W has an initial displacement of $u_0 = 1.0$ in and an initial velocity of $u_0 = 20$ in/sec, determine the displacement and the velocity 1 sec later.

Problem 1.3
Determine the natural frequency for horizontal motion of the steel frame in Fig. P1.3. Assume the horizontal girder to be infinitely rigid and neglect the mass of the columns.

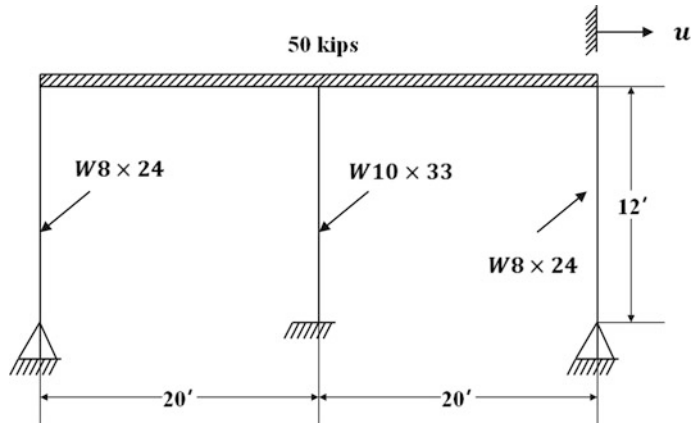

Fig. P1.3

Problem 1.4
Calculate the natural frequency in the horizontal mode of the steel frame in Fig. P1.4 for the following cases:

(a) The horizontal member is assumed to be rigid.
(b) The horizontal member is flexible and made of steel sections-- W 8 × 24.

Hint: *When the girder stiffness needs to be considered to determine the effective stiffness of column fixed on the ground, the following formula is useful.*

$$k_e = k_c(left) + k_c(right)$$
$$= \frac{24E_cI_c}{h^3}\frac{(1+6\gamma)}{(4+6\gamma)}$$

where,

$$\gamma = \frac{I_g/L}{I_c/h}$$

I_g and L are the moment of inertia and span length for girder.

I_c and h are the moment of inertia and height of column.

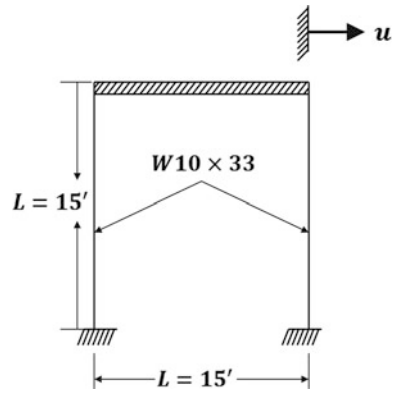

Fig. P1.4

Problem 1.5

Determine the natural frequency of the fixed beam in Fig. P1.5 carrying a concentrated weight W *at* its center. Neglect the mass of the beam.

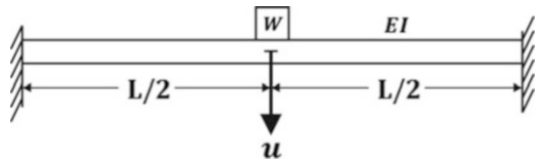

Fig. P1.5

Problem 1.6

The numerical values for Problem 1.5 are given as: $L = 120$ in. $EI = 10^9$ (lb.in^2), $W = 5000$ lb, with initial conditions $u_0 = 0.5$ in and $v_0 = 15$ in/sec. Determine the displacement, velocity, and acceleration of W *at* $t = 2$ sec later. Plot the responses (i.e., displacement, velocity, and acceleration) using MATLAB and determine the maximum amplitude.

Problem 1.7

Consider the simple pendulum of weight W illustrated in Fig. P1.7. If the cord length is L, determine the motion of the pendulum. The initial angular displacement and initial angular velocity are θ_0 and $\dot{\theta}_0$, respectively (Assume the angle θ is small).

Note: A simple pendulum is a particle of concentrated weight that oscillates in a vertical arc and is supported by a weightless cord. The only forces acting are those of gravity and the cord tension (i.e., frictional resistance is neglected).

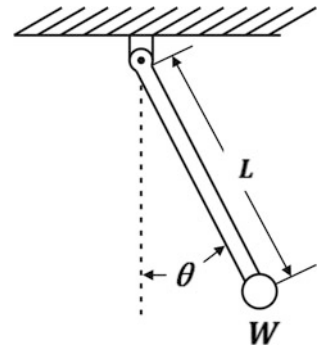

Fig. P1.7

Problem 1.8

A diver standing at the end of a diving board that cantilevers 2 ft oscillates at a frequency 2 cps. Determine the flexural rigidity EI of the diving board. The weight of the diver is 180 lb. (Neglect the mass of the diving board).

Problem 1.9

A bullet weighing 0.2 lb is fired at a speed of 100 ft/sec into a wooden block weighing $W = 50$ lb and supported by a spring of stiffness 300 lb/in (Fig. P1.9). Determine the displacement $u(t)$ and velocity $u(t)$ of the block after t sec.

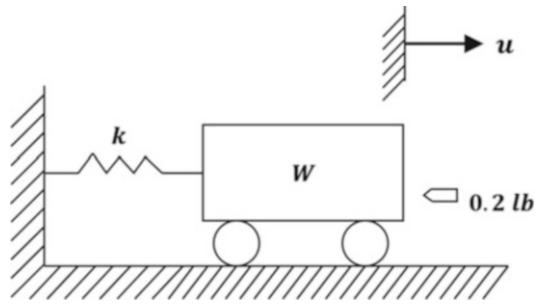

Fig. P1.9

Problem 1.10

An elevator weighing 500 lb is suspended from a spring having a stiffness of $k = 600$ lb/in. A weight of 300 lb is suspended through a cable to the elevator as shown schematically in Fig. P1.10. Determine the equation of motion of the elevator if the cable of the suspended weight suddenly breaks.

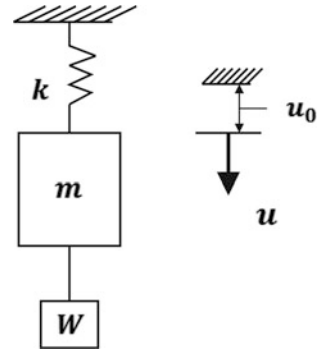

Fig. P1.10

Problem 1.11

Write the differential equation of motion for the inverted pendulum shown in Fig. P1.11 and determine its natural frequency. Assume small oscillations, and neglect the mass of the rod.

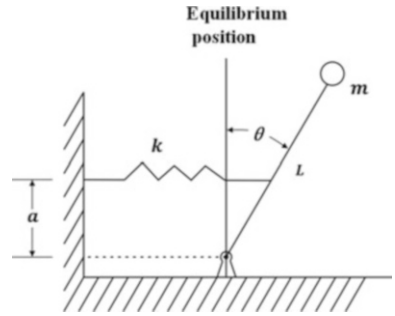

Fig. P1.11

Problem 1.12

Show that the natural frequency for the system of Problem 1.11 may be expressed as

$$f = f_0\sqrt{1 - \frac{W}{W_{cr}}}$$

where $W = mg$, W_{cr} is the critical buckling weight, and f_0 is the natural frequency neglecting the effect of gravity.

Problem 1.13

A vertical pole of length L and flexural rigidity EI carries a mass m at its top, as shown in Fig. P1.13. Neglecting the weight of the pole, derive the differential equation for small horizontal vibrations of the mass, and find the natural frequency. Assume that the effect of gravity is small and neglect nonlinear effects.

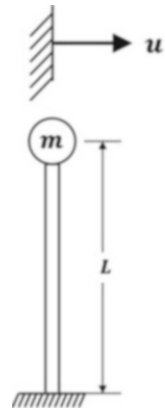

Fig. P1.13

Problem 1.14

Show that the natural frequency for the system in Problem 1.13 may be expressed as

$$f = f_0\sqrt{1 - \frac{W}{W_{cr}}}$$

where f_0 is the natural frequency calculated neglecting the effect of gravity and W_{cr} is the critical buckling weight.

Problem 1.15

Determine an expression for the natural frequency of the weight W in each of the cases shown in Fig. P1.15. The beams are uniform of cross-sectional moment of inertia I and modulus of elasticity E. Neglect the mass of the beams.

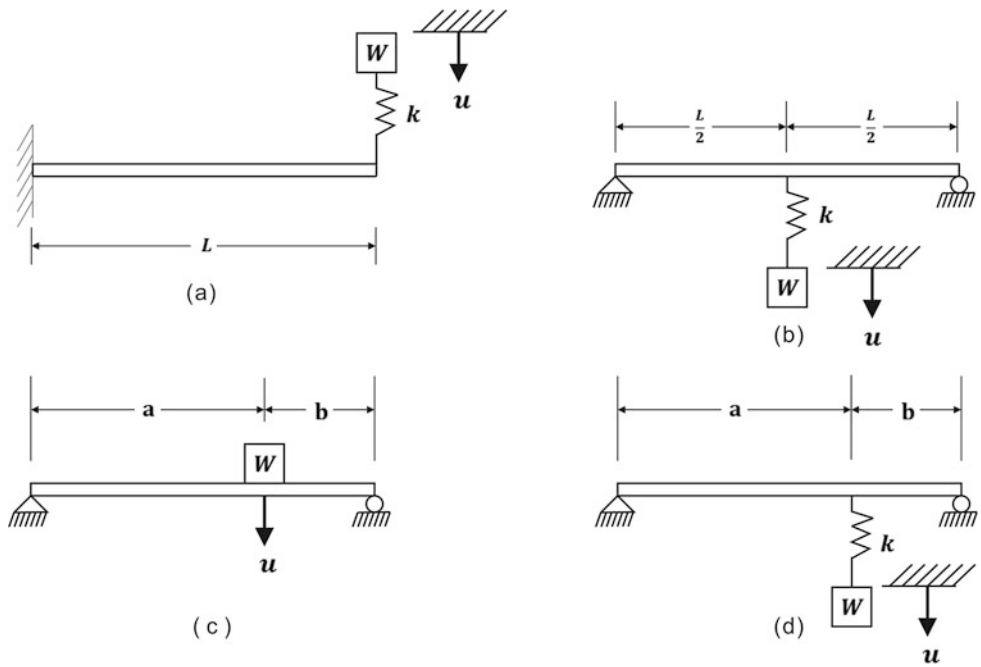

Fig. P1.15

Problem 1.16

A system is modeled by two freely vibrating masses m_1 and m_2 interconnected by a spring having a constant k as shown in Fig. P1.16. Determine for this system the differential equation of motion for the relative displacement $u_r = u_2 - u_1$ between the two masses. Also determine the corresponding natural frequency of the system.

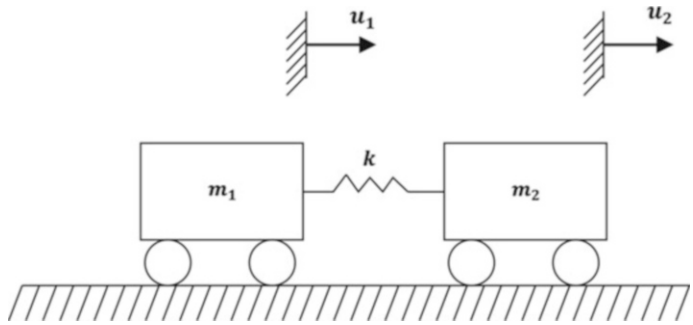

Fig. P1.16

Problem 1.17

Calculate the natural frequency for the vibration of the mass m shown in Fig. P1.17. Member AE is rigid with a hinge at C and a supporting spring of stiffness k at D. (Problem contributed by Professors Vladimir N. Alekhin and Alesksey A. Antipin of the Urals State Technical University, Russia.)

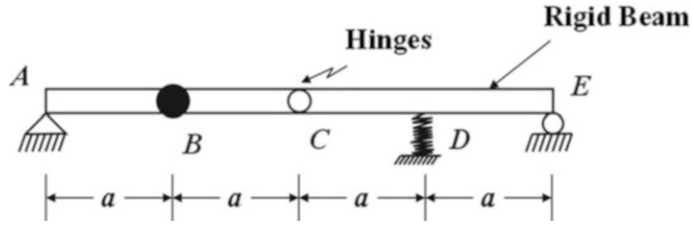

Fig. P1.17

Problem 1.18

Determine the natural frequency of vibration in the vertical direction for the rigid foundation (Fig. P1.18) transmitting a uniformly distributed pressure on the soil having a resultant force $Q = 2000$ kN. The area of the foot of the foundation is $A = 10$ m^2. The coefficient of elastic compression of the soil is $k = 25{,}000$ kN/m^3. (Problem contributed by Professors Vladimir N. Alekhin and Alesksey A. Antipin of the Urals State Technical University, Russia.)

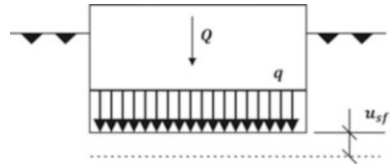

Fig. P1.18

Problem 1.19

Calculate the natural frequency of free vibration of the chimney on elastic foundation (Fig. P1.19), permitting the rotation of the structure as a rigid body about the horizontal axis *x*-*x*. The total weight of the structure is *W* with its center of gravity at a height *h* from the base of the foundation. The mass moment of inertia of the structure about the axis *x*-*x* is *I* and the rotational stiffness of the soil is *k* (resisting moment of the soil per unit rotation). (Problem contributed by Professors Vladimir N. Alekhin and Alesksey A. Antipin of the Urals State Technical University, Russia.)

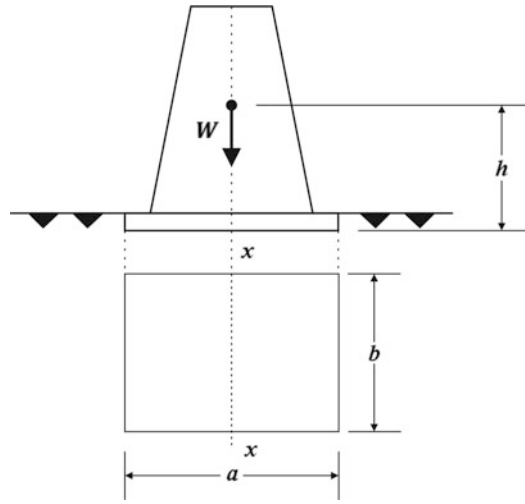

Fig. P1.19

2 Damped Single Degree-of-Freedom System

We have seen in the preceding chapter that the simple oscillator under idealized conditions of no damping, once excited, will oscillate indefinitely with a constant amplitude at its natural frequency. However, experience shows that it is not possible to have a device that vibrates under these ideal conditions. Forces designated as frictional or damping forces are always present in any physical system undergoing motion. These forces dissipate energy; more precisely, the unavoidable presence of these frictional forces constitute a mechanism through which the mechanical energy of the system, kinetic or potential energy, is transformed to other forms of energy such as heat. The mechanism of this energy transformation or dissipation is quite complex and is not completely understood at this time. In order to account for these dissipative forces in the analysis of dynamic systems, it is necessary to make some assumptions about these forces, on the basis of experience.

2.1 Viscous Damping

In considering damping forces in the dynamic analysis of structures, it is usually assumed that these forces are proportional to the magnitude of the velocity, and opposite to the direction of motion. This type of damping is know as viscous damping; it is the type of damping force that could be developed in a body restrained in its motion by a surrounding viscous fluid.

There are situations in which the assumption of viscous damping is realistic and in which the dissipative mechanism is approximately viscous. Nevertheless, the assumption of viscous damping is often made regardless of the actual dissipative characteristics of the system. The primary reason for such wide use of this assumed type of damping is that it leads to a relatively simple mathematical analysis.

2.2 Equation of Motion

Let us assume that we have modeled a structural system as a simple oscillator with viscous damping, as shown in Fig. 2.1a. In this figure m and k are, respectively, the mass and spring constant of the oscillator and c is the viscous damping coefficient.

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M. Paz, Y. H. Kim, *Structural Dynamics*, https://doi.org/10.1007/978-3-319-94743-3_2

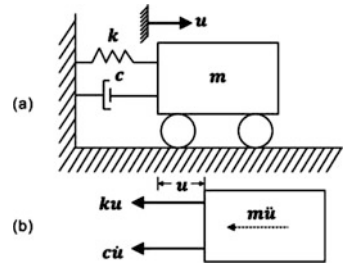

Fig. 2.1 (**a**) Viscous damped oscillator. (**b**) Free body diagram

We proceed, as in the case of the undamped oscillator, to draw the free body diagram (FBD) and apply Newton's Law to obtain the differential equation of motion. Figure 2.1b shows the FBD of the damped oscillator in which the inertial force $m\ddot{u}$ is also shown, so that we can use D'Alembert's Principle. The summation of forces in the u direction gives the differential equation of motion

$$m\ddot{u} + c\dot{u} + ku = 0 \tag{2.1}$$

The reader may verify that a trial solution $u = A \sin \omega t$ or $u = B \cos \omega t$ will not satisfy Eq. (2.1). However, the exponential function $u = Ce^{pt}$ does satisfy this equation. Substitution of this function into Eq. (2.1) results in the equation

$$mCp^2e^{pt} + cCpe^{pt} + kCe^{pt} = 0$$

which, after cancellation of the common factors, reduces to an equation called the characteristic equation for the system, namely

$$mp^2 + cp + k = 0 \tag{2.2}$$

The roots of this quadratic equation are

$$\begin{matrix} P_1 \\ P_2 \end{matrix} = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}} \tag{2.3}$$

thus the general solution of Eq. (2.1) is given by the superposition of the two possible solutions, namely

$$u(t) = C_1e^{p_1t} + C_2e^{p_2t} \tag{2.4}$$

where C_1 and C_2 are constant of integration to be determined from the initial conditions.

The final form of Eq. (2.4) depends on the sign of the expression under the radical in Eq. (2.3). Three distinct cases may occur; the quantity under the radical may either be zero, positive or negative. The limiting case in which the quantity under the radical is zero is treated first. The damping present in this case is called critical damping.

2.3 Critically Damped System

For a system oscillating with critical damping ($c = c_{cr}$), the expression under the radical in Eq. (2.3) is equal to zero, that is

$$\left(\frac{c_{cr}}{2m}\right)^2 - \frac{k}{m} = 0 \tag{2.5}$$

or

$$c_{cr} = 2\sqrt{km} \tag{2.6}$$

where c_{cr} designates the critical damping value. Since the natural frequency of the undamped system is given by $\omega = \sqrt{k/m}$, the critical damping coefficient given by Eq. (2.6) may also be expressed in alternative expressions as

$$c_{cr} = 2m\omega \quad \text{or} \quad c_{\text{cr}} = \frac{2k}{\omega} \tag{2.7}$$

In a critically damped system the roots of the characteristic equation are equal, and from Eq. (2.3), they are

$$p_1 = p_2 = -\frac{c_{cr}}{2m} \tag{2.8}$$

Since the two roots are equal, the general solution given by Eq. (2.4) would provide only one independent constant of integration, hence, one independent solution, namely

$$u_1(t) = C_1 e^{-(c_{cr}/2m)t} \tag{2.9}$$

Another independent solution may be found by using the function

$$u_2(t) = C_2 t u_1(t) = C_2 t e^{-(c_{cr}/2m)t} \tag{2.10}$$

$u_2(t)$, as the reader may verify, also satisfies the differential Eq. (2.1). The general solution for a critically damped system is then given by the superposition of these two solutions,

$$u(t) = (C_1 + C_2 t)e^{-(c_{cr}/2m)t} \tag{2.11}$$

2.4 Overdamped System

In an overdamped system, the damping coefficient is greater that the value for critical damping, namely

$$c > c_{cr} \tag{2.12}$$

Therefore, the expression under the radical of Eq. (2.3) is positive; thus the two roots of the characteristic equation are real and distinct, and consequently the solution is given directly by Eq. (2.4). It should be noted that for the overdamped or the critically damped system, the resulting motion is not oscillatory; the magnitude of the oscillations decays exponentially with time to zero. Figure 2.2 depicts graphically the response for the simple oscillator with critical damping. The response of the overdamped system is similar to the motion of the critically damped system of Fig. 2.2, but in the return toward the neutral position requires more time as the damping is increased.

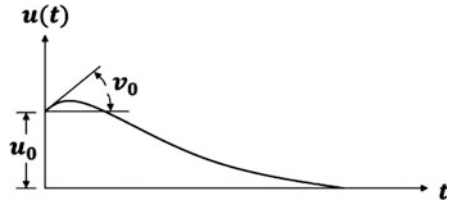

Fig. 2.2 Free-vibration response with critical damping

2.5 Underdamped System

When the value of the damping coefficient is less than the critical value ($c < c_{cr}$), which occurs when the expression under the radical is negative, the roots of the characteristic Eq. (2.3) are complex conjugates, so that

$$\begin{matrix} P_1 \\ P_2 \end{matrix} = -\frac{c}{2m} \pm i\sqrt{\frac{k}{m} - \left(\frac{c}{2m}\right)^2} \tag{2.13}$$

where $i = \sqrt{-1}$ is the imaginary unit.

For this case, it is convenient to make use of Euler's equations which relate exponential and trigonometric functions, namely,

$$\begin{aligned} e^{ix} &= \cos x + i \sin x \\ e^{-ix} &= \cos x - i \sin x \end{aligned} \tag{2.14}$$

The substitution of the roots p_1 and p_2 from Eq. (2.13) into Eq. (2.4) together with the use of Eq. (2.14) gives the following convenient form for the general solution of the underdamped system:

$$u(t) = e^{-(c/2m)t}(A \cos \omega_D t + B \sin \omega_D t) \tag{2.15}$$

where A and B are redefined constants of integration and ω_D, the damping frequency of the system, is given by

$$\omega_D = \sqrt{\frac{k}{m} - \left(\frac{c}{2m}\right)^2} \tag{2.16}$$

or

$$\omega_D = \omega\sqrt{1 - \xi^2} \tag{2.17}$$

This last result is obtained after substituting, in Eq. (2.16), the expression for the undamped natural frequency

$$\omega = \sqrt{\frac{k}{m}} \tag{2.18}$$

and defining the damping ratio of the system as

$$\xi = \frac{c}{c_{cr}} \tag{2.19}$$

where the critical camping coefficient c_{cr} is given by Eq. (2.6).

Finally, when the initial conditions of displacement and velocity, u_0 and v_0, are introduced, the constants of integration can be evaluated and substituted into Eq. (2.15), giving

$$u(t) = e^{-\xi\omega t}\left(u_0 \cos \omega_D t + \frac{v_0 + u_0 \xi\omega}{\omega_D} \sin \omega_D t\right) \tag{2.20}$$

Alternatively, this expression can be written as

$$u(t) = Ce^{-\xi\omega t} \cos(\omega_D t - \alpha) \tag{2.21}$$

where

$$C = \sqrt{u_0^2 + \frac{(v_0 + u_0\xi\omega)^2}{\omega_D^2}} \tag{2.22}$$

and

$$\tan\alpha = \frac{(v_0 + u_0\xi\omega)}{\omega_D u_0} \tag{2.23}$$

A graphical record of the response of an underdamped system with initial displacement u_0 but starting with zero velocity ($v_0 = 0$) is shown in Fig. 2.3. It may be seen in this figure that the motion is oscillatory, but not periodic. The amplitude of vibration is not constant during the motion but decreases for successive cycles; nevertheless, the oscillations occur at equal intervals of time. This time interval is designated as the damped period of vibration and is given from Eq. (2.17) by

$$T_D = \frac{2\pi}{\omega_D} = \frac{2\pi}{\omega\sqrt{1-\xi^2}} \tag{2.24}$$

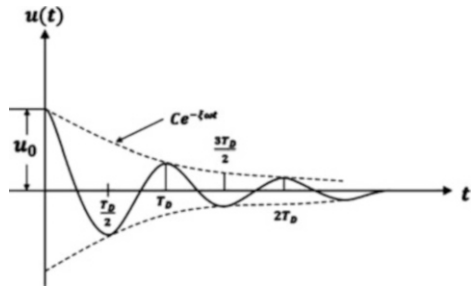

Fig. 2.3 Free-vibration response for undamped system

The value of the damping coefficient for real structures is much less than the critical damping coefficient and usually ranges between 2 and 10% of the critical damping value. Substituting for the extreme value $\xi = 0.10$ into Eq. (2.17),

$$\omega_D = 0.995\omega \tag{2.25}$$

It can be seen that the frequency of vibration for a system with as much as a 10% damping ratio is essentially equal to the undamped natural frequency. Thus in practice, the natural frequency for a damped system may be taken to be equal to the undamped natural frequency.

2.6 Logarithmic Decrement

A practical method for determining experimentally the damping coefficient of a system is to initiate free vibration, obtain a record of the oscillatory motion, such as the one shown in Fig. 2.4, and measure the rate of decay of the amplitude of motion. The decay may be conveniently expressed by the logarithmic decrement δ which is defined as the natural logarithm of the ratio of any two successive peak amplitudes, u_1 and u_2, in the free vibration, that is

$$\delta = \ln \frac{u_1}{u_2} \tag{2.26}$$

The evaluation of damping from the logarithmic decrement follows. Consider the damped vibration motion represented graphically in Fig. 2.4 and given analytically by Eq. (2.21) as

$$u(t) = Ce^{-\xi\omega t} \cos(\omega_D t - \alpha) \qquad \text{((2.21) repeated)}$$

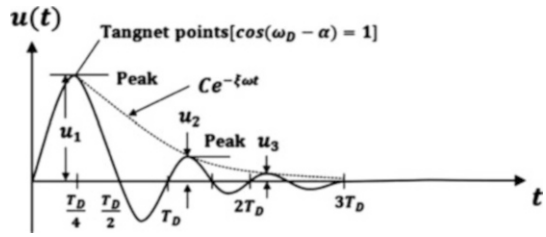

Fig. 2.4 Curve showing peak displacements and displacements at the points of tangency

We note from this equation that when the cosine factor is unity, the displacement is on points of the exponential curve $u(t) = Ce^{-\xi\omega t}$ as shown in Fig. 2.4. However, these points are near but not equal to the positions of maximum displacement. The points on the exponential curve appear slightly to the right of the points of peak or maximum amplitude. For most practical problems, the discrepancy is negligible and the displacement curve at points where the cosine is equal to one may be assumed to coincide at the peak amplitude with the curve $u(t) = Ce^{-\xi\omega t}$, so that we may write for two consecutive peaks u_1 at time t_1 and u_2 at time T_D seconds later.

$$u_1 = Ce^{-\xi\omega t_1}$$

and

$$u_2 = Ce^{-\xi\omega(t_1+T_D)}$$

Dividing these two peak amplitudes and taking the natural logarithm, we obtain

$$\delta = \ln \frac{u_1}{u_2} = \xi\omega T_D \tag{2.27}$$

or by substituting T_D, the damped period, from Eq. (2.24),

$$\delta = \frac{2\pi\xi}{\sqrt{1-\xi^2}} \tag{2.28}$$

As we can see, the damping ratio ξ can be calculated from Eq. (2.28) after determining experimentally the amplitudes of two successive peaks of the system in free vibration. For small values of the damping ratio, Eq. (2.28) can be approximated by

$$\delta = 2\pi\xi \tag{2.29}$$

Alternatively, the logarithmic decrement may be calculated as the ratio of two consecutive peak accelerations, which are easier to measure experimentally than displacements. In this case, taking the first and the second derivatives in Eq. (2.21), we obtain

$$\dot{u}(t) = Ce^{-\xi\omega t}[-\xi\omega\cos(\omega_D t - \alpha) - \omega_D \sin(\omega_D t - \alpha)]$$
$$\ddot{u}(t) = Ce^{-\xi\omega t}\{[-\xi\omega\cos(\omega_D t - \alpha) - \omega_D \sin(\omega_D t - \alpha)](-\xi\omega) + [\xi\omega\omega_D \sin(\omega_D t - \alpha) - \omega_D^2 \cos(\omega_D t - \alpha)]\}$$

At a time t_1, when $\cos(\omega_D t_1 - \alpha) = 1$ and $\sin(\omega_D t_1 - \alpha) = 0$

$$\ddot{u}_1 = Ce^{-\xi\omega t_1}\{\xi^2\omega^2 - \omega_D^2\}$$

and at time $t_2 = t_1 + T_D$, corresponding to a period later, when again the cosine function is equal to one and the sine function is equal to zero,

$$\ddot{u}_2 = Ce^{-\xi\omega(t_1+T_D)}\{\xi^2\omega^2 - \omega_D^2\}$$

The ratio of the acceleration at times t_1 and t_2 is then

$$\frac{\ddot{u}_1}{\ddot{u}_2} = e^{\xi\omega T_D}$$

and taking natural logarithmic results in the logarithmic decrement in terms of the accelerations as

$$\delta = \ln\frac{\ddot{u}_1}{\ddot{u}_2} = \xi\omega T_D$$

which is identical to the expression for the logarithmic decrement given by Eq. (2.27) in terms of displacement.

Illustrative Example 2.1

A vibrating system consisting of a weight of $W = 10$ lb and a spring with stiffness $k = 20$ lb/in is viscously damped so that the ratio of two consecutive amplitudes is 1.00 to 0.85. Determine:

1. The natural frequency of the undamped system.
2. The logarithmic decrement.
3. The damping ratio.
4. The damping coefficient.
5. The damped natural frequency.

Solution:

1. The undamped natural frequency of the system in radians per second is

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\left(20\,\text{lb/in} \times \frac{386\,\text{in/sec}^2}{10\,\text{lb}}\right)} = 27.78\ \text{rad/sec}$$

or in cycles per second

$$f = \frac{\omega}{2\pi} = 4.42 \text{ cps}$$

2. The logarithmic decrement is given by Eq. (2.26) as

$$\delta = \ln \frac{u_1}{u_2} = \ln \frac{1.00}{0.85} = 0.163$$

3. The damping ratio from Eq. (2.29) is approximately equal to

$$\xi = \frac{\delta}{2\pi} = \frac{0.163}{2\pi} = 0.026$$

4. The damping coefficient is obtained from Eqs. (2.6 and 2.19) as

$$c = \xi c_{cr} = 2 \times 0.026 \sqrt{(10 \times 20)/386} = 0.037 \frac{\text{lb. sec}}{\text{in}}$$

5. The natural frequency of the damped system is given by Eq. (2.17), so that

$$\omega_D = \omega \sqrt{1 - \xi^2}$$

$$\omega_D = 27.78 \sqrt{1 - (0.026)^2} = 27.77 \text{ rad/sec}$$

Ilustrative Example 2.2

A platform of weight $W = 4000$ lb is being supported by four equal columns that are clamped to the foundation as well as to the platform. Experimentally it has been determined that a static force of $F = 1000$ lb appliled horizontally to the platform produces a displacement of $\Delta = 0.10$ in. It is estimated that damping in the structures is of the order of 5% of the critical damping. Determine for this structure the following:

1. Undamped natural frequency.
2. Absolute damping coefficient
3. Logarithmic decrement.
4. The number of cycles and the time required for the amplitude of motion to be reduced from an initial value of 0.1 to 0.01 in.

Solution:

1. The stiffness coefficient (force per unit displacement) is computed as

$$k = \frac{F}{\Delta} = \frac{1000}{0.1} = 10,000 \text{ lb/in}$$

and the undamped natural frequency

$$\omega = \sqrt{\frac{kg}{W}} = \sqrt{\frac{10,000 \times 386}{4000}} = 31.06 \text{ rad/sec}$$

2. The critical damping is

$$c_{cr} = 2\sqrt{km} = 2\sqrt{10,000 \times 4000/386} = 643.8 \frac{\text{lb. sec}}{\text{in}}$$

and the absolute damping

$$c = \xi c_{cr} = 0.05 \times 643.8 = 32.19 \frac{\text{lb. sec}}{\text{in}}$$

3. Approximately, the logarithmic decrement is

$$\delta = \ln\left(\frac{u_0}{u_1}\right) = 2\pi\xi = 2\pi(0.05) = 0.314$$

and the ratio of two consecutive amplitudes

$$\frac{u_0}{u_1} = 1.37$$

4. The ratio between the first amplitude u_0 and the amplitude u_k after k cycles may be expressed as

$$\frac{u_0}{u_k} = \frac{u_0}{u_1} \cdot \frac{u_1}{u_2} \cdots \frac{u_{k-1}}{u_k}$$

Then taking the natural logarithm, we obtain

$$\ln \frac{u_0}{u_k} = \delta + \delta + \ldots + \delta = k\delta$$

$$\ln \frac{0.1}{0.01} = 0.314k$$

$$k = \frac{\ln 10}{0.314} = 7.73 \rightarrow 8 \text{ cycles}$$

The damped frequency ω_D is given by

$$\omega_D = \omega\sqrt{1 - \xi^2} = 31.06\sqrt{1 - (0.05)^2} = 31.02 \text{ rad/sec}$$

and the damped period T_D by

$$T_D = \frac{2\pi}{\omega_D} = \frac{2\pi}{31.02} = 0.2025 \text{ sec}$$

Then the time for eight cycles is

$$t(8 \text{ cycles}) = 8T_D = 1.62 \text{ sec}$$

Illustrative Example 2.3

A machine weighing 1000 lb is mounted through springs having a total stiffness $k = 2000$ lb/in to a simple supported beam as shown in Fig. 2.5a. Determine using the analytical model shown in Fig. 2.5b the equivalent mass m_E, the equivalent spring constant k_E, and the equivalent damping coefficient c_E for the system assumed to have 10% of the critical damping. Neglect the mass of the beam.

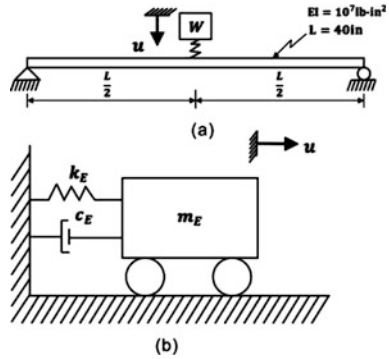

Fig. 2.5 (**a**) System for Ilustrative Example 2.3. (**b**) Analytical model

Solution:
The spring constant k_b for a uniform simple supported beam is obtained from the deflection δ resulting for a force P applied at the center of the beam:

$$\delta = \frac{PL^3}{48EI}$$

Hence,

$$\begin{aligned} k_b &= \frac{P}{\delta} = \frac{48EI}{L^3} \\ &= \frac{48 \times 10^7}{40^3} = 7500 \text{ lb/in} \end{aligned}$$

The equivalent spring constant is then calculated using Eq. (1.5) for two springs in a series:

$$\begin{aligned} \frac{1}{k_E} &= \frac{1}{k} + \frac{1}{k_b} \\ &= \frac{1}{2000} + \frac{1}{7500} \\ k_E &= 1579 \text{ lb/in} \end{aligned} \qquad \text{(Ans)}$$

The equivalent mass is:

$$m_E = \frac{W}{g} = \frac{1000}{386} = 2.59 \left(\frac{\text{lb. sec}^2}{\text{in}} \right)$$

The critical damping is calculated from Eq. (2.6):

$$\begin{aligned} c_{cr} &= 2\sqrt{k_E m_E} \\ &= 2\sqrt{1579 \times 2.59} = 127.92 \left(\frac{\text{lb. sec}}{\text{in}} \right) \end{aligned} \qquad \text{(Ans)}$$

The damping coefficient c_E is then calculated from Eq. (2.19):

$$\begin{aligned} \xi &= \frac{c_E}{c_{cr}} \\ c_E &= \xi c_{cr} = 0.10 \times 127.92 = 12.79 \left(\frac{\text{lb. sec}}{\text{in}} \right) \end{aligned} \qquad \text{(Ans)}$$

2.7 Response of SDF Using MATLAB Program

Using finding values in illustrative Example 2.1, plot the responses with following damping ratios: 0.01, 0.03, 0.5, and 0.9. The initial conditions are the displacement of 1 in. and the velocity of 0 in./sec.

```
clear all
clc


%%%%-GIVEN VALUES-%%%%
m = 10/386;                                          %Mass (lb.sec^2/in.)
k = 20;                                              %Stiffness (lb/in.)
c = 0.037;                                           %Damping coefficient. (lb.sec/in.)

%%%Initial conditions%%%
u0=1;                                                %Initial displ.
v0=0;                                                %Initial velocity

%%%Natural period and natural frequency%%%
omega = sqrt(k/m);                                   %Natural frequency

%%%Critial damping coefficient%%%
c_cr = 2*sqrt(k*m);                                  %Critical damping coeff.
xi = c/c_cr;                                         %Damping ratio

%%%Damping ratio%%%
zeta = [0.01, xi, 0.5, 0.9];                         %Damping ratios

%%%-ESTIMATION-%%%
for i=1:4                                            %Calculating for each damping ratios

omegad=omega*sqrt(1-zeta(:,i)^2);                    %Damped natural frequency

t= 0:0.01:6;
arg1=exp(-zeta(:,i)*omega*t);
C=sqrt(((v0+u0*zeta(:,i)*omega)/omegad)^2+v0^2);     %Max. amplitude
phi=atan((v0+u0*zeta(:,i)*omega)/(omegad*v0));       %Phase angle


Max = C.*arg1;                                       %Envelope for decaying the peaks
u(:,i)=Max.*cos(omegad*t-phi);                       %Response u(t)

end

%%% Create Plots

figure
subplot(2,2,1)                                       %Plot u(t) at 0.01 damping ratio
plot(t,u(:,1))
title('\xi=0.01')
%Create xlabel
xlabel ('Time(sec)');
%Create ylabel
ylabel ('Displacement (in.)');

subplot(2,2,2)                                       %Plot u(t) of Prob.2.1
plot(t,u(:,2))
title(['\xi=' num2str(xi,1) ])
%Create xlabel
xlabel ('Time(sec)');
%Create ylabel
ylabel ('Displacement (in.)');

subplot(2,2,3)                                       %Plot u(t) at 0.5 damping ratio
plot(t,u(:,3))
title('\xi=0.5')
%Create xlabel
xlabel ('Time(sec)');
%Create ylabel
ylabel ('Displacement (in.)');

subplot(2,2,4)                                       %Plot u(t) at 0.9 damping ratio
plot(t,u(:,4))
title('\xi=0.9')
%Create xlabel
xlabel ('Time(sec)');
%Create ylabel
ylabel ('Displacement (in.)');
```

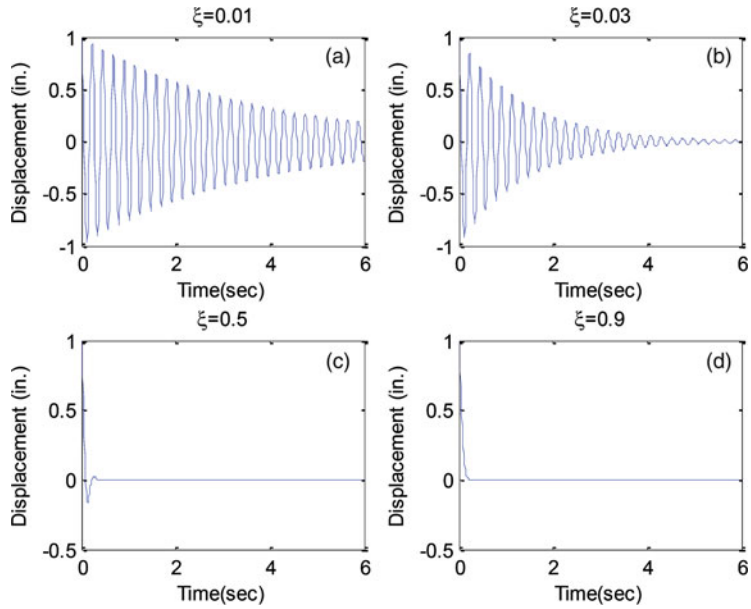

Fig. 2.6 Responses of damped SDF with varied damping ratios damping ratios: (a) 0.01, (b) 0.03, (c) 0.5, (d) 0.9

2.8 Summary

Real structures dissipate energy while undergoing vibratory motion. The most common and practical method for considering this dissipation of energy is to assume that it is due to viscous damping forces. These forces are assumed to be proportional to the magnitude of the velocity but acting in the direction opposite to the motion. The factor of proportionality is called the viscous damping coefficient. It is expedient to express this coefficient as a fraction of the critical damping in the system ($\xi = c/c_{cr}$). The critical damping may be defined as the least value of the damping coefficient for which the system will not oscillate when disturbed initially, but it will simply return to the equilibrium position.

The differential equation of motion for the free vibration of a damped single degree-of-freedom system is given by

$$m\ddot{u} + c\dot{u} + ku = 0 \qquad \text{(Eq. (2.1) repeated)}$$

The analytical expression for the solution of this equation depends on the magnitude of the damping ratio. Three cases are possible:

1. Critically damped system ($\xi = 1$).
2. Overdamped system ($\xi > 1$).
3. Underdamped system ($\xi < 1$).

For the underdamped system ($\xi < 1$), the solution of the differential equation of motion may be written as

$$u(t) = e^{-\xi\omega t}\left[u_0 \cos \omega_D t + \frac{v_0 + u_0\xi\omega}{\omega_D} \sin \omega_D t\right] \quad \text{(Eq. (2.20) repeated)}$$

in which

$\omega = \sqrt{k/m}$ is the undamped frequency
$\omega_D = \omega\sqrt{1-\xi^2}$ is the damped frequency
$\xi = \frac{c}{c_{cr}}$ is the damping ratio
$c_{cr} = 2\sqrt{km}$ is the critical damping

and u_0 and v_0 are, respectively, the initial displacement and velocity.

A common method of determining the damping present in a system is to evaluate experimentally the logarithmic decrement, which is defined as the natural logarithm of the ratio of two consecutive peaks for the displacement or acceleration, in free vibration, that is,

$$\delta = \ln\frac{u_1}{u_2} \quad \text{or} \quad \delta = \ln\frac{\ddot{u}_1}{\ddot{u}_2} \quad \text{(Eq. (2.26) repeated)}$$

The damping ratio in structural systems is usually less than 10% of the critical damping ($\xi < 0.1$). For such systems, the damped frequency is approximately equal to the undamped frequency.

2.9 Problems

Problem 2.1
Repeat Problem 1.2 assuming that the system has 15% of critical damping.

Problem 2.2
Repeat Problem 1.6 assuming that the system has 10% of critical damping.

Problem 2.3
The amplitude of vibration of the system shown in Fig. P2.3 is observed to decrease 5% on each consecutive cycle of motion. Determine the damping coefficient c of the system $k = 200$ lb/in and $m = 10$ lb.sec^2/in.

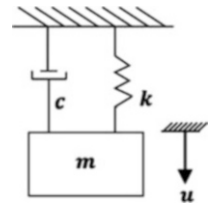

Fig. P2.3

Problem 2.4
It is observed experimentally that the amplitude of free vibration of a certain structure, modeled as a single degree-of-freedom system, decreases in 10 cycles from 1 in to 0.4 in. What is the percentage of critical damping?

Problem 2.5
Show that the displacement for critical and overcritical damped systems with initial displacement u_0 and velocity v_0 may be written as

$$u = e^{-\omega t}[u_0(1+\omega t) + v_0 t] \qquad \text{for } \xi = 1$$

$$u = e^{-\xi\omega t}\left[u_0 \cosh\omega_D' t + \frac{v_0 + u_0\xi\omega}{\omega_D'}\sinh\omega_D' t\right] \qquad \text{for } \xi > 1$$

where $\omega_D' = \omega\sqrt{\xi^2 - 1}$.

a) Write the MATLAB coding for these two cases.
b) Run the MATLAB file with the same cases in Problem 2.1 with damping ratios of 1 and 2.

Problem 2.6
A structure is modeled as a damped oscillator having a spring constant $k = 30$ kip/in and undamped natural frequency $\omega = 25$ rad/sec. Experimentally it was found that a force of 1 kip produced a relative velocity of 1.0 in/sec in the damping element. Determine:

a) The damping ratio ξ.
b) The damped period T_D.
c) The logarithmic decrement δ.
d) The ratio between two consecutive amplitudes.

Problem 2.7
In Fig. 2.4 it is indicated that the tangent points to the displacement curve corresponds to $\cos(\omega_D t - \alpha) = 1$. Therefore the difference in $\omega_D t$ between any two consecutive tangent points is 2π. Show that the difference in $\omega_D t$ between any two consecutive peaks is also 2π.

Problem 2.8
Show that for a damped system in free vibration the logarithmic decrement may be written as

$$\delta = \frac{1}{k}\ln\frac{u_i}{u_{i+k}}$$

where k is the number of cycles separating the two measured peak amplitudes u_i and u_{i+k}.

Problem 2.9
It has been estimated that damping in the system of Problem 1.11 is 10% of the critical value. Determine the damped frequency f_D of the system and the absolute value of the damped coefficient c.

Problem 2.10
A single degree-of-freedom system consists of a mass with a weight of 386 lb and a spring of stiffness $k = 3000$ lb/in. By testing the system it was found that a force of 100 lb produces a relative velocity of 12 in/sec. Find:

a) The damping ratio, ξ.
b) The damped frequency of vibration, f_D.
c) Logarithmic decrement, δ.
d) The ratio of two consecutive amplitudes.

Problem 2.11
Solve Problem 2.10 when the damping coefficient is $c = 2$ lb.sec/in.

Problem 2.12
For each of the systems considered in Problem 1.15, determine the equivalent spring constant k_E and the equivalent damping coefficient c_E in the analytical model shown in Fig. P2.12. Assume that the damping in these systems is equal to 10% of critical damping.

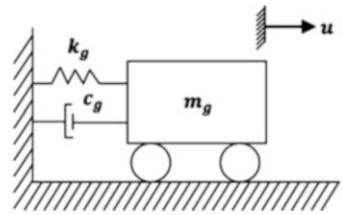

Fig. P2.12

Problem 2.13
A vibration generator with two weights each of 30 lb with an eccentricity of 10 in rotating about vertical axis in opposite directions is mounted on the roof of a one – story builsing with a roof that weighs 300 kips. It is observed that the maximum lateral acceleration of 0.05 g occurs when the vibrator generator is rotating at 400 rpm. Determine the equivalent viscous damping in the structure.

Problem 2.14
A system modeled by two freely vibrating masses m_1 and m_2 is interconnected by a spring and a damper element as shown in Fig. P2.14. Determine for this system the differential equation of motion in terms of relative motion of the masses $u_r = u_2 - u_1$.

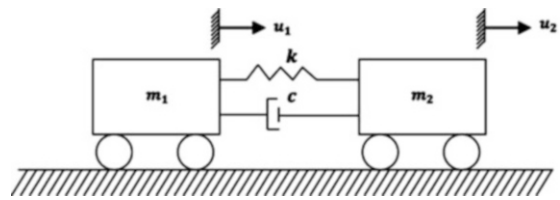

Fig. P2.14

Problem 2.15

Determine the relative motion $u_r = u_2 - u_1$ for the system shown in Fig. P2.14 in terms of the natural frequency ω, damped frequency, ω_D and relative damping, ξ. *Hint:* Define the equivalent mass as $M = m_1\, m_2/(m_1 + m_2)$.

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Response of One-Degree-of-Freedom System to Harmonic Loading

3

In this chapter, we will study the motion of structures idealized as single-degree-of-freedom systems excited harmonically, that is, structures subjected to forces or displacements whose magnitudes may be represented by a sine or cosine function of time. This type of excitation results in one of the most important motions in the study of mechanical vibrations as well as in applications to structural dynamics. Structures are very often subjected to the dynamic action of rotating machinery which produces harmonic excitations due to the unavoidable presence of mass eccentricities in the rotating parts of such machinery. Furthermore, even in those cases when the excitation is not a harmonic function, the response of the structure may be obtained using the Fourier Method, as the superposition of individual responses to the harmonic components of external excitation. This approach will be dealt with in Chap. 20 as a special topic.

3.1 Harmonic Excitation: Undamped System

The impressed force $F(t)$ acting on the simple oscillator in Fig. 3.1 is assumed to be harmonic and equal to $F_0 \sin \overline{\omega} t$ where F_0 is the peak amplitude and $\overline{\omega}$ is the frequency of the force in radians per second.

The differential equation obtained by summing all the forces in the free body diagram of Fig. 3.1b is

$$m\ddot{u} + ku = F_0 \sin \overline{\omega} t \tag{3.1}$$

The solution of Eq. (3.1) can be expressed as

$$u(t) = u_c(t) + u_P(t) \tag{3.2}$$

where $u_c(t)$ is the complementary solution satisfying the homogeneous equation, that is, Eq. (3.1) with the left hand-side set equal to zero; and $u_p(t)$ is the particular solution based on the solution satisfying the nonhomogeneous differential Eq. (3.1). The complementary solution, $u_c(t)$, is given by Eq. (1.17) as

$$u_c(t) = A \cos \omega t + B \sin \omega t \tag{3.3}$$

where

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M. Paz, Y. H. Kim, *Structural Dynamics*, https://doi.org/10.1007/978-3-319-94743-3_3

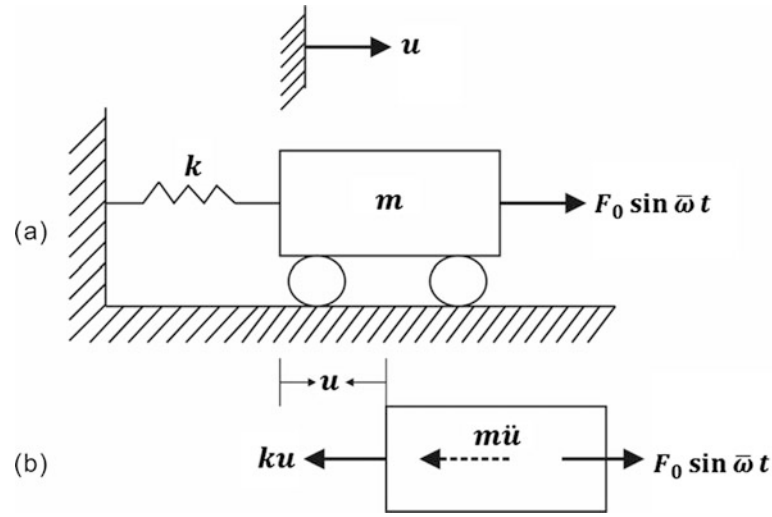

Fig. 3.1 (**a**) Undamped oscillator harmonically excited, (**b**) Free body diagram

$$\omega = \sqrt{k/m}$$

The nature of the forcing function in Eq. (3.1) suggests that the particular solution be taken as

$$u_p(t) = U \sin \overline{\omega} t \tag{3.4}$$

where U is the amplitude of the particular solution. The substitution of Eq. (3.4) into Eq. (3.1) followed by cancellation of common factors gives

$$-m\overline{\omega}^2 U + kU = F_0$$

or

$$U = \frac{F_0}{k - m\overline{\omega}^2} = \frac{F_0/k}{1 - r^2} \tag{3.5}$$

in which r represents the ratio (frequency ratio) of the applied forced frequency to the natural frequency of vibration of the system, that is,

$$r = \frac{\overline{\omega}}{\omega} \tag{3.6}$$

Combining Eqs. (3.3) through (3.5) with Eq. (3.2) yields

$$U(t) = A \cos \omega t + B \sin \omega t + \frac{F_0/k}{1 - r^2} \sin \overline{\omega} t \tag{3.7}$$

If the initial conditions for the displacement and for the velocity at time $t = 0$ are taken as zero ($u_0 = 0$, $v_0 = 0$), the constants of integration determined from Eq. (3.7) are:

$$A = 0 \qquad \text{and} \qquad B = -r \frac{F_0/k}{1 - r^2}$$

which, upon substitution in Eq. (3.7), results in

$$u(t) = \frac{F_0/k}{1-r^2}\left(\sin\overline{\omega}t - r\sin\omega t\right) \tag{3.8}$$

As we can see from Eq. (3.8), the response is given by the superposition of two harmonic terms of different frequencies. The resulting motion is not harmonic; however, in the practical case, damping forces will always be present in the system and will cause the last term, i.e., the free frequency term in Eq. (3.8), to eventually vanish. For this reason, this term is said to represent the transient response. The forcing frequency term in Eq. (3.8), namely

$$u(t) = \frac{F_0/k}{1-r^2}\sin\overline{\omega}t \tag{3.9}$$

is referred to as the steady-state response. It is clear from Eq. (3.8) that in the case of no damping in the system, the transient will not vanish and the response is then given by Eq. (3.8). It can also be seen from Eq. (3.8) or Eq. (3.9) that when the forcing frequency is equal to natural frequency ($r = 1.0$), the amplitude of the motion becomes infinitely large. A system acted upon by an external excitation of frequency coinciding with the natural frequency is said to be at resonance. In this circumstance, the amplitude will increase gradually to infinite. However, materials that are commonly used in practice are subjected to strength limitations and in actual structures failures occur long before extremely large amplitudes can be attained.

3.2 Harmonic Excitation: Damped System

Now consider the case of the one-degree-of-freedom system in Fig. 3.2a vibrating under the influence of viscous damping. The differential equation of motion is obtained by equating to zero the sum of the forces in the free body diagram of Fig. 3.2b. Hence

$$m\ddot{u} + c\dot{u} + ku = F_0\sin\overline{\omega}t \tag{3.10}$$

The complete solution of this equation again consists of the complementary solution $u_c(t)$ and the particular solution $u_p(t)$.The complementary solution is given for the underdamped case ($c < c_{cr}$) by Eqs. (2.15) after using (2.19) as

$$u_c(t) = e^{-\xi\omega t}(A\cos\omega_D t + B\sin\omega_D t) \tag{3.11}$$

The particular solution may be found by substituting u_p in this case assumed to be of the form

$$u_p(t) = C_1\sin\overline{\omega}t + C_2\cos\overline{\omega}t \tag{3.12}$$

into Eq. (3.10) and equating the coefficients of the sine and cosine functions. The unknowns C_1 and C_2 are found with plugging $u_p(t)$ into $m\ddot{u}_p + c\dot{u}_p + ku_p = F_0\sin\overline{\omega}t$.

$$C_1 = \frac{F_0}{k}\left(\frac{1-r^2}{(1-r^2)^2 + (2\xi r)^2}\right) \quad \text{and} \quad C_2 = \frac{F_0}{k}\left(\frac{-2\xi r}{(1-r^2)^2 + (2\xi r)^2}\right) \tag{3.13}$$

In addition, here we follow a more elegant approach using Euler's relation, namely

$$e^{i\overline{\omega}t} = \cos\overline{\omega}t + i\sin\overline{\omega}t$$

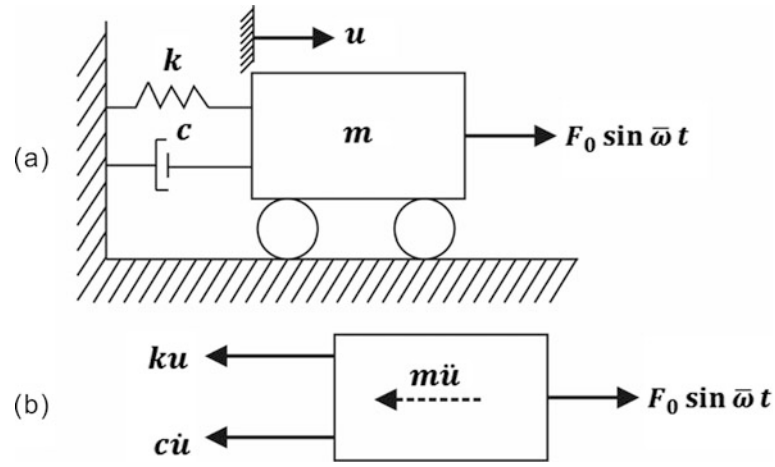

Fig. 3.2 (**a**) Damped oscillator harmonically excited, (**b**) Free body diagram

For this purpose, the reader should realize that we can write Eq. (3.10) as

$$m\ddot{u} + c\dot{u} + ku = F_0 e^{i\bar{\omega}t} \tag{3.14}$$

with the understanding that only the imaginary component of $F_0 e^{i\bar{\omega}t}$, i.e., the force component of $F_0 \sin \bar{\omega} t$, is acting and, consequently, the response will then consist only of the imaginary part of the total solution of Eq. (3.14). In other words, we obtain the solution of Eq. (3.14) which has real and imaginary components, and disregard the real component.

It is reasonable to expect that the particular solution of Eq. (3.14) will be of the form

$$u_p = Ce^{i\bar{\omega}t} \tag{3.15}$$

Substitution of Eq. (3.14) into Eq. (3.13) and cancellation of the factor $e^{i\bar{\omega}t}$ gives

$$-m\bar{\omega}^2 C + ic\bar{\omega}C + kC = F_0$$

or

$$C = \frac{F_0}{k - m\bar{\omega}^2 + ic\bar{\omega}}$$

and

$$u_p = \frac{F_0 e^{i\bar{\omega}t}}{k - m\bar{\omega}^2 + ic\bar{\omega}} \tag{3.16}$$

By using polar coordinate form, the complex denominator in Eq. (3.16) may be written as

$$u_p = \frac{F_0 e^{i\bar{\omega}t}}{\sqrt{\left(k - m\bar{\omega}^2\right)^2 + \left(c\bar{\omega}\right)^2} e^{i\theta}}$$

or

$$u_p = \frac{F_0 e^{i\left[\overline{\omega} t - \theta\right]}}{\sqrt{\left(k - m\overline{\omega}^2\right)^2 + \left(c\overline{\omega}\right)^2}} \tag{3.17}$$

where

$$\tan\theta = \frac{c\overline{\omega}}{k - m\overline{\omega}^2} \tag{3.18}$$

The response to the force in $F_0 \sin\overline{\omega} t$ (the imaginary component of $F_0 e^{i\overline{\omega} t}$) is then the imaginary component of Eq. (3.17), namely

$$u_P = \frac{F_0 \sin\left(\overline{\omega} t - \theta\right)}{\sqrt{\left(k - m\overline{\omega}\right)^2 + \left(c\overline{\omega}\right)^2}} \tag{3.19}$$

or

$$u_p = U \sin\left(\overline{\omega} t - \theta\right) \tag{3.20}$$

where

$$U = \frac{F_0}{\sqrt{\left(k - m\overline{\omega}\right)^2 + \left(c\overline{\omega}\right)^2}}$$

is the amplitude of the steady-state motion. Equations (3.19) and (3.18) may conveniently be written in terms of dimensionless ratios as

$$u(t) = \frac{u_{st} \sin\left(\overline{\omega} t - \theta\right)}{\sqrt{\left(1 - r^2\right)^2 + \left(2\xi r\right)^2}} \tag{3.21}$$

and

$$\tan\theta = \frac{2\xi r}{1 - r^2} \tag{3.22}$$

where $u_{\mathrm{st}} = F_0/k$ is seen to be the static deflection of the spring acted upon by the force F_0, $\xi = c \,/\, c_{cr}$ the damping ratio, and $r = \overline{\omega}/\omega$ the frequency ratio. The total response is then obtained by combining the complementary solution (transient response) from Eq. (3.11) and the particular solution (steady-state response) from Eq. (3.21), that is,

$$u(t) = e^{-\xi\omega t}\left(A\cos\omega_D t + B\sin\omega_D t\right) + \frac{u_{st} \sin\left(\overline{\omega} t - \theta\right)}{\sqrt{\left(1 - r^2\right)^2 + \left(2r\,\xi\right)^2}} \tag{3.23}$$

The reader should be warned that the constants of integration A and B must be evaluated from initial conditions using the total response given by Eq. (3.23) and not from just the transient component of the response given in Eq. (3.11). By examining the transient component of response, it may be seen that the presence of the exponential factor $e^{-\xi\omega t}$ will cause this component to vanish, leaving only the steady-state motion which is given by Eq. (3.21).

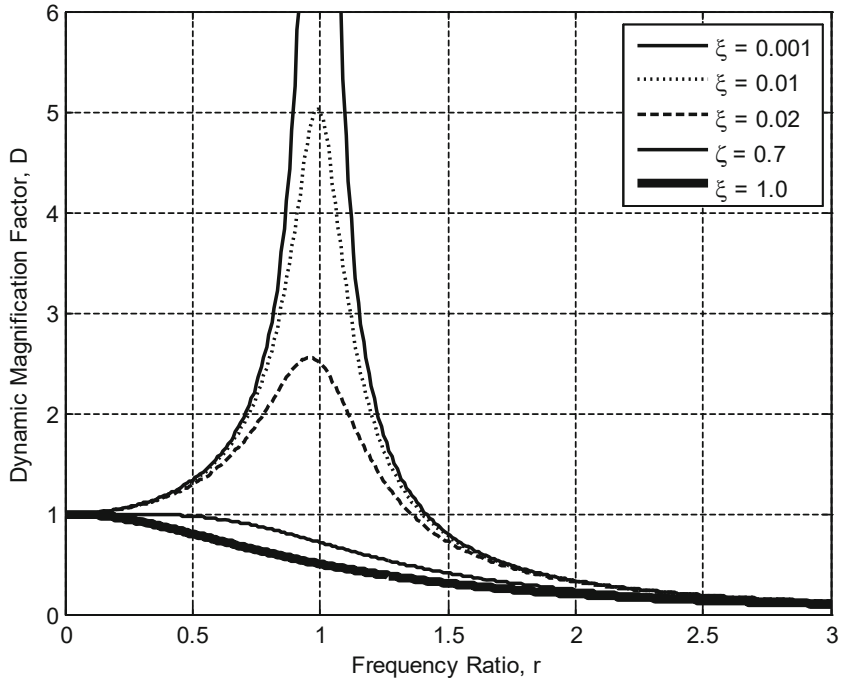

Fig. 3.3 Dynamic magnification factor as a function of the frequency ratio for various amounts of damping

The ratio of the steady-state amplitude of $u_p(t)$ to the static deflection u_{st} defined above is known as the dynamic magnification factor D, and is given from Eqs. (3.20) and (3.21) by

$$D = \frac{U}{u_{st}} = \frac{1}{\sqrt{(1-r^2)^2 + (2r\xi)^2}} \tag{3.24}$$

It may be seen from Eq. (3.24) that the dynamic magnification factor varies with the frequency ratio r and the damping ratio ξ. Parametric plots of the dynamic magnification factor are shown in Fig. 3.3. The phase angle θ, given in Eq. (3.22), also varies with the same quantities as it is shown in the plots of Fig. 3.4. We note in Fig. 3.3 that for a lightly damped system, the peak amplitude occurs at a frequency ratio very close to r = 1; that is, the dynamic magnification factor has its maximum value virtually at resonance ($r = 1$). It can also be seen from Eq. (3.24) that at resonance the dynamic magnification factor is inversely proportional to the damping ratio, that is,

$$D(r = 1) = \frac{1}{2\xi} \tag{3.25}$$

Although the dynamic magnification factor evaluated at resonance is close to its maximum value, it is not exactly the maximum response for a damped system. However, for moderate amounts of damping, the difference between the approximate value of Eq. (3.25) and the exact maximum is negligible.

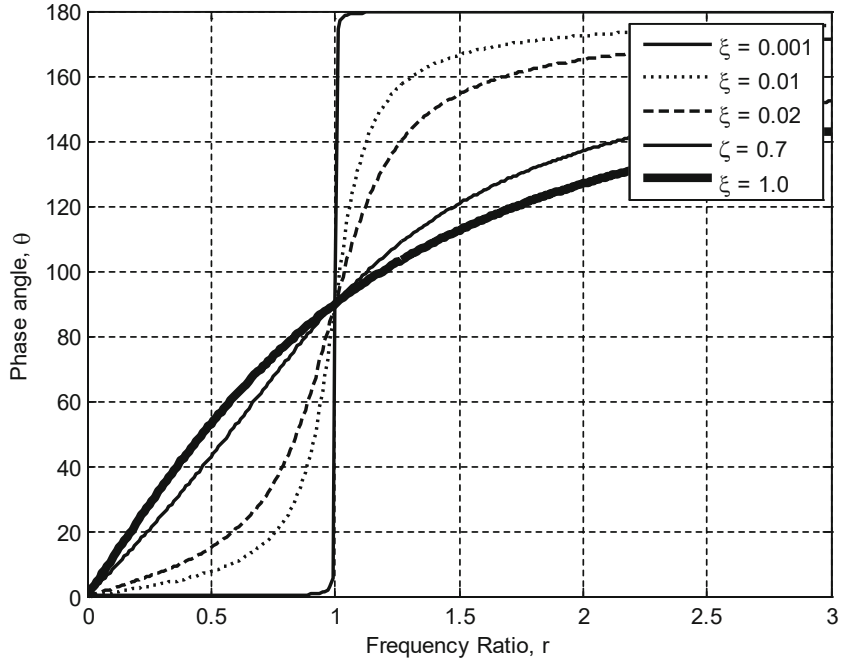

Fig. 3.4 Phase angle θ as a function of frequency ratio for various damping values

Illustrative Example 3.1

A simple beam supports at its center a machine having a weight $W = 16{,}000$ lb. The beam is made of two standard S8 $\times$ 23 sections with a clear span $L = 12$ ft and total cross-sectional moment of inertia $I = 2 \times 64.2 = 128.4$ in^4. The motor runs at 300 rpm, and its rotor is out of balance to the extent of $W' = 40$ lb at an eccentricity of $e_0 = 10$ in. What will be the amplitude of the steady-state response if the equivalent viscous damping for the system is assumed 10% of the critical?

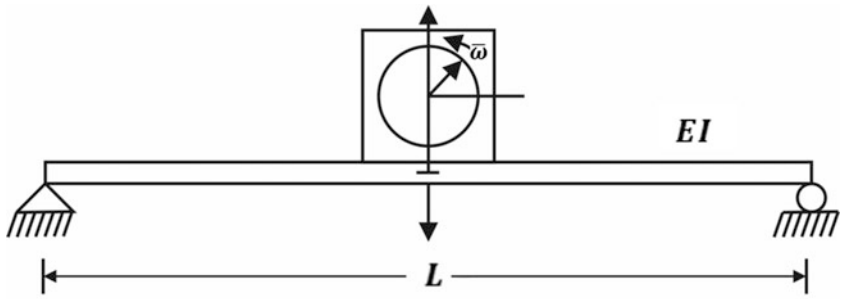

Fig. 3.5 Diagram for beam-machine system of Illustrative Example 3.1

Solution:

This dynamic system may be modeled by the damped oscillator. The distributed mass of the beam will be neglected in comparison with the large mass of the machine. Figs. 3.5 and 3.6 show, respectively, the schematic diagram of the beam-machine system and the adapted model.

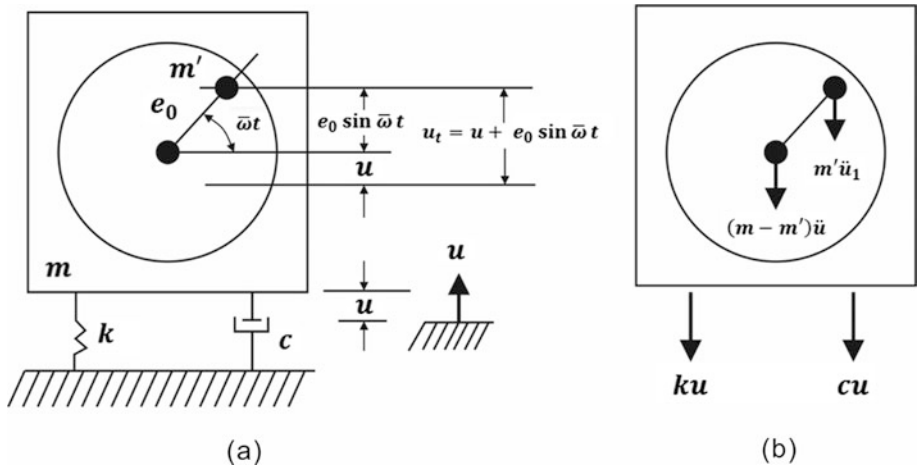

Fig. 3.6 (**a**) Analytical model for Illustrative Example 3.1; (**b**) Free body diagram

The force at the center of a simply supported beam necessary to deflect this point one unit (i.e., the stiffness coefficient) is given by the formula

$$k = \frac{48EI}{L^3} = \frac{48 \times 30 \times 10^6 \times 128.4}{(144)^3} = 61,920 \text{ lb/in}$$

The natural frequency of the system (neglecting the mass of the beam) is

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{61,920}{16,000/386}} = 38.65 \text{ rad/ sec},$$

The forced frequency is

$$\overline{\omega} = \frac{300 \times 2\pi}{60} = 31.41 \text{ rad/ sec}$$

and the frequency ratio

$$r = \frac{\overline{\omega}}{\omega} = \frac{31.41}{38.65} = 0.813$$

Referring the Fig. 3.6, let m be the total mass of the motor and m' the unbalanced rotating mass. Then, if u is the vertical displacement from the equilibrium position of the non-rotating mass (m - m'), the displacement u_1 of the eccentric mass m' as shown in Fig. 3.6 is

$$u_1 = u + e_0 \sin \overline{\omega} t \qquad \text{(a)}$$

The equation of motion in then obtained by summing forces along the vertical direction in the free body diagram of Fig. 3.6b, where the inertial forces of both the nonrotating mass and of the eccentric mass are also shown. This summation yields

$$(m - m')\ddot{u} + m'\ddot{u}_1 + c\dot{u} + ku = 0 \tag{b}$$

in which $m' = W'/g$ is the eccentric mass.

Substitution of $\ddot{u}_1$ obtained from Eq. (a) gives

$$(m - m')\ddot{u} + m'\left(\ddot{u} - e_0\overline{\omega}^2 \sin\overline{\omega}t\right) + c\dot{u} + ku = 0 \tag{b}$$

and with a rearrangement of terms

$$m\ddot{u} + c\dot{u} + ku = m'e_0\overline{\omega}^2 \sin\overline{\omega}t \tag{c}$$

This last equation is of the same form as the equation of motion (3.10) for the damped oscillator excited harmonically by a force of amplitude

$$F_0 = m'e_0\overline{\omega}^2 \tag{d}$$

Substituting in Eq. (d) the numerical values for this example, we obtain

$$F_0 = (40)(10)(31.41)^2/386 = 1022 \text{ lb}$$

From Eq. (3.20), the amplitude of the steady-state resulting motion is then

$$U = \frac{1022/61,920}{\sqrt{\left(1 - 0.813^2\right)^2 + (2 \times 0.813 \times 0.1)^2}}$$

$$U = 0.044 \text{ in} \tag{Ans}$$

Illustrative Example 3.2

The steel frame shown in Fig. 3.7 supports a rotating machine that exerts a horizontal force at the girder level $F(t) = 200 \sin 5.3t$ lb. Assuming 5% of critical damping, determine: (a) the steady-state amplitude of vibration and (b) the maximum dynamic stress in the columns. Assume the girder is rigid.

Solution:

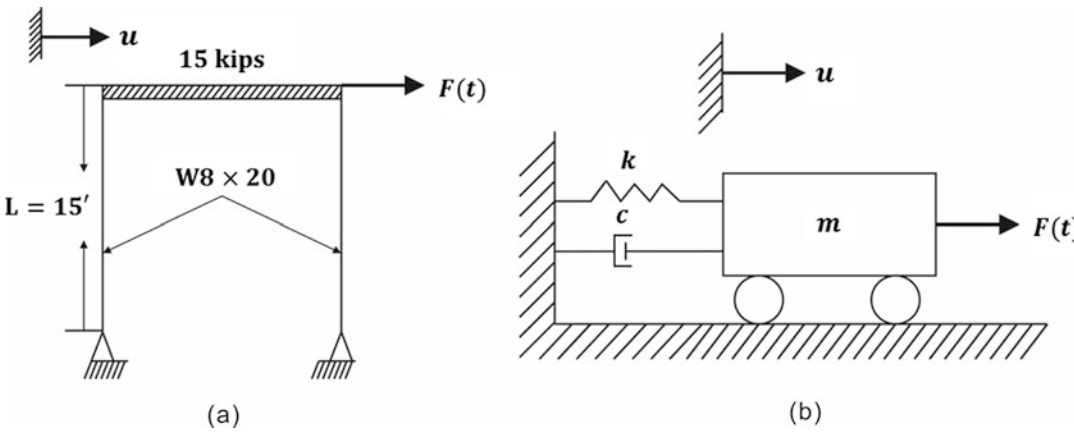

Fig. 3.7 (**a**) Diagram of the frame for Illustrative Example 3.2 (**b**) Analytical model

This structure may be modeled for dynamic analysis as the damped oscillator shown in Fig. 3.7b. The parameters in this model are computed as follows[1]:

$$k = \frac{3E(2I)}{L^3} = \frac{3 \times 30 \times 10^6 \times 2 \times 69.2}{(12 \times 15)^3} = 2136 \text{ lb/in}$$

$$\xi = 0.05$$

$$u_{st} = \frac{F_0}{k} = \frac{200}{2136} = 0.0936 \text{ in}$$

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{2136 \times 386}{15,000}} = 7.41 \text{ rad/ sec}$$

$$r = \frac{\overline{\omega}}{\omega} = \frac{5.3}{7.41} = 0.715$$

The steady-state amplitude from Eqs. (3.19) and (3.20) is

$$U = \frac{u_{st}}{\sqrt{(1 - r^2)^2 + (2r\xi)^2}} = 0.189 \text{ in}$$

Then, the maximum shear force is the columns is

$$V_{\max} = \frac{3EI\,U}{L^3} = 201.8 \text{ lb}$$

The maximum bending moment

$$M_{\max} = V_{\max} L = 36,324 \text{ lb} \cdot \text{in}$$

and the maximum stress

$$\sigma_{\max} = \frac{M_{\max}}{I/c} = \frac{36,324}{17} = 2136 \text{ psi} \quad \text{(Ans)}$$

in which I / c is the section modulus.

3.3 Evaluation of Damping at Resonance

We have seen in Chap. 2 that the free-vibration decay curve permits the evaluation of damping of a single-degree-of freedom system by simply calculating the logarithm decrement and using Eq. (2.28) or Eq. (2.29). Another technique for determining damping is based on observations of steady-state harmonic response, which requires harmonic excitations of the structure in a range of frequencies in the neighborhood of resonance. With the application of a harmonic force $F_0 \sin \overline{\omega} t$ at closely spaced values of frequencies, the response curve for the structure can be plotted, resulting in displacement amplitudes as a function of the applied frequencies. A typical response curve for such a moderately

[1] *A unit displacement at the top of pinned supported columns requires a force equal to 3 EI/L^3.

damped structure is shown in Fig. 3.8. It is seen from Eq. (3.24) that, at resonance, the damping ratio is given by

$$\xi = \frac{1}{2D(r=1)} \tag{3.25}$$

where $D(r=1)$ is the dynamic magnification factor evaluated at resonance. In practice, the damping ratio ξ is determined from the dynamic magnification factor evaluated at the maximum amplitude, namely

$$\xi = \frac{1}{2D_m} \tag{3.26}$$

where

$$D_m = \frac{U_m}{u_{st}}$$

and U_m is the maximum amplitude. The error involved in evaluating the damping ratio ξ using the approximate Eq. (3.26) is not significant in ordinary structures. This method of determining the damping ratio requires only some simple equipment to vibrate the structure in a range of frequencies that span the resonance frequency and a transducer for measuring amplitudes; nevertheless, the evaluation of the static displacement $u_{st} = F_0 / k$ may present a problem since, frequently, it is difficult to apply a static lateral load to the structure.

3.4 Bandwidth Method (Half-Power) to Evaluate Damping

An examination of the response curves in Fig. 3.3 shows that the shape of these curves is controlled by the amount of damping present in the system; in particular, the bandwidth, that is, the difference between two frequencies corresponding to the same response amplitude, is related to the damping in the system. A typical frequency amplitude curve obtained experimentally for a moderately damped structure is shown in Fig. 3.8. In the evaluation of damping it is convenient to measure the bandwidth at $1/\sqrt{2}$ of the peak amplitude as shown in this figure. The frequencies corresponding in this bandwidth f_1 and f_2 are also referred to as half-power points and are shown in Fig. 3.8. The values of the frequencies for this bandwidth can be determined by setting the response amplitude in Eq. (3.20) equal to $1/\sqrt{2}$ times the resonant amplitude given by Eq. (3.24), that is

$$\frac{u_{st}}{\sqrt{(1-r^2)^2 + (2r\xi)^2}} = \frac{1}{\sqrt{2}} \frac{u_{st}}{2\xi}$$

Squaring both sides and solving for the frequency ratio results in

$$r^2 = 1 - 2\xi^2 \pm 2\xi\sqrt{1+\xi^2}$$

or by neglecting ξ^2 in the square root term

$$r_1^2 \cong 1 - 2\xi^2 - 2\xi$$
$$r_2^2 \cong 1 - 2\xi^2 + 2\xi$$
$$r_1 \cong 1 - \xi - \xi^2$$
$$r_2 \cong 1 + \xi - \xi^2$$

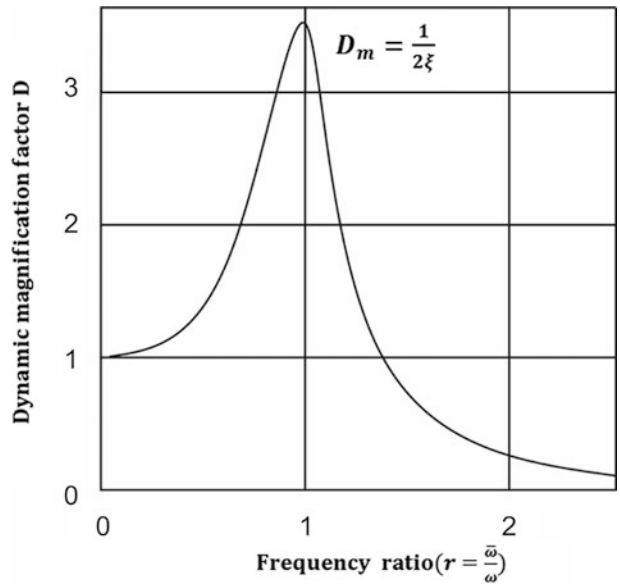

Fig. 3.8 Frequency response curve for moderately damped system

Finally, the damping ratio is given approximately by half the difference between these half-power frequency ratios, namely

$$\xi = \frac{1}{2}(r_2 - r_1)$$

or

$$\xi = \frac{1}{2}\frac{\omega_2 - \omega_1}{\omega} = \frac{f_2 - f_1}{f_2 + f_1} \tag{3.27}$$

since

$$\frac{\omega_2 - \omega_1}{2\omega} = \frac{f_2 - f_1}{2f} \qquad \text{and} \qquad f \approx \frac{f_1 + f_2}{2}$$

Illustrative Example 3.3
Experimental data for the frequency response of a single degree-of-freedom system are plotted in Fig. 3.9. Determine the damping ratio of this system.

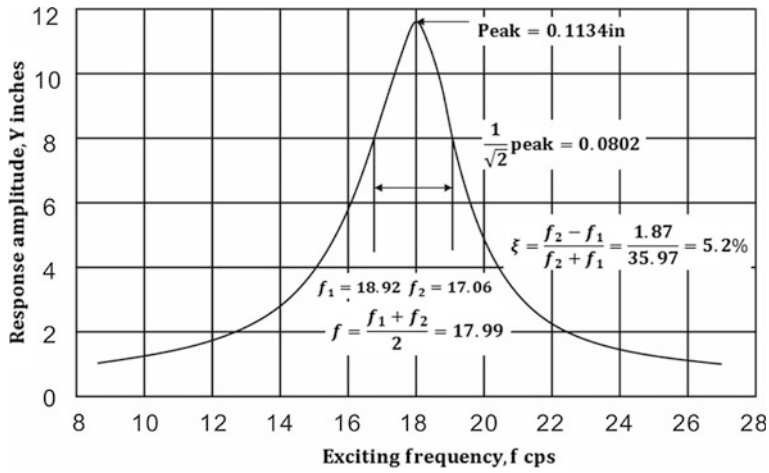

Fig. 3.9 Experimental frequency response curve of Illustrative Example 3.3

Solution:
From Fig. 3.9, the peak amplitude is 0.1134 in. hence the amplitude at half-power is equal to

$$0.1134/\sqrt{2} = 0.0802 \text{ in}$$

The frequencies at this amplitude obtained from Fig. 3.9 are

$$f_1 = 17.05 \; cps$$
$$f_2 = 18.92$$

the damping ratio is then calculated from Eq. (3.27) as

$$\xi \cong \frac{f_2 - f_1}{f_2 + f_1}$$
$$\xi \cong \frac{18.92 - 17.05}{18.92 + 17.05} = 5.2\% \qquad \text{(Ans)}$$

3.5 Energy Dissipated by Viscous Damping

The energy E_D dissipated by viscous damping during one cycle of harmonic vibration of frequency $\overline{\omega}$ is equal to the work done by the damping force $c\dot{u}$ during a differential displacement du integrated over one period of vibration $T = 2\pi/\overline{\omega}$.

Hence,

$$E_D = \int_0^{2\pi/\overline{\omega}} (c\dot{u})\, du = \int_0^{2\pi/\overline{\omega}} (c\dot{u})\frac{du}{dt}\, dt = \int_0^{2\pi/\overline{\omega}} c\dot{u}^2\, dt \qquad (3.28)$$

The velocity $\dot{u} = \dot{u}(t)$ for the damped oscillator acted upon by the harmonic force, $F = F_0 \sin \overline{\omega} t$, is given by the derivative of Eq. (3.19) as

$$\dot{u}(t) = U\overline{\omega}\cos\left(\overline{\omega}t - \theta\right) \tag{3.29}$$

which substituted in Eq. (3.28) gives

$$E_D = cU^2\overline{\omega}^2 \int_0^{2\pi/\overline{\omega}} \cos^2\left(\overline{\omega}t - \theta\right)dt = \pi c\overline{\omega}U^2$$

$$E_D = 2\pi\xi rkU^2 \tag{3.30}$$

where as previously defined

$$\xi = \frac{c}{c_{cr}}, \qquad r = \frac{\overline{\omega}}{\omega}, \qquad \omega = \sqrt{\frac{k}{m}}, \qquad \text{and} \qquad c_{cr} = 2\sqrt{km} \tag{3.31}$$

Equation (3.30) shows that the energy dissipated by viscous damping is proportional to the square of the amplitude of the motion U. It can be shown (see Problem 3.1) that during one cycle, the work W_F of the external force $F = F_0 \sin\overline{\omega}t$ is precisely the equal to the energy E_D dissipated by the damping force as expressed by Eq. (3.30).

3.6 Equivalent Viscous Damping

As mentioned in the introductory sections of Chap. 2, the mechanism by which structures dissipate energy during vibratory motion is usually assumed to be viscous. This assumption provides the enormous advantage that the differential equation of motion remains linear for damped dynamic systems vibrating in the elastic range. Only for some exceptionally situations such as the use of frictional devices installed in buildings to ameliorate damage resulting from strong motion earthquakes, viscous damping is usually assumed to account for frictional or damping forces in structural dynamics. The numerical value assigned to the damping coefficient is based on values obtained experimentally and the determination of an equivalent viscous damping.

The concept of equivalent viscous damping is based on test results obtained using harmonic forces. Thus, in reference to the experimental frequency response plot in Fig. 3.9, the equivalent damping ξ_{eq} may be based on the maximum relative amplitude of motion, $D_m = U_m/u_{st}$, or on the bandwidth corresponding to frequencies f_1 to f_2 at amplitude equal to $D_m/\sqrt{2}$. Thus, the equivalent viscous damping ξ_{eq} may be calculated from Eq. (3.26) as

$$\xi_{eq} = \frac{u_{st}}{2U_m} \tag{3.32}$$

or from Eq. (3.27) as

$$\xi_{eq} = \frac{f_2 - f_1}{f_2 + f_1} \tag{3.33}$$

Alternatively, the equivalent viscous damping ξ_{eq} may also be evaluated experimentally using the expression for the logarithmic decrement, δ, from Eq. (2.28) or approximately from Eq. (2.29) as

$$\xi_{eq} = \frac{\delta}{2\pi} \tag{3.34}$$

However, the most common definition of equivalent viscous damping is based on equating the energy dissipated, in a vibratory cycle of the actual structure, to the energy dissipated in an equivalent

viscous system. Hence, equating the energy, E^*, dissipated in a cycle of harmonic vibration determined from experiment to the energy E_D, dissipated by an equivalent viscous system given by Eq. (3.30) we have

$$E^* = 2\pi \xi_{eq} rkU^2$$

and

$$\xi_{eq} = \frac{E^*}{2\pi rkU^2}$$

or

$$\xi_{eq} = \frac{1}{4\pi r}\frac{E^*}{E_s} \tag{3.35}$$

in which E_s, the strain energy stored at maximum displacement if the system were elastic, which is given by

$$E_s = \frac{1}{2}kU^2 \tag{3.36}$$

In the determination of the energy, E^* dissipated per cycle and the elastic energy, E_s stored at the maximum displacement, an experiment is conducted by vibrating the structure at the resonant frequency for which $r = \overline{\omega}/\omega = 1$. At this frequency, damping in the system has a maximum effect. With appropriate test equipment and measuring instrumentation, the restoring force and displacement during a cycle of vibration are measured to obtain a plot of the type shown in Fig. 3.10. The area enclosed in the loop during one cycle of vibration is equal to the energy dissipated, E^* and the triangular area corresponding to the amplitude U is equal to the strain energy, E_s. Consequently, the equivalent viscous damping is evaluated by Eq. (3.35) from the experimental results E^* and E_s, with $r = 1$, obtained from resisting force-displacement plot.

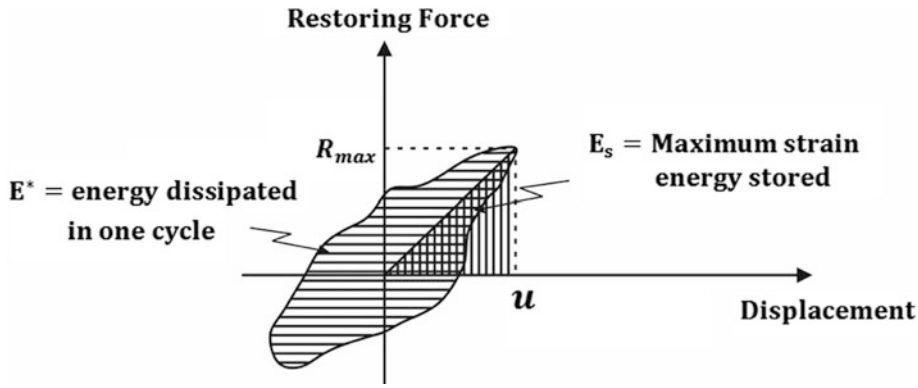

Fig. 3.10 Restoring force vs. displacement during a cycle of vibration showing the energy dissipated E^* (area within the loop) and the maximum energy stored (triangular area under maximum displacement)

Illustrative Example 3.4

Laboratory tests on a structure modeled by the damped spring-mass system [Fig. 3.11a], are conducted to evaluate equivalent viscous damping using (a) peak amplitude and (b) energy dissipated. The experimental restoring force-displacement plot at resonance is shown in Fig. 3.11b.

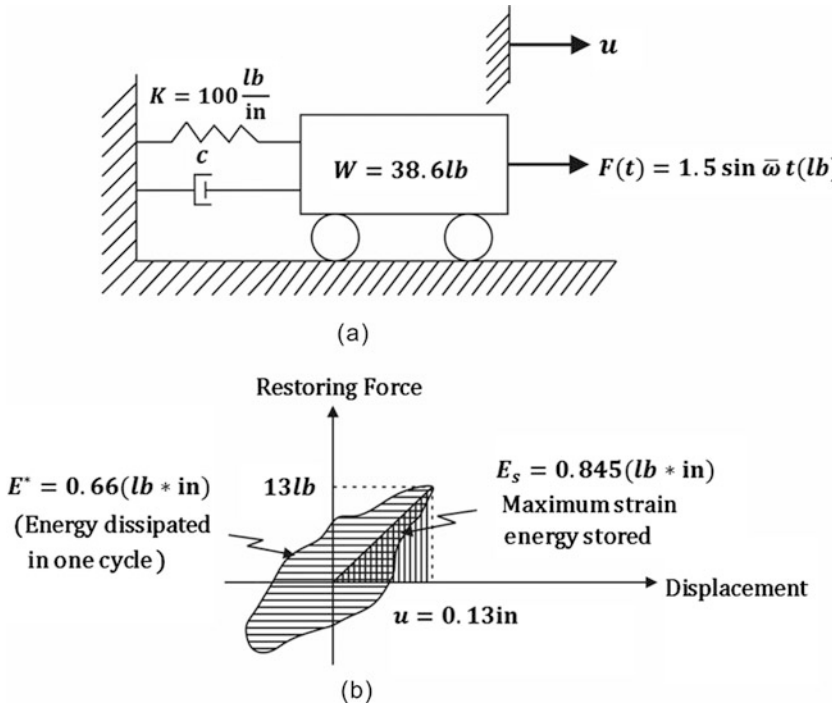

Fig. 3.11 (**a**) Analytical model for Illustrative Example 3.4; (**b**) Force-displacement plot

Solution:
The static deflection, the mass, and the natural frequency are:

$$u_{st} = \frac{F_0}{k} = \frac{1.5}{100} = 0.015 \text{ in} \qquad m = \frac{W}{g} = \frac{38.6}{386} = 0.1 \left(\text{lb} \cdot \sec^2/\text{in}\right)$$

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{100}{0.1}} = 31.62 \text{ rad/ sec}$$

(a) Equivalent viscous damping calculated from peak amplitude:

$$\xi_{eq} = \frac{u_{st}}{2U_m} = \frac{0.015}{(2))(0.13)} = 0.0576 = 5.7\% \qquad \text{by Eq. (3.32)}$$

(b) Equivalent viscous damping calculated from energy dissipated $E^* = 0.66$ (lb.in) and elastic energy at maximum displacement, $E_s = 0.845$ (lb.in), as shown in Fig. 3.11: (at resonance, $r = 1.0$)

$$\xi_{eq} = \frac{1}{4\pi r}\frac{E^*}{E_s} = \frac{0.66}{4\pi(1.0)0.845} = 0.0621 = 6.21\% \qquad \text{by Eq. (3.35)}$$

3.7 Response to Support Motion

3.7.1 Absolute Motion

There are many actual cases where the foundation or support of a structure is subjected to time varying motion. Structures subjected to ground motion by earthquakes or other excitations such as explosions

or dynamic action of machinery are examples in which support motions may have to be considered in the analysis of dynamic response. Let us consider in Fig. 3.12 the case where the support of the simple oscillator modeling the structure is subjected to a harmonic motion given by the expression

$$u_s(t) = u_0 \sin \overline{\omega} t \tag{3.37}$$

where u_0 is the maximum amplitude and $\overline{\omega}$ is the frequency of the support motion. The differential equation of motion is obtained by setting equal to zero the sum of the forces (including the inertial force) in the corresponding free body diagram shown in Fig. 3.12b. The summation of the forces in the horizontal direction gives

$$m\ddot{u} + c\left(\dot{u} - \dot{u}_s\right) + k(u - u_s) = 0 \tag{3.38}$$

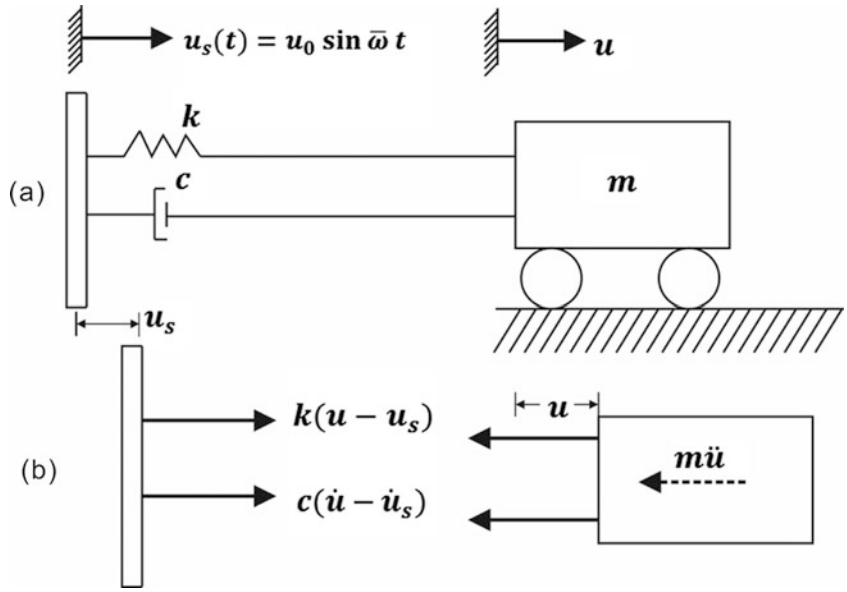

Fig. 3.12 (**a**) Damped simple oscillator harmonically excited through its support, (**b**) Free body diagram including inertial force

The substitution of Eq. (3.37) into Eq. (3.38) and the rearrangement of terms result in

$$m\ddot{u} + c\dot{u} + ku = ku_0 \sin \overline{\omega} t + c\overline{\omega} u_0 \cos \overline{\omega} t \tag{3.39}$$

The two harmonic terms of frequency $\overline{\omega}$ in the right-hand side of this equation may be combined and Eq. (3.39) rewritten [similarly to Eqs. (1.20) and (1.23)] as

$$m\ddot{u} + c\dot{u} + ku = F_0 \sin\left(\overline{\omega} t + \beta\right) \tag{3.40}$$

where

$$F_0 = u_0 \sqrt{k^2 + \left(c\overline{\omega}\right)^2} = u_0 k \sqrt{1 + (2r\xi)^2} \tag{3.41}$$

and

$$\tan \beta = c\overline{\omega}/k = 2r\xi \tag{3.42}$$

It is apparent that the differential Eq. (3.40) is of the same form as Eq. (3.10) for the oscillator excited by the harmonic force $F_0 \sin\left(\overline{\omega} t + \beta\right)$. Consequently, the steady-state solution of Eq. (3.40) is given as before by Eqs. (3.19) and (3.20), except for the addition of the angle β in the argument of the sine function, that is

$$u(t) = \frac{F_0/k \sin\left(\overline{\omega} t + \beta - \theta\right)}{\sqrt{\left(1 - r^2\right)^2 + \left(2r\xi\right)^2}} \tag{3.43}$$

or substituting F_0 from Eq. (3.41)

$$\frac{u(t)}{u_0} = \frac{\sqrt{1 + \left(2r\xi\right)^2}}{\sqrt{\left(1 - r^2\right)^2 + \left(2r\xi\right)^2}} \sin\left(\overline{\omega} t + \beta - \theta\right) \tag{3.44}$$

Equation (3.44) is the expression for the relative transmission of the support motion to the oscillator. This is an important problem in vibration isolation in which equipment must be protected from harmful vibrations of the supporting structure. The degree of relative isolation is known as transmissibility and is defined as the ratio of the amplitude of motion U of the oscillator to the amplitude u_0, the motion of the support. From Eq. (3.44), the transmissibility T_r is then given by

$$T_r = \frac{U}{u_0} = \frac{\sqrt{1 + \left(2r\xi\right)^2}}{\sqrt{\left(1 - r^2\right)^2 + \left(2r\xi\right)^2}} \tag{3.45}$$

Analogously to the motion transmitted, we may find the acceleration transmitted from the foundation to the mass. The acceleration transmitted to the mass is given by the second derivative of $u(t)$ in Eq. (3.44) as

$$\ddot{u}(t) = \frac{-\overline{\omega}^2 u_0 \sqrt{1 + \left(2r\xi\right)^2} \sin\left(\overline{\omega} t + \beta - \theta\right)}{\sqrt{\left(1 - r^2\right)^2 + \left(2r\xi\right)^2}} \tag{3.46}$$

while the acceleration $\ddot{u}_s(t)$ of the foundation is obtained from Eq. (3.37)

$$\ddot{u}_s(t) = -u_0 \overline{\omega}^2 \sin \overline{\omega} t \tag{3.47}$$

The acceleration transmissibility, T_r, is then given by the ratio of the amplitudes of the acceleration in Eqs. (3.46) and (3.47). Hence,

$$T_r = \frac{\sqrt{1 + \left(2r\xi\right)^2}}{\sqrt{\left(1 - r^2\right)^2 + \left(2r\xi\right)}} \tag{3.48}$$

It may be seen that the transmissibility of acceleration given by Eq. (3.48) is identical to Eq. (3.45), the transmissibility of displacements. Hence, the same expression will give either displacement or acceleration transmissibility.

A plot of transmissibility as a function of the frequency ratio and damping ratio is shown in Fig. 3.13. The curves in this figure are similar to the curves in Fig. 3.3, representing the frequency response of the damped oscillator. The major difference between these two sets of curves is that all of the curves in Fig. 3.13 pass through the same point at a frequency ratio $r = \sqrt{2}$. It can be seen in Fig. 3.13 that damping tends to reduce the effectiveness of vibration isolation for frequencies greater than this ratio, that is, for r greater than $\sqrt{2}$.

Plot Figs. 3.3, 3.4 and 3.13

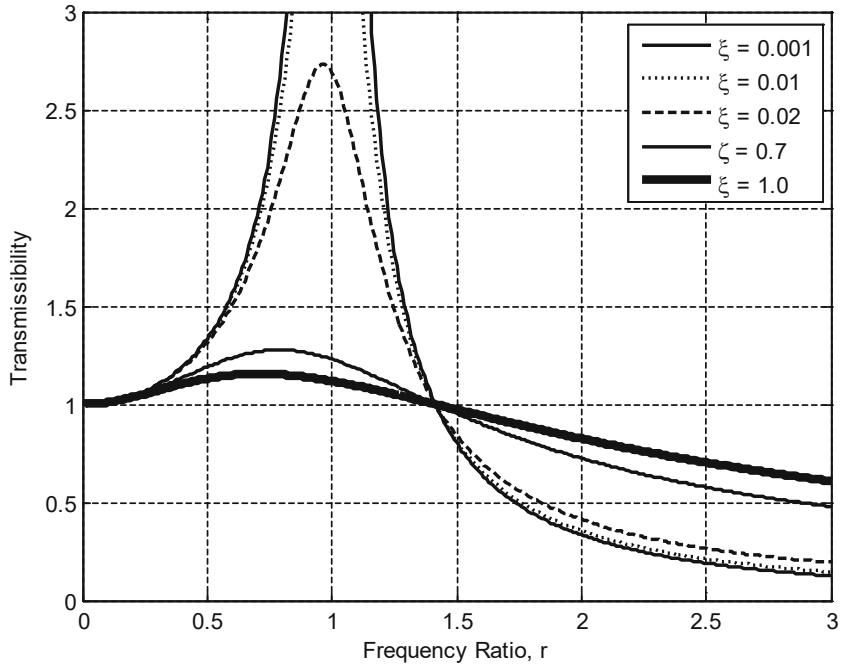

Fig. 3.13 Transmissibility versus frequency ratio for vibration isolation

```
clear all
clc
%%%%-GIVEN VALUES-%%%%
r = 0:0.01:3.0;                         %Frequency ratio ranging 0 to 3 with 0.01 intervals
xi = [0.001, 0.1, 0.2, 0.7, 1.0];       %Damping ratios of 0.001, 0.01, 0.02, 0.7, and 1.0

%%%-ESTIMATION-%%%
for i = 1:5
    z = xi(i);
    denom1 = (1-r.*r).^2;
    denom2 = (2*z*r).^2;
    denom = sqrt(denom1+denom2);
        D(i,:) = 1./denom;                  %Dynamic amplification factor (Eq. 3.24)

    denom3 = 2*xi(i).*r;
    denom4 = (1-r.^2);
    theta(i,:) = atand(denom3./denom4);     %Phase angle (Eq. 3.22)

    denom5 = sqrt(1+denom3.^2);
    T(i,:) =  denom5./denom;                %Transmissibility (Eq. 3.45)
end

%%%Create figures (Figs. 3.3, 3.4, and 3.13)
figure1 = figure;
```

```
%%%Fig. 3.3
    axes1= axes('Parent', figure1);
xlim(axes1, [0 3]);                    %set x limits for the plot
ylim(axes1, [0 6]);                    %set y limits for the plot

    box (axes1,'on');
    grid (axes1,'on');
    hold (axes1, 'all');

    plot1 = plot(r, D, 'Parent', axes1, 'LineWidth', 2, 'Color', [0 0 0]);
    set(plot1(1),'LineStyle','-','DisplayName','\xi = 0.001');
    set(plot1(2),'LineStyle',':','DisplayName','\xi = 0.01');
    set(plot1(3),'LineStyle','--','DisplayName','\xi = 0.02');
    set(plot1(4),'DisplayName','\zeta = 0.7');
    set(plot1(5),'LineWidth',4,'DisplayName','\xi = 1.0');

xlabel ('Frequency Ratio, r');
ylabel ('Dynamic Magnification Factor, D');
    legend (axes1, 'show');
    grid on

figure2 = figure;

%%%Fig. 3.4
    axes1= axes('Parent', figure2);
xlim(axes1, [0 3]);                     %set x limits for the plot
ylim(axes1, [0 180]);                   %set y limits for the plot

    box (axes1,'on');
    grid (axes1,'on');
    hold (axes1, 'all');

    theta(theta<0)=theta(theta<0)+180;
    plot2 = plot(r, theta, 'Parent', axes1, 'LineWidth', 2, 'Color', [0 0 0]);
    set(plot2(1),'LineStyle','-','DisplayName','\xi = 0.001');
    set(plot2(2),'LineStyle',':','DisplayName','\xi = 0.01');
    set(plot2(3),'LineStyle','--','DisplayName','\xi = 0.02');
    set(plot2(4),'DisplayName','\zeta = 0.7');
    set(plot2(5),'LineWidth',4,'DisplayName','\xi = 1.0');

xlabel ('Frequency Ratio, r');
ylabel ('Phase angle, \theta');
    legend (axes1, 'show');
    grid on

figure3 = figure;

%%%Fig. 3.13
    axes1= axes('Parent', figure3);
xlim(axes1, [0 3]);                    %set x limits for the plot
ylim(axes1, [0 3]);                    %set y limits for the plot

    box (axes1,'on');
    grid (axes1,'on');
    hold (axes1, 'all');

    plot2 = plot(r, T, 'Parent', axes1, 'LineWidth', 2, 'Color', [0 0 0]);
    set(plot2(1),'LineStyle','-','DisplayName','\xi = 0.001');
    set(plot2(2),'LineStyle',':','DisplayName','\xi = 0.01');
    set(plot2(3),'LineStyle','--','DisplayName','\xi = 0.02');
    set(plot2(4),'DisplayName','\zeta = 0.7');
    set(plot2(5),'LineWidth',4,'DisplayName','\xi = 1.0');

xlabel ('Frequency Ratio, r');
ylabel ('Transmissibility');
legend (axes1, 'show');
grid on
```

Illustrative Example 3.5

A delicate instrument weighing 100 lb is to be mounted on a rubber pad to the floor of a test laboratory where the vertical acceleration is 0.1 g at a frequency of $f = 10$ cps. It has been determined experimentally that the ratio of the stiffness, k, to the damping coefficient, c, is equal to 100 (1/sec) for the type of rubber pad material used in the isolation. What is the stiffness of the isolation required to reduce to 0.01 g the acceleration transmitted to the instrument?

Solution:

Setting the acceleration transmissibility given by Eq. (3.48) equal to $0.01g \,/\, 0.1g = 0.1$, we have

$$T_r = \frac{\sqrt{1 + (2r\xi)^2}}{\sqrt{(1-r)^2 + (2r\xi)^2}} = 0.1 \tag{a}$$

Beginning with an assumed value $\xi = 0.10$ for the damping ratio and squaring both sides of Eq. (a):

$$\frac{1 + 0.04r^2}{(1 - r^2)^2 + 0.04r^2} = 0.01$$

Then solving this resulting quadratic equation yields

$$\begin{aligned}
r^2 &= 13.346 \\
r &= \overline{\omega}/\omega = 3.653 \\
\overline{\omega} &= 2\pi f = 2\pi\, 10 = 62.83 \text{ rad/ sec} \\
\omega &= \overline{\omega}/r = 17.20 \text{ rad/ sec} \\
m &= 100/386 = 0.259 \text{ lb} \cdot \text{ sec}^2/\text{in} \\
k &= m\omega^2 = 0.259 \times 17.20^2 = 76.64 \text{ lb/in}
\end{aligned}$$

Now, we check the value of damping contained in the rubber spring:

$$k/c = 100$$

or

$$c = k/100 = 76.22/100 = 0.766 \text{ (lb} \cdot \text{ sec /in)}$$

Critical damping:

$$c_{cr} = 2\sqrt{km} = 2\sqrt{76.64 \times 0.259} = 8.91 \text{ (lb} \cdot \text{ sec /in)}$$

Then, the calculated damping ratio is

$$\xi = c/c_{cr} = 0.766/8.91 = 0.086$$

which is somewhat less that the assumed value $\xi = 0.10$. If desired, an iterative cycle could be performed introducing $\xi = 0.086$ in Eq. (a) and repeating the calculations.

3.7.2 Relative Motion

Equation (3.43) provides the absolute response of the damped oscillator to a harmonic motion of its base. Alternatively, we can solve the differential Eq. (3.38) in terms of the relative motion between the mass m and the support given by

$$u_r = u - u_s \tag{3.49}$$

which substituted into Eq. (3.38) results in

$$m\ddot{u}_r + c\dot{u}_r + ku_r = F_{eff} \sin \overline{\omega}\, t \tag{3.50}$$

where $F_{eff} = -m\ddot{u}_s$ may be interpreted as the amplitude of the effective force acting on the mass of the oscillator with the displacement indicated by coordinate u_r. Using Eq. (3.37) to obtain $\ddot{u}_s$ and substituting in Eq. (3.50) results in

$$m\ddot{u}_r + c\dot{u}_r + ku_r = mu_0\overline{\omega}^2 \sin \overline{\omega} t \tag{3.51}$$

Again, Eq. (3.51) is of the same form as Eq. (3.10) with $F_0 = mu_0\overline{\omega}^2$. Then, from Eqs. (3.19) and (3.20), the steady-state response in terms of relative motion is given by

$$\frac{u_r(t)}{u_0} = \frac{r^2 \sin\left(\overline{\omega} t - \theta\right)}{\sqrt{(1-r^2)^2 + (2r\xi)^2}} \tag{3.52}$$

where θ is given in Eq. (3.21),

$$r^2 = \frac{\overline{\omega}^2}{\omega^2} \quad \text{and} \quad \omega^2 = \frac{k}{m}.$$

The maximum relative amplitude of the displacement U_r,

$$U_r = \frac{u_0 r^2}{\sqrt{(1-r^2)^2 + (2r\xi)^2}} \tag{3.53}$$

Illustrative Example 3.6

If the frame of Illustrative Example 3.2 (Fig. 3.7) is subjected to a sinusoidal ground motion $u_s(t) = 0.2 \sin 5.3t$, determine: (a) the transmissibility of motion to the girder, (b) the maximum shearing force in the supporting columns, and (c) maximum stresses in the columns.

Solution:

(a) The parameters for this system are calculated in Illustrative Example 3.2 as

$$k = 2136 \text{ lb/in}$$
$$\xi = 0.05$$
$$u_0 = 0.2 \text{ in}$$
$$u_{st} = 0.0936 \text{ in}$$
$$\omega = 7.41 \text{ rad/ sec}$$
$$\overline{\omega} = 5.3 \text{ rad/ sec}$$
$$\text{r} = 0.715$$

The transmissibility from Eq. (3.45) is

$$T_r = \sqrt{\frac{1 + (2r\xi)^2}{(1-r^2)^2 + (2r\xi)^2}} = 2.1 \tag{Ans}$$

(b) The maximum relative amplitude of the displacement U_r is from Eq. (3.53)

$$U_r = \frac{u_0 r^2}{\sqrt{(1-r^2)^2 + (2r\xi)^2}} = 0.206 \text{ in}$$

Then the maximum shear force in each column is

$$V_{\max} = \frac{kU_r}{2} = 219.8 \text{ lb} \qquad \text{(Ans)}$$

(c) The maximum bending moment

$$M_{\max} = V_{\max} L = 39,507 \text{ lb} \cdot \text{in}$$

and the corresponding stress

$$\sigma_{\max} = \frac{M_{\max}}{I/c} = \frac{39,567}{17} = 2327 \text{ psi} \qquad \text{(Ans)}$$

in which I / c is the section modulus.

Illustrative Example 3.7

A machine having a total weight of 1800 lb, including its foundation, is to be isolated from the vibration of the ground, which is $f = 22.8$ cps, due to other machines operating nearby. Determine the stiffness of a rubber isolation spring to limit the transmitted vibration to 1/10: (a) neglect damping and (b) consider damping given by the expression $c = k/170$ obtained experimentally [units of c (lb.sec/in) and of k (lb/in)].

Solution:

(a) $\xi = 0$

By Eq. (3.45) with $\xi = 0$

$$T_r = \frac{U}{u_0} = \frac{1}{\pm(1-r^2)} = 0.1$$

$$-1 + r^2 = 10 \qquad r^2 = 11$$

$$r = 3.3166 = \frac{\overline{\omega}}{\omega}$$

$$\overline{\omega} = 2\pi f = 143.24 \text{ rad/ sec}$$

$$m = \frac{1800}{386} = 4.663 \text{ lb} \cdot \text{ sec}^2/\text{in}$$

$$\omega = \frac{\overline{\omega}}{r} = \frac{143.24}{3.3166} = 43.188 \text{ rad/ sec}$$

Then,

$$k = \omega^2 m = (43.188)^2 (4.663) = 8698 \text{ lb/in} \qquad \text{(Ans.)}$$

(b) Assume $\xi = 0.10$
(c) Squaring Eq. (3.45) and substituting $\xi = 0.10$ gives

$$T_r^2 = \frac{1 + (0.2r)^2}{(1 - r^2)^2 + (0.2r)^2} = \left(\frac{1}{10}\right)^2$$

which results in the following quadratic equation in r^2:

$$r^4 - 5.96r^2 - 99 = 0$$

$$r^2 = 13.366 \qquad \mathrm{r} = \frac{\overline{\omega}}{\omega} = 3.65$$

and

$$\omega = \frac{\overline{\omega}}{r} = \frac{143.24}{3.65} = 39.244 \ (\mathrm{rad/\ sec}\ )$$

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{k}{4.663}} = 39.244 \ (\mathrm{rad/\ sec}\ )$$

giving

$$k = 7181.3 \ \mathrm{lb/in}$$

The damping is then determined from

$$c = k/170 = 7181.3/170$$

as

$$c = 42.24 \ \mathrm{lb \cdot\ sec\ /in}$$

and the damping ratio as

$$c_{cr} = 2\sqrt{km} = 2\sqrt{(7181.3)(4.663)} = 348.34 \ (\ \mathrm{lb \cdot\ sec\ /in})$$

$$\xi = \frac{42.24}{384.34} = 0.1212 = 12\%$$

which is slightly higher than the value $\xi = 0.10$, initially assumed. Now, repeating the calculations for $\xi = 0.11$ gives

$$r^4 - 6.7916 - 99 = 0$$

$$r^2 = 13.909 \qquad \mathrm{r} = \frac{\overline{\omega}}{\omega} = 3.73$$

$$\omega = \frac{143.24}{3.73} = 38.40 = \sqrt{\frac{k}{m}}$$

$$k = 6876 \ \mathrm{lb/in} \quad \text{and} \quad c = \frac{k}{170} = 40.45 \ \mathrm{lb \cdot\ sec\ /in} \qquad \text{(Ans)}$$

$$c_{cr} = 2\sqrt{km} = 2\sqrt{6876 \times 4.663} = 358.14 \ \mathrm{lb \cdot\ sec\ /in}$$

and

$$\xi = \frac{40.45}{358.14} = 0.113 = 11\%$$

This calculated value, $\xi = 11\%$ for the damping ratio, agrees with the last value tried. Therefore, the required spring constant for the damped isolation is $k = 6876$ lb/in as calculated above.

3.8 Force Transmitted to the Foundation

In the preceding section, we determined the response of the structure to a harmonic motion of its foundation. In this section we shall consider a similar problem of vibration isolation; the problem now, however, is to find the force transmitted to the foundation. Consider again the damped oscillator with a harmonic force $F(t) = F_0 \sin \overline{\omega} t$ acting on its mass as shown in Fig. 3.2. the differential equation of motion is

$$m\ddot{u} + c\dot{u} + ku = F_0 \sin \overline{\omega} t$$

with the steady-state solution, Eq. (3.19),

$$u = U \sin\left(\overline{\omega} t - \theta\right)$$

where U and θ, are given, respectively, by Eqs. (3.20) and (3.21) as

$$U = \frac{F_0/k}{\sqrt{(1-r^2)^2 + (2r\xi)^2}} \tag{3.53}$$

and

$$\tan\theta = \frac{2\xi r}{1-r^2}$$

The force transmitted to the support through the spring is ku and through the damping element is $c\dot{u}$. Hence the total force transmitted F_T is

$$F_T = ku + c\dot{u} \tag{3.54}$$

Differentiating Eq. (3.19) and substituting in Eq. (3.54) yields

$$F_T = U\left[k \sin\left(\overline{\omega} t - \theta\right) + c\overline{\omega} \cos\left(\overline{\omega} t - \theta\right)\right]$$

or

$$F_T = U\sqrt{k^2 + c^2\overline{\omega}^2} \sin\left(\overline{\omega} t - \theta + \beta\right) \tag{3.55}$$

$$F_T = Uk\sqrt{1 + (2r\xi)^2} \sin\left(\overline{\omega} t - \phi\right) \tag{3.56}$$

in which

$$\tan\beta = \frac{c\overline{\omega}}{k} = 2\xi r \tag{3.57}$$

and

$$\phi = \theta - \beta \tag{3.58}$$

Then, from Eqs. (3.53) and (3.56), the maximum force A_T transmitted to the foundation is

$$A_T = F_0 \sqrt{\frac{1 + (2\xi r)^2}{(1 - r^2)^2 + (2r\xi)^2}} \tag{3.59}$$

In this case, the transmissibility T_r is defined as the ratio between the amplitude of the force transmitted to the foundation and the amplitude of the applied force. Hence from Eq. (3.59)

$$T_r = \frac{A_T}{F_0} = \sqrt{\frac{1 + (2\xi r)^2}{(1 - r^2)^2 + (2r\xi)^2}} \tag{3.60}$$

It is interesting to note that, both, the transmissibility of motion from the foundation to the structure, Eq. (3.45), and the transmissibility of the force from the structure to the foundation, Eq. (3.60), are given by exactly the same function. Hence the curves of transmissibility in Fig. 3.13 represent either type of transmissibility. An expression for the total phase angle ϕ in Eq. (3.56) may be determined by taking the tangent function to both members of Eq. (3.58), so that

$$\tan\phi = \frac{\tan\theta - \tan\beta}{1 + \tan\theta \tan\beta}$$

Then, the substitution of $\tan\theta$ and $\tan\beta$, respectively, from Eqs. (3.21) and (3.57) results in

$$\tan\phi = \frac{2\xi r^3}{1 - r^2 + 4\xi^2 r^2} \tag{3.61}$$

Illustrative Example 3.8

A machine of weight $W = 3860$ lb is mounted on a simple supported steel beam as shown in Fig. 3.15a. A piston that moves up and down in the machine produces a harmonic force of magnitude $F_0 = 7000$ lb at a frequency of $\overline{\omega} = 60$ rad/sec. Neglecting the weight of the beam and assuming 10% of the critical damping, determine: (a) the amplitude of the motion of the machine, (b) the force transmitted to the beam supports, and (c) the corresponding phase angle.

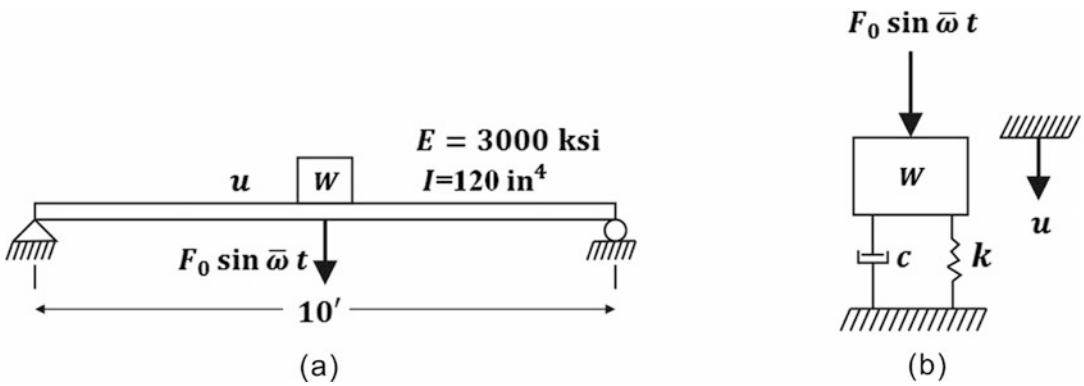

Fig. 3.14 (**a**) Beam-machine system for Illustrative Example 3.8. (**b**) Analytical Model

Solution:
The damped oscillator in Fig. 3.14b is used to model the system. The following parameters are calculated:

$$k = \frac{48EI}{L^3} = 10^5 \text{ lb/in}$$

$$\omega = \sqrt{\frac{k}{m}} = 100 \text{ rad/ sec}$$

$$\xi = 0.1$$

$$r = \frac{\overline{\omega}}{\omega} = 0.6$$

$$u_{st} = \frac{F_0}{k} = 0.07 \text{ in}$$

(a) From Eq. (3.20), the amplitude of motion is

$$U = \frac{u_{st}}{\sqrt{(1-r^2)^2 + (2r\xi)^2}} = 0.1075 \text{ in} \qquad \text{(Ans)}$$

with a phase angle from Eq. (3.21)

$$\theta = \tan^{-1} \frac{2r\xi}{1-r^2} = 10.6^0$$

(b) From Eq. (3.60), the transmissibility is

$$T_r = \frac{A_T}{F_0} = \sqrt{\frac{1 + (2r\xi)^2}{(1-r^2)^2 + (2r\xi)^2}} = 1.547$$

Hence the amplitude of the force transmitted to the foundation is

$$A_T = F_0 T_r = 10,827 \text{ lb} \qquad \text{(Ans)}$$

(c) The corresponding phase angle from Eq. (3.61) is

$$\phi = \tan^{-1} \frac{2\xi r^3}{1 - r^2 + (2r\xi)^2} = 3.78^\circ \qquad \text{(Ans)}$$

3.9 Seismic Instruments

When a system of the type shown in Fig. 3.15 is used for the purpose of vibration measurement, the relative displacement between the mass and the base is recorded. Such an instrument is called a

seismograph and it can be designed to measure either the displacement or the acceleration of the base. The peak relative response U/u_0 of the seismograph depicted in Fig. 3.15, for harmonic motion of the base, is given from Eq. (3.52) by

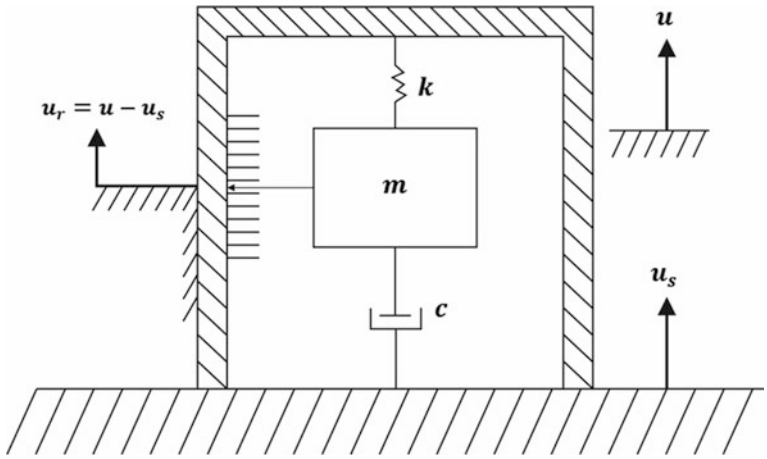

Fig. 3.15 Model of a seismograph

$$\frac{U_r}{u_0} = \frac{r^2}{\sqrt{(1-r^2)^2 + (2r\xi)^2}} \tag{3.62}$$

A plot of this equation as a function of the frequency ratio and damping ratio is shown in Fig. 3.16. It may be seen from this figure that the response is essentially constant for frequency ratios $r > 1$ and damping ratio $\xi = 0.5$. Consequently, the response of a properly damped instrument of this type is essentially proportional to the base-displacement amplitude for high frequencies of motion of the base. The instrument will thus serve as a displacement meter for measuring such motions. The range of applicability of the instrument is increased by reducing the natural frequency, i.e., by reducing the spring stiffness or increasing the mass.

Now consider the response of the same instrument to a harmonic acceleration of the base $\ddot{u}_s = \ddot{u}_0 \sin \overline{\omega} t$. The equation of motion of this system is obtained from Eq. (3.50) as

$$m\ddot{u} + c\dot{u} + ku = -m\ddot{u}_0 \sin \overline{\omega} t \tag{3.63}$$

The steady-state response of this system expressed as the dynamic magnification factor is then given from Eq. (3.23) by

$$D = \frac{U}{m\ddot{u}_0/k} = \frac{1}{\sqrt{(1-r^2)^2 + (2r\xi)^2}} \tag{3.64}$$

This equation is represented graphically in Fig. 3.3. In this case, it can be seen from this figure that for a damping ratio $\xi = 0.7$, the value of the response is nearly constant in the frequency range $0 < r < 0.6$. Thus, it is clear from Eq. (3.64) that the response indicated by this instrument will be directly proportional to the base-acceleration amplitude for frequencies up to about six-tenths of the natural frequency. Its range of applicability will be increased by increasing the natural frequency, that is, by increasing the stiffness of the spring or by decreasing the mass of the oscillator. Such an instrument is an accelerometer.

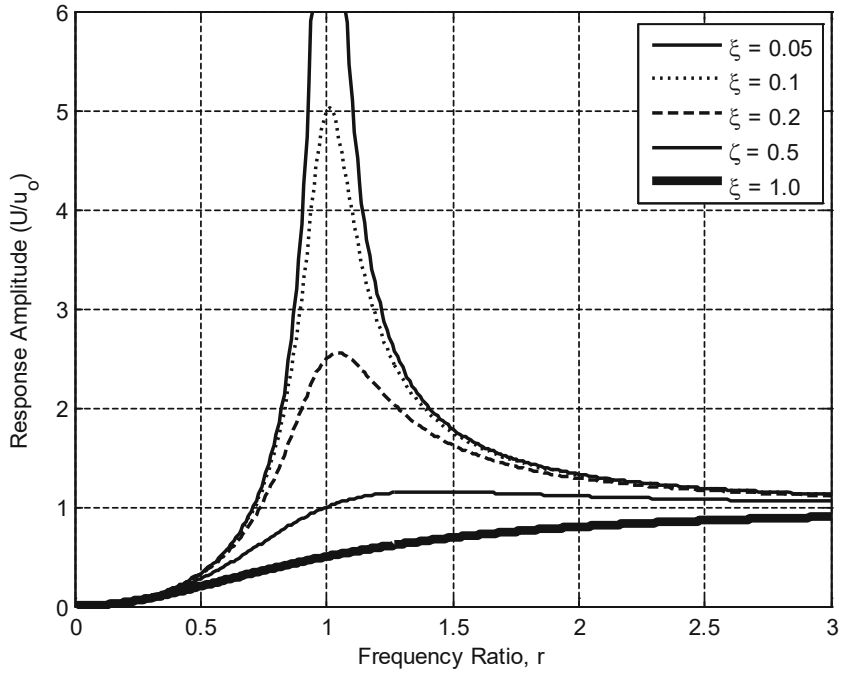

Fig. 3.16 Response of seismograph to harmonic motion of the base

3.10 Response of One-Degree-of-Freedom System to Harmonic Loading Using MATLAB

Illustrative Example 3.9

The steel frame shown in Fig. 3.17 supports a rotating machine that exerts a horizontal force at the girder level $F(t) = 200 \sin 2.3t$ (lb). The frequency of force is equal to 2.3 rad/sec. Assume 5% of critical damping and determine: (a) The maximum displacement of a total vibration between 0 to 10 sec, and (b) The maximum dynamic stress in the columns. Assume the girder is rigid. (This is the same structure in Illustrative Example 3.2) (Fig. 3.18).

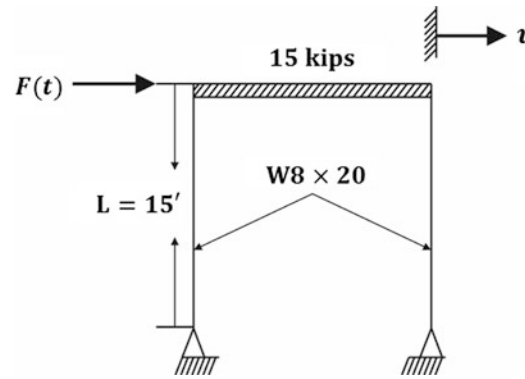

Fig. 3.17 Diagram of the frame of Illustrative Example 3.9

Solution:
The parameters for this structure were previously calculated in Illustrative Example 3.2 as:

Mass:	$m = 15 \times 1000/386$ (lb.sec^2/in)
Stiffness:	$k = \frac{3E(2I)}{L^3} = \frac{3 \times 30 \times 10^6 \times 2 \times 69.2}{(12 \times 15)^3} = 2136$ lb/in
Damping ratio:	$\xi = 0.05$
Amplitude Harmonic Force:	$F_0 = 200$ (lb)
Force time function:	$F(t) = \sin 2.3t$
Period (of the force)	$2.3T = 2\pi$. Therefore, $T = 2.73$ sec
Time Step (Select 20 steps):	0.01 sec.

For running the files, two files are needed to save in the same folder and run main file. Ex3_9.m file will use SDOF.m file to solve partial differential equation to calculate the response. The original framework of MATLAB code is well presented in Anderson and Naeim (2012).

The approach is transform one "second order differential equation" to two "first order differential equations."

$$\frac{d\dot{u}}{dt} = -2\xi\omega\dot{u} - \omega^2 u$$
$$u = u_1;$$
$$\dot{u}_1 = u_2 = \frac{du_1}{dt}$$
$$\frac{du_2}{dt} = -2\xi\omega u_2 - \omega^2 u_1$$

$$\rightarrow \quad u_2 = \frac{du_1}{dt} \rightarrow u(2)\ [MATLAB]$$
$$\frac{du_2}{dt} = -2\xi\omega u_2 - \omega^2 u_1 \rightarrow -2\xi\omega u(2) - \omega^2 u(1)\ [MATLAB]$$

Two first order differential equations are solved simultaneously in SDPF.m file.

Matlab file: Ex3_9.m

```
close all
clear
clc

%%%%-GIVEN VALUES-%%%%
%%% Initial Conditions
tspan = 0:0.01:50;                %0 to 5 secs with the interval of 0.01 sec
IC = [0 0]';                      %Initial conditions (u0=0, v0=0)

%%%Plot the displacement using ODE45
%%%Estimate response using ODE45 funtion embedded in MATALB
[t, u] = ode45(@SDOF, tspan, IC);
plot (t, u(:,1));

%%%Create xlabel
xlabel ('t(sec)');
ylabel ('u(in.)');

%%%Display maximum value of displacement response
umax=max(u(:,1))
```

MATLAB file Name: SDOF.m

```
function  u = SDOF(t, u)

%%%%-GIVEN VALUES-%%%%
m =15*1000/386;                     %Mass (lb.sec^2/in.)
k= 3*30*10^6*2*69.2/(12*15)^3;      %Stiffness (lb/in.)
xi = 0.05;                          %Damping ratio.

%%%Define the forcing function
F = 200*sin(2.3*t);                 %Force as a function of time, t

omega =sqrt(k/m);                   %Natural Frequency

%%%%-ESTIMATION-%%%%
u = [u(2); -omega^2*u(1)-2*xi*omega*u(2)+F/m];
```

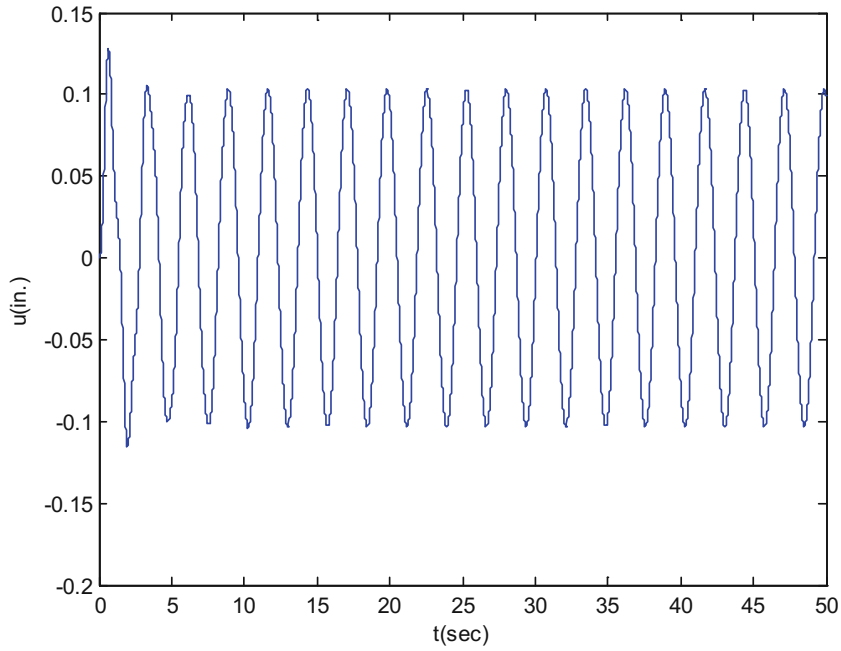

Fig. 3.18 Response of Illustrative Example 3.9

Illustrative Example 3.10

The steel frame In Illustrative Example 3.9 is supports a rotating machine that exerts a horizontal force at the girder level $F(t) = 200 \sin 7.41t$ (lb). Use MATLAB to determine the response. The solution is found using the MATLAB function based ODE 4 (A fourth-order Runge-Kutta method.). Other functions of MATLAB codes are also introduced in Anderson and Naeim (2012). More information on the numerical method can be explained in Chap. 4. In this example, the solution can be easily obtained from changes of two files (Fig. 3.19).

Note: The frequency of force is equal to the natural frequency.

Matlab file: Ex3_10.m

```
close all
clear
clc

%%%%-GIVEN VALUES-%%%%
%%% Initial Conditions
tspan = 0:0.01:50;                %0 to 5 secs with the interval of 0.01 sec
IC = [0 0]';                      %Initial conditions (u0=0, v0=0)

%%% Plot the displacement using ODE45
%%%Estimate response using ODE45 funtion embedded in MATALB
[t, u] = ode45(@SDOF1, tspan, IC);
figure (1)
plot (t, u(:,1));

% Create xlabel
xlabel ('t(sec)');
ylabel ('u(in.)');

k= 3*30*10^6*2*69.2/(12*15)^3;
F_0 = 200;

% Display maximum value of displacement response
umax=max(u(:,1))
```

MATLAB file Name: SDOF1.m

```
function  u = SDOF1(t, u)

%%%%-GIVEN VALUES-%%%%
m =15*1000/386;                  %Mass (lb.sec^2/in.)
k= 3*30*10^6*2*69.2/(12*15)^3;   %Stiffness (lb/in.)
xi = 0.05;                       %Damping coefficient. (lb.sec/in.)

%%%Define the forcing function
F_0 = 200;
F = F_0*sin(7.41*t);             %Force as a function of time, t

omega =sqrt(k/m)                 %Natural frequency

u_st = F_0/k;                    %Static displacement

%%%%-ESTIMATION-%%%%
u = [u(2); -omega^2*u(1)-2*xi*omega*u(2)+F/m];
```

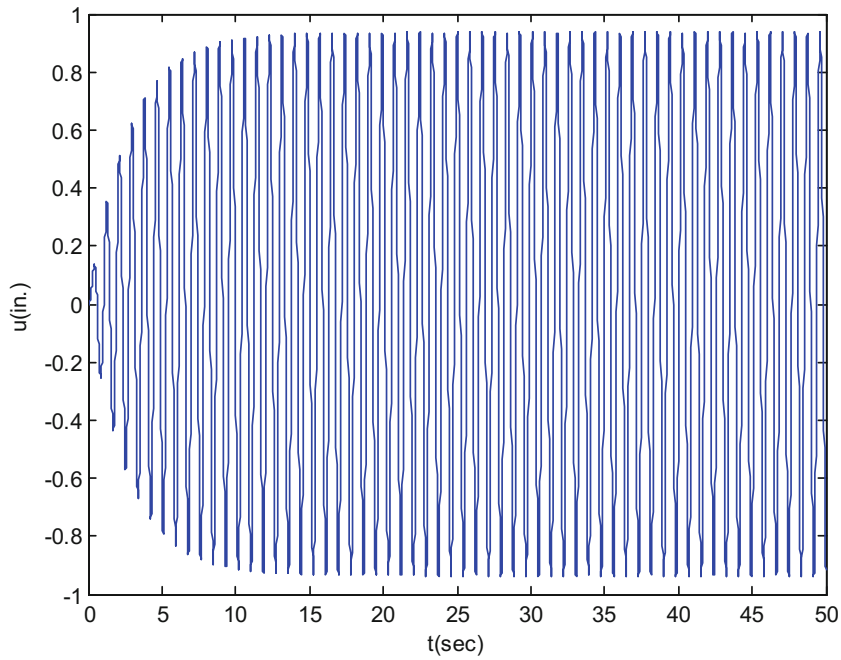

Fig. 3.19 Response of Illustrative Example 3.10

Illustrative Example 3.11
The steel frame in Illustrative Example 3.9 is now subjected to an acceleration at its base given by the function $F(t) = 0.3 \sin 5.3t$ in/sec^2. Use MATLAB to determine the response.

3.11 Summary

In this chapter, we have determined the response of a single-degree-of-freedom system subjected to harmonic loading. This type of loading is expressed as a sine, cosine, or as an exponential function

and can be easily handled mathematically for the undamped or damped structure. The differential equation of motion for a linear single-degree-of-freedom system is the second-order differential equation

$$m\ddot{y} + c\dot{u} + ku = F_0 \sin \overline{\omega} t \qquad (3.10) \text{ repeated}$$

or

$$\ddot{u} + 2\xi\omega\dot{u} + \omega^2 u = \frac{F_0}{m} \sin \overline{\omega} t$$

in which $\overline{\omega}$ is the forced frequency,

$$\xi = \frac{c}{c_{cr}} \quad \text{isthedampingratio}$$

and

$$\omega = \sqrt{\frac{k}{m}} \quad \text{is the natural frequency}$$

The general solution of Eq. (3.10) is obtained as the combination of the complementary (transient) and the particular (steady-state) solutions, namely

$$u = \underbrace{e^{-\xi\omega t}(A \cos \omega_D t + B \sin \omega_D t)}_{\text{transient solution}} + \underbrace{\frac{F_0/k \sin\left(\overline{\omega} t - \theta\right)}{\sqrt{(1 - r^2)^2 + (2r\xi)^2}}}_{\text{steady state solution}}$$

in which

$$r = \frac{\overline{\omega}}{\omega} \qquad \text{is the frequcency ration,}$$

$$\omega_{\mathrm{D}} = \omega\sqrt{1 - \xi^2} \qquad \text{is the damped natural frequency,}$$

$$\theta = \tan^{-1}\left(\frac{2r\xi}{1 - r^2}\right) \qquad \text{is the phase angle}$$

and

A and B are constants of integration which can be determined from the initial conditions.

The transient part of the solution vanishes rapidly to zero because of the negative exponential factor, thus leaving only the steady-state solution. Of particular significance is the condition of resonance $\left(r = \overline{\omega}/\omega = 1\right)$ for which the amplitude of motion become very large for the damped system and tend to become infinity for the undamped system.

The response of the structure to support or foundation motion can be obtained in terms of the absolute motion of the mass or of its relative motion with respect to the support. In this latter case, the equation assumes a simpler and more convenient form, namely

$$m\ddot{u}_r + c\dot{u}_r + ku_r = F_{eff}(t) \qquad (3.50) \text{ repeated}$$

in which

$$u_r = u - u_s \quad \text{is the relative displcaement}$$

and

$$F_{eff}(t) = -m\ddot{u}_s(t) \quad \text{is the effective force}$$

For harmonic excitation of the foundation, the solution of Eq. (3.50) in terms of the relative motion is of the same form as the solution if Eq. (3.10) in which the force is acting on the mass.

In this chapter, we have also shown that the equivalent damping in the system may be evaluated experimentally either from the peak amplitude or from the bandwidth obtained from a plot of the amplitude-frequency curve when the system is forced to harmonic vibration. Most commonly, equivalent viscous damping is evaluated by equating the experimentally measured energy dissipated in the system during a vibratory cycle at the resonant frequency to the theoretically calculated energy that the system, assumed viscously damped, would dissipate in a cycle. This approach leads to the following expression for the equivalent viscous damping:

$$\xi_{eq} = \frac{1}{4\pi r}\frac{E^*}{E_s} \qquad (3.35)\ \text{repeated}$$

in which

E^* = energy dissipated in the system during a cycle of harmonic vibration at resonance
E_s = strain energy stored at maximum displacement if the system were elastic
r = ratio of forced vibration frequency to the natural frequency of the system

Two related problems of vibrating isolation were discussed in this chapter: (1) the motion transmissibility, that is, the relative motion transmitted from the foundation to the structure; and (2) the force transmissibility which is the relative magnitude of the force transmitted from the structure to the foundation. For both of these problems, the transmissibility is given by

$$T_r = \sqrt{\frac{1 + (2r\xi)^2}{(1 - r^2)^2 + (2r\xi)^2}}$$

3.12 Analytical Problem

Problem 3.1

Demonstrate that during one cycle in harmonic vibration, the work W_F of the external force $F = F_0 \sin \overline{\omega} t$ is equal to the energy E_D dissipated by the damping force as expressed by Eq. (3.30).

$$E_D = 2\pi \xi r k U^2 \qquad (3.30)\ \text{repeated}$$

Solution:

During one cycle, the work of the external force $F = F_0 \sin \overline{\omega} t$ is

$$W_F = \int_0^{2\pi/\overline{\omega}} F_0 \sin \overline{\omega} t \ \ dy = \int_0^{2\pi/\overline{\omega}} F_0 \sin \overline{\omega} t \frac{du}{dt} dt$$

$$= \int_0^{2\pi/\overline{\omega}} F_0 \sin \overline{\omega} t \ \ \dot{u}(t)\, dt$$

in which $\dot{u}(t)$ is given by Eq. (3.29). Hence,

$$\begin{aligned} W_F &= \int \left[F_0 \sin \overline{\omega}\ t\right][\overline{\omega} U \cos \overline{\omega}(t-\theta)dt] \\ &= \pi F_0 U \sin\theta \end{aligned} \tag{a}$$

To demonstrate that work, W_F, of the exciting force given by Eq. (a) is equal to the energy dissipated, E_D, by the viscous force in Eq. (3.30), we need to substitute the sine of the phase angle θ into Eq. (a).

Thus from Eq. (3.21), we have:

$$\begin{aligned} \tan\theta &= \frac{2\xi r}{1-r^2} \\ \frac{\sin\theta}{\cos\theta} &= \frac{2\xi r}{1-r^2} \\ \frac{\sin^2\theta}{\sin^2\theta + \cos^2\theta} &= \frac{(2\xi r)^2}{(1-r^2)^2 + (2r\xi)^2} \end{aligned} \tag{3.21 repeated}$$

Therefore

$$\sin\theta = \frac{2\xi r}{\sqrt{(1-r^2)^2 + (2r\xi)^2}}$$

Then using Eq. (3.20),

$$U = \frac{u_{st}}{\sqrt{(1-r^2)^2 + (2r\xi)^2}} \tag{3.20 repeated}$$

$$\sin\theta = \frac{2\xi U}{u_{st}}$$

which substituted in Eq. (a) yields

$$W_F = \pi F_0 U^2 \frac{2\xi r}{F_0/k}$$

$$W_F = 2\pi\xi\ r k U^2 \tag{b}$$

Thus, the work of external force, W_F, expressed by Eq. (b), is equal to the energy, E_D, dissipated per cycle by the viscous damping force as given by Eq. (3.30).

3.13 Problems

Problem 3.2

An electric motor of total weight $W = 1000$ lb is mounted at the center of a simply supported beam as shown in Fig. P3.2). The unbalance in the rotor is $W'e = 1$ lb.in. Determine the steady-state amplitude of vertical motion of the motor for a speed of 900 rpm. Assume that the damping in the system is 10% of the critical damping. Neglect the mass of the supporting beam.

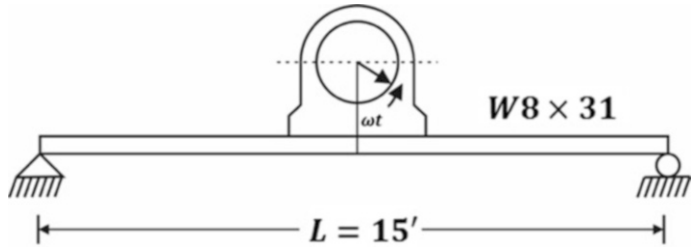

Fig. P3.2

Problem 3.3
Determine the maximum force transmitted to the supports of the beam in Problem 3.2.

Problem 3.4
Determine the steady-state amplitude for the horizontal motion of the steel frame in Fig. P3.4. Assume the horizontal girder to be infinitely rigid and neglect both the mass of the columns and damping. Using MATLAB program, determine the response of frame structure from 0 to 20 s.

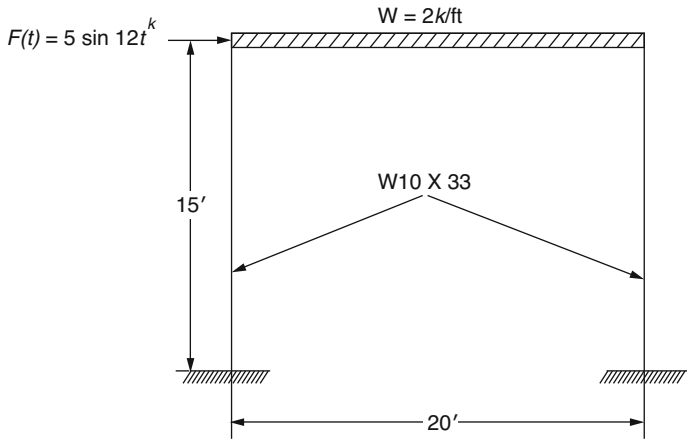

Fig. P3.4

Problem 3.5
Solve for Problem 3.4 assuming that the damping in the system is 8% of the critical damping.

Problem 3.6
For Problem 3.5 determine: (a) the maximum force transmitted to the foundation and (b) the transmissibility.

Problem 3.7
A delicate instrument is to be spring mounted to the floor of a test laboratory where it has been determined that the floor vibrates vertically with harmonic motion of amplitude 0.1 at 10 cps. If the instrument weighs 100 lb, determine the stiffness of the isolation springs required to reduce the vertical motion amplitude of the instrument to 0.01 in. Neglect damping.

Problem 3.8

Consider the water tower shown in Fig. P3.8 which is subjected to ground motion produced by a passing train in the vicinity of the tower. The ground motion is idealized as a harmonic acceleration of the foundation of the tower with an amplitude of 0.1 g at a frequency of 10cps. Determine the motion of the tower relative to the motion of its foundation. Assume an effective damping coefficient of 10% of the critical damping in the system.

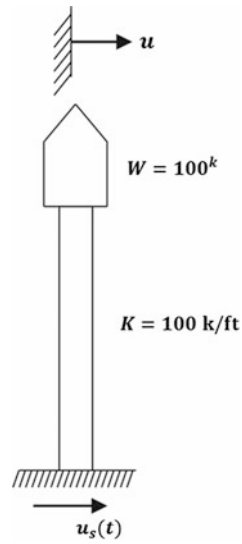

Fig. P3.8

Problem 3.9

Determine the transmissibility in Problem 3.8.

Problem 3.10

An electric motor of total weight $W = 3330$ lb is mounted on a simple supported beam with overhang as shown in Fig. P3.10. The unbalance of the rotor is $W'e = 50$ lb.·in. (a) Find the amplitudes of forced vertical vibration of the motor for speeds 800, 1000, and 1200 rpm. (b) Draw a plot of the amplitude versus rpm Assume damping equal to 10% of critical damping using MATLAB.

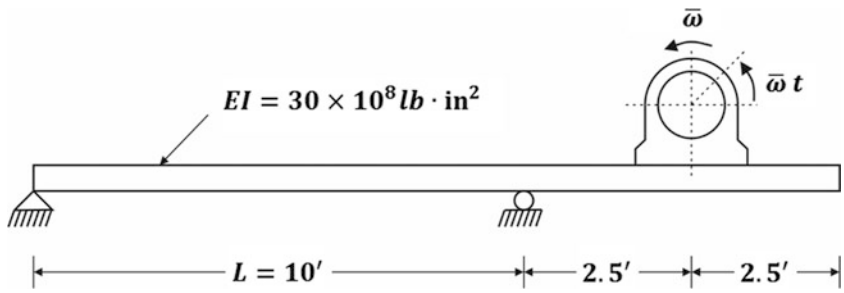

Fig. P3.10

Problem 3.11
Estimate the damping in a single-degree-of-freedom system that is excited by a harmonic force. The peak displacement amplitude at resonance was measured equal to 3 in and equal to 0.2 in at one-tenth of the natural frequency of the system.

Problem 3.12
Determine the damping in a system in which during a vibration test under a harmonic force it was observed that at a frequency 10% higher than the resonant frequency, the displacement amplitude was exactly one-half of the resonant amplitude.

Problem 3.13
Determine the natural frequency, amplitude of vibration, and maximum normal stress in the simple supported beam carrying an engine of weight $W = 30$ KN (Fig. P3.13). The engine rotates at 400 rpm and induces a vertical force $F(t) = 8 \sin \overline{\omega} t$. ($E = 210 \times 10^9$ N/m^2, $I = 8950 \times 10^{-8}$ m^4, $S = 597 \times 10^6$ m^3)
(Problem contributed by Vladimir N. Alekhin and Aleksey A. Antipin of the Urals State University, Russia)

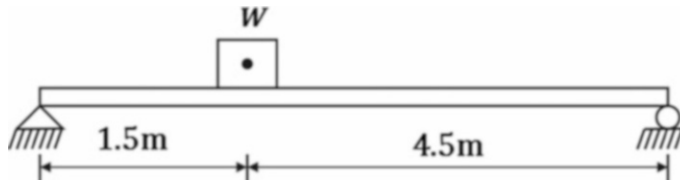

Fig. P3.13

Problem 3.14
A machine of mass m rests on an elastic floor as shown in Fig. P3.14. In order to find the natural frequency of the vertical motion, a mechanical shaker of mass m_s is bolted to the machine and run at various speeds until the resonant frequency f_r is found. Determine the natural frequency f_n of the floor-machine system in terms of f_r and the given data.

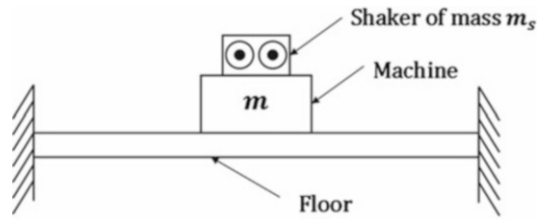

Fig. P3.14

Problem 3.15
Determine the frequency at which the peak amplitude of a damped oscillator will occur. Also, determine the peak amplitude and corresponding phase angle.

Problem 3.16

A structure modeled as a damped spring-mass system (Fig. P3.16) with $mg = 2520$ lb, $k = 89{,}000$ lb/in, and $c = 112$ lb. in/sec is subjected to a harmonic exciting force. Determine: (a) the natural frequency, (b) the damping ratio, (c) the amplitude of the exciting force when the peak amplitude of the vibrating mass is measured to be 0.37 in. and (d) the amplitude of the exciting force when the amplitude measured is at the peak frequency assumed to be the resonant frequency.

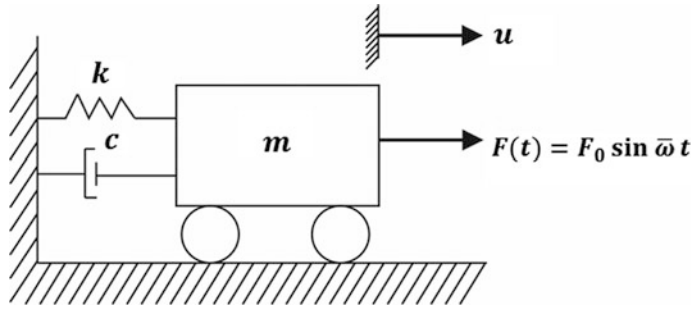

Fig. P3.16

Problem 3.17

A structural system modeled as a damped oscillator is subjected to the harmonic excitation produced by an eccentric rotor. The spring constant k and the mass m are known but not the damping and the amount of unbalance in the rotor, From measured amplitudes U_r at resonance and U_1 at a frequency ratio $r_1 \neq 1$, determine expressions to calculate the damping ratio ξ and the amplitude of the exciting force F_r at resonance.

Problem 3.18

A system is modeled by two vibrating masses m_1 and m_2 interconnected by a spring k and damper element c (Fig. P3.18). For harmonic force $F = F_0 \sin \overline{\omega} t$ acting on mass m_2 determine: (a) equation of motion in terms of the relative motion of the two masses, $u_r = u_2 - u_1$; (b) the steady-state solution of the relative motion.

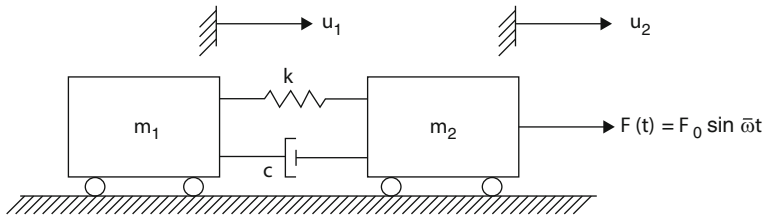

Fig. P3.18

Response to General Dynamic Loading

4

In the preceding chapter we studied the response of a single degree-of-freedom system with harmonic loading. Through this type of loading is important, real structures are often subjected to loads that are not harmonic. In this chapter we shall study the response of the single degree-of-freedom system to a general type of force. We shall see that the response can be obtained in terms of an integral that for some simple load functions can be evaluated analytically. For the general case, however, it will be necessary to resort to a numerical integration procedure.

4.1 Duhamel's Integral – Undamped System

An impulsive loading is a load which is applied during a short duration of time. The corresponding impulse of this type of load is defined as the product of the force and the time of its duration.

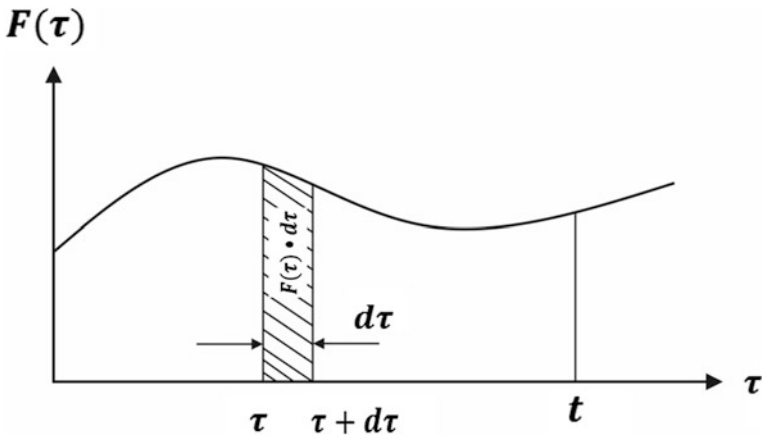

Fig. 4.1 General load function as impulsive loading

For example, the impulse of the force $F(\tau)$ depicted in Fig. 4.1 at time τ during the time interval $d\tau$ is represented by the shaded area and it is equal to $F(\tau)d\tau$. This impulse acting on a body of mass m produces a change in velocity that can be determined from Newton's Law of Motion, namely

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M. Paz, Y. H. Kim, *Structural Dynamics*, https://doi.org/10.1007/978-3-319-94743-3_4

$$m\frac{dv}{d\tau} = F(\tau)$$

Rearrangement yields

$$dv = \frac{F(\tau)d\tau}{m} \tag{4.1}$$

where $F(\tau)d\tau$ is the impulse and dv is the incremental velocity. This incremental velocity may be considered to be an initial velocity of the mass at time τ. Let us now consider this impulse $F(\tau)d\tau$ acting on the structure represented by the undamped oscillator. At time τ the oscillator will experience a change in velocity given by Eq. (4.1). This change in velocity is then introduced in Eq. (1.20) as the initial velocity v_0 together with the initial displacement $u_0 = 0$ at time τ producing a displacement $du(t)$ at a later time t given by

$$du(t) = \frac{F(\tau)d\tau}{m\omega}\sin\omega(t-\tau) \tag{4.2}$$

The loading function may then be regarded as a series of short impulses at successive incremental times $d\tau$, each producing its own differential response at time t of the form given by Eq. (4.2). Therefore, we conclude that the total displacement at time t due to the continuous action of the force $F(\tau)$ is given by the summation or integral of the differential displacements $du(t)$ from time $t = 0$ to time t, that is,

$$u(t) = \frac{1}{m\omega}\int_0^t F(\tau)\sin\omega(t-\tau)\,d\tau \tag{4.3}$$

The integral in this equation is known as Duhamel's integral. Equation (4.3) represents the total displacement produced by the exciting force $F(\tau)$ acting on the undamped oscillator; it includes both the steady-state and the transient components of the motion corresponding to zero initial conditions, $u_0 = 0$ and $v_0 = 0$. If the function $F(\tau)$ cannot be expressed analytically, the integral of Eq. (4.3) can always be evaluated approximately by suitable numerical methods. To include the effect of initial displacement u_0 and initial velocity v_0 at time $t = 0$, it is only necessary to add to Eq. (4.3) the solution given by Eq. (1.20) for the effects due to the initial conditions. Thus the total displacement of an undamped single degree-of-freedom system with an arbitrary load is given by

$$u(t) = u_0\cos\omega t + \frac{v_0}{\omega}\sin\omega t + \frac{1}{m\omega}\int_0^t F(\tau)\sin\omega(t-\tau)\,d\tau \tag{4.4}$$

Applications of Eq. (4.4) for some simple forcing functions for which it is simple to obtain the explicit integration of Eq. (4.4) are presented below.

4.1.1 Constant Force

Consider the case of a constant force of magnitude F_0 applied suddenly to the undamped oscillator at time $t = 0$ as shown in Fig. 4.2.

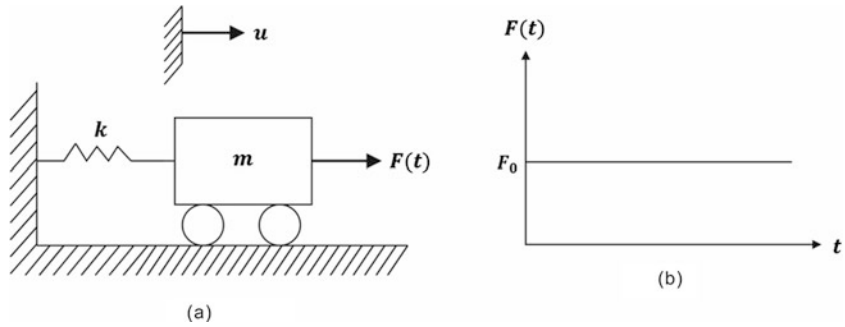

Fig. 4.2 Undamped oscillator acted upon by constant force

For both initial displacement and initial velocity equal to zero, the application of Eq. (4.4) to this case gives

$$u(t) = \frac{1}{m\omega} \int_0^t F_0 \sin \omega(t - \tau)\, d\tau$$

and the integration yields

$$\begin{aligned} u(t) &= \frac{F_0}{m\omega^2} |\cos \omega(t - \tau)|_0^t \\ u(t) &= \frac{F_0}{k}(1 - \cos \omega t) = u_{st}(1 - \cos \omega t) \end{aligned} \tag{4.5}$$

where $u_{st} = \frac{F_0}{k}$ is the static displacement due to a force of magnitude F_0. The response for such a suddenly applied constant load is shown in Fig. 4.3. It will be observed that this solution is very similar to the solution for the free vibration of the undamped oscillator. The major difference is that the coordinate axis t has been shifted by an amount equal to $u_{st} = \frac{F_0}{k}$. Also, it should be noted that the maximum displacement $2u_{st}$ is exactly twice the displacement that the force F_0 would produce if it were applied statically. We have found an elementary but important result; the maximum displacement of a linear elastic system for a constant force applied suddenly is twice the displacement caused by the same force applied statically (slowly). This result for displacement is also true for the internal forces and stresses in the structure.

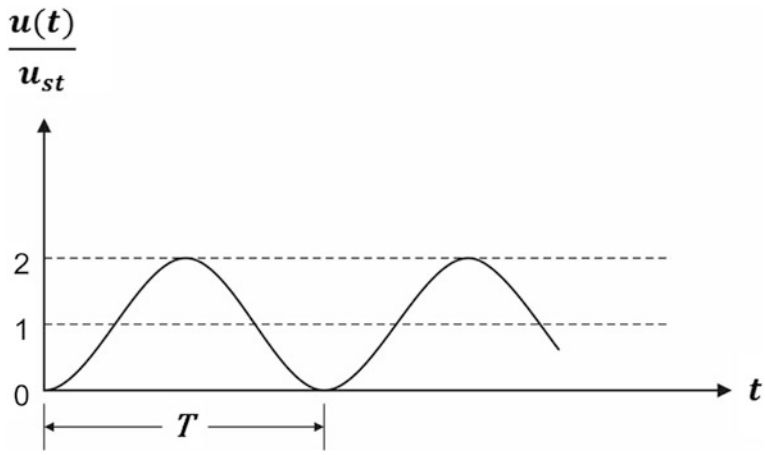

Fig. 4.3 Response of an undamped single degree-of-freedom system to a suddenly applied constant force

4.1.2 Rectangular Load

Let us consider a second case, that of a constant force F_0 suddenly applied but only during a limited time duration, t_d as shown in Fig. 4.4:

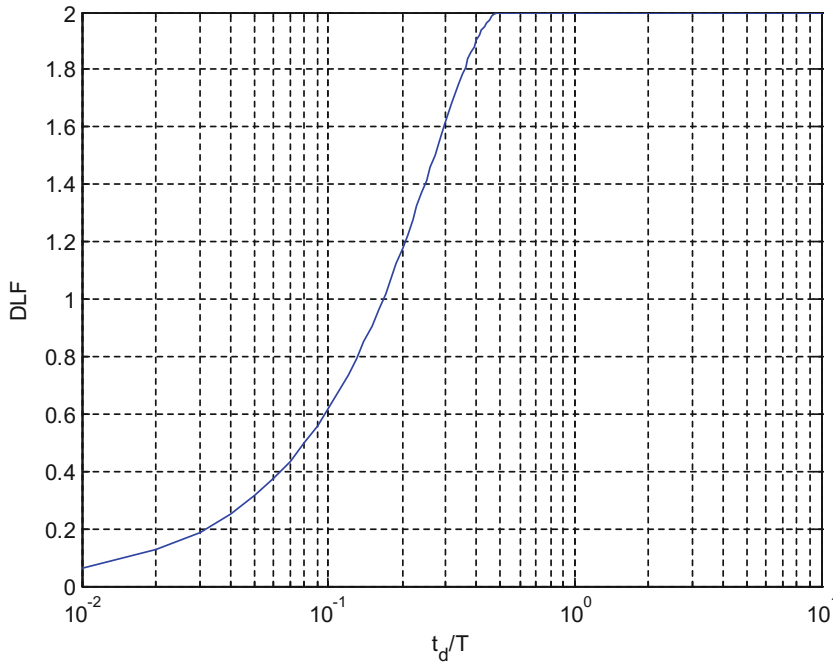

Fig. 4.4 Maximum dynamic load factor for the undamped oscillator acted on by a rectangular force

Up to the time t_d, the solution given by Eq. (4.5) applies and at that time the displacement and velocity are

$$u_d = \frac{F_0}{k}(1 - \cos \omega t_d)$$

and

$$v_d = \frac{F_0}{k}\omega \sin \omega t_d$$

For the response after time t_d we apply Eq. (1.20) for free vibration, taking as the initial conditions the displacement and velocity at t_d. After replacing t by $t - t_d$ and u_0 and v_0 for u_d and v_d, respectively, we obtain

$$u(t) = \frac{F_0}{k}(1 - \cos \omega t_d)\cos \omega(t - t_d) + \frac{F_0}{k} \sin \omega t_d \sin \omega(t - t_d)$$

which can be reduced to

$$u(t) = \frac{F_0}{k}\{\cos\omega(t - t_d) - \cos\omega t\} \tag{4.6}$$

If the dynamic load factor (*DLF*) is defined as the displacement at any time t divided by the static displacement $u_{st} = \frac{F_0}{k}$, we may write Eqs. (4.5 and 4.6) as

$$DLF = 1 - \cos\omega t, \qquad t \le t_d$$

and

$$DLF = \cos\omega(t - t_d) - \cos\omega t, \qquad t \ge t_d \tag{4.7}$$

It is often convenient to express time as a dimensionless parameter by simply using the natural period instead of the natural frequency $\left(\omega = \frac{2\pi}{T}\right)$. Hence, Eqs. (4.7) may be written as

$$DLF = 1 - \cos 2\pi\frac{t}{T}, \qquad t \le t_d$$

and

$$DLF = \cos 2\pi\left(\frac{t}{T} - \frac{t_d}{T}\right) - \cos 2\pi\frac{t}{T}, \quad t \ge t_d \tag{4.8a}$$

or using the trigonometric identity

$$\cos\alpha - \cos\beta = -2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$$

as

$$DLF = \left(2\sin\frac{\pi t_d}{T}\right)\sin\left[2\pi\left(\frac{t}{T} - \frac{t_d}{2T}\right)\right], \quad t \ge t_d \tag{4.8b}$$

The use of dimensionless parameters in Eq. (4.8a) serves to emphasize the fact that the ratio of duration of the time, that the constant force is applied, to the natural period rather than the actual value of either quantity is the important parameter. The maximum dynamic load factor $(DLF)_{max}$ obtained by maximizing Eq. (4.8b), is plotted in Fig. 4.4. It is observed from this figure that the maximum dynamic load factor for loads of duration $\left(\frac{t_d}{T} > 0.5\right)$ is the same as if the load duration had been infinite.

In general, the maximum response occurs during the application of the load, except for loadings of very short duration $\left(\frac{t_d}{T} < 0.4\right)$. In such cases, the maximum response may occur during the free vibration after the cessation of the load; it is then necessary to extend the loading time for a duration of about one period, in which the magnitude of the load is set equal to zero.

Charts, as shown in Fig. 4.4, which give the maximum response of a single degree-of-freedom system for a given loading function, are called response spectral charts. Response spectral charts for impulsive loads of short duration are often presented for the undamped system. For the short duration of the load, damping does not have a significant effect on the response of the system. The maximum dynamic load factor usually corresponds to the first peak of response and the amount of damping normally found in structures is not sufficient to appreciably decrease this value.

4.1.3 Triangular Load

We consider now a system represented by the undamped oscillator, initially at rest and subjected to a force $F(t)$ that has an initial value F_0 and that decreases linearly to zero at time t_d (Fig. 4.5). The response may be computed by Eq. (4.4) in two intervals. For the first interval, $\tau \leq t_d$ the force is given by

$$F(\tau) = F_0\left(1 - \frac{\tau}{t_d}\right)$$

and the initial conditions by

$$u_0 = 0, \ v_0 = 0$$

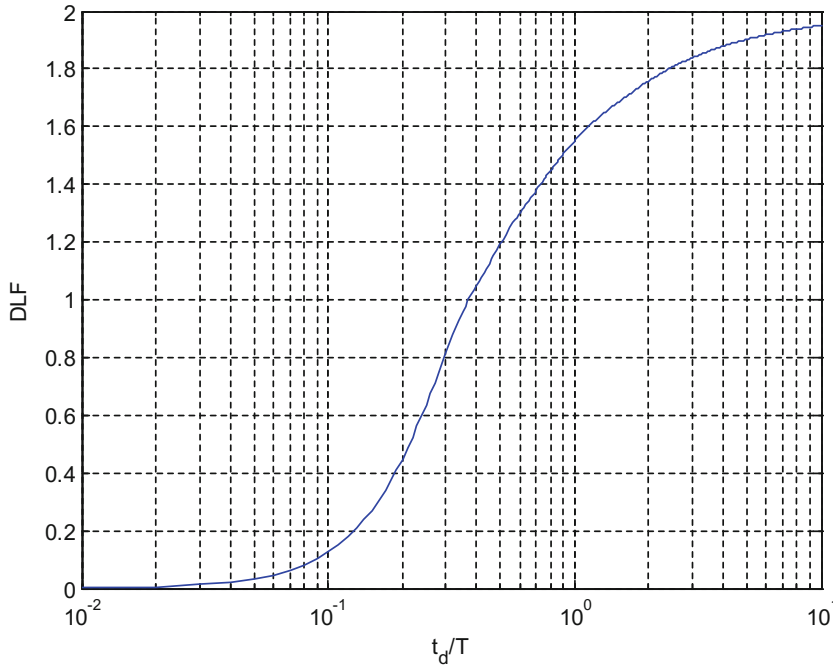

Fig. 4.5 Maximum dynamic load factor for the undamped oscillator acted upon by a triangular force

The substitution of these values in Eq. (4.4) and integration gives

$$u = \frac{F_0}{k}(1 - \cos\omega t) + \frac{F_0}{kt_d}\left(\frac{\sin\omega t}{\omega} - t\right) \tag{4.9}$$

or in terms of the dynamic load factor and dimensionless parameters

$$DLF = \frac{u}{u_{st}} = 1 - \cos\left(\frac{2\pi t}{T}\right) + \frac{\sin\left(\frac{2\pi t}{T}\right)}{\left(\frac{2\pi t_d}{T}\right)} - \frac{t}{t_d} \quad 0 \leq t \leq t_d \tag{4.10}$$

which defines the response before time t_d.

For the second interval ($t \geq t_d$), we obtain from Eq. (4.9) the displacement and velocity at time t_d as

$$u_d = \frac{F_0}{k}\left(\frac{\sin \omega t_d}{\omega t_d} - \cos \omega t_d\right)$$

and

$$v_d = \frac{F_0}{k}\left(\omega \sin \omega t_d + \frac{\cos \omega t_d}{t_d} - \frac{1}{t_d}\right) \tag{4.11}$$

These values may be considered as the initial conditions at time $t = t_d$ for this second interval. Replacing in Eq. (1.20) t by $t - t_d$ and u_0 and v_0 respectively, by u_d and v_d and noting that $F(\tau) = 0$ in this interval we obtain the response as

$$u = \frac{F_0}{k\omega t_d}\{\sin \omega t - \sin \omega(t - t_d)\} - \frac{F_0}{k}\cos \omega t$$

and upon dividing by $u_{st} = \frac{F_0}{k}$ gives

$$DLF = \frac{1}{\omega t_D}\{\sin \omega t - \sin \omega(t - t_d)\} - \cos \omega t \tag{4.12}$$

In terms of the dimensionless time parameter, this last equation may be written as

$$DLF = \frac{1}{\omega T_D}\left\{\sin 2\pi \frac{t}{T} - \sin 2\pi\left(\frac{t}{T} - \frac{t_d}{T}\right)\right\} - \cos 2\pi \frac{t}{T} \quad t \geq t_d \tag{4.13}$$

The plot of the maximum dynamic load factor as a function of the relative time duration t_d/T for the undamped oscillator is shown in Fig. 4.5. As would be expected, the maximum value of the dynamic load factor approaches 2 as t_d/T becomes large; that is, the effect of the decay of the force is negligible for the time required for the system to reach the maximum peak.

We have studied the response of the undamped oscillator for two simple impulse loadings; the rectangular pulse and the triangular pulse. Extensive charts have been prepared by the U. S. Army Corps of Engineers,[1] and are available for a variety of loading pulses. As already mentioned, the evaluation of Duhamel's Integral Method for a general forcing requires the use of numerical methods (See Problems 4.1).

Illustrative Example 4.1

A one-story building, shown in Fig. 4.6, is modeled as a 15-ft high frame with two steel columns fixed at the base and a rigid beam supporting a total weight of $W = 5000$ lb. Each column has a moment of inertia $I = 69.2$ in^2 and a section modulus $S = \frac{I}{c} = 17\,\text{in}^3\left(E = 30 \times 10^6\,\text{psi}\right)$.

Determine the maximum response of the frame to a rectangular impulse of amplitude 3000 lb and duration $t_d = 0.1$ s applied horizontally to the top member of the frame. The response of interest is the horizontal displacement at the top of the frame and the bending stress in the columns.

[1] U. S. Army Corps of Engineers, *Design of Structures to Resist the Effects of Atomic Weapons*, Manuals 415, 415, 3rd 416, March 15, 1957; Manuals 417 and 419, January 15, 1958; Manuals 418, 420, 421, January 15, 1960.

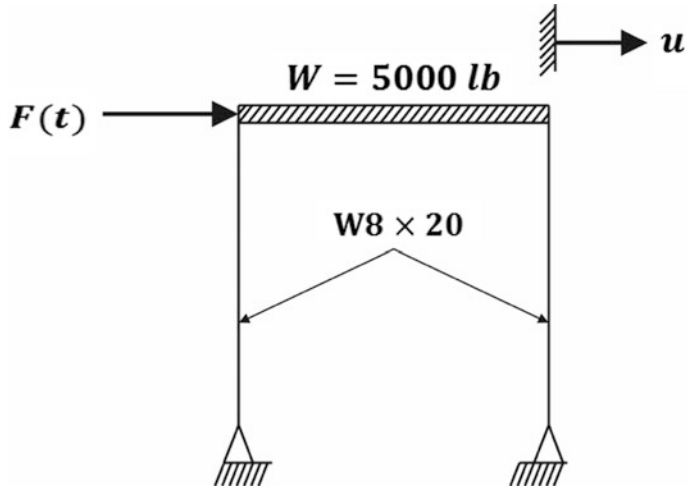

Fig. 4.6 Idealized frame for Illustrative Example 4.1

Solution:

1. Natural period.

$$k = \frac{12E(2I)}{L^3} = \frac{12 \times 30 \times 10^6 \times 2 \times 69.2}{(15 \times 12)^3} = 8544 \text{ lb/in}$$

$$m = \frac{5000}{386} = 12.9534 \text{ lb. sec}^2/\text{in}$$

$$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{12.9534}{8544}} = 0.2446 \text{ sec}$$

2. Maximum displacement.

$$\frac{t_d}{T} = \frac{0.1}{0.2446} = 0.408$$

$$DLF_{\max} = \frac{u_{\max}}{u_{st}} = 1.9 \text{ (from Fig.4.4)}$$

$$u_{st} = \frac{F_0}{k} = \frac{3000}{8544} = 0.3511 \text{ in}$$

$$u_{\max} = (1.9)(0.3511) = 0.667 \text{ in} \quad \text{(Ans)}$$

3. Maximum bending stress.
The bending moment M in the columns is given by

$$M = V_{\max}\frac{L}{2} = ku_{\max}\frac{L}{2}$$

$$= \frac{6EI}{L^2}u_{\max} = \frac{6 \times 30 \times 10^6 \times 69.2}{(15 \times 12)^2}0.667 = 256,424 \text{ (lb.in)}$$

and the maximum stress σ_{max} by

$$\sigma_{max} = \frac{M}{S} = \frac{256,424}{17} = 15,083\,\text{psi} \qquad \text{(Ans)}$$

Alternatively, the maximum response can be estimated using MATLAB file.

```
clear all
close all
clc


%%%%-GIVEN VALUES-%%%%
T = 0.2446;                %Natural period
omega = 2*pi/T;            %Natural frequency
F_0=3000;                  %Maximum force
k=8544;                    %Stiffness

t = 0:0.001:3;             %0 to 3 secs with the interval of 0.001 sec
td_T = 0.408;              %td/T ratio


%%%-ESTIMATION-%%%
td = td_T*T;               %Duration of force
u_st = F_0/k;              %Static displacement

%%%Estimate response with two functions (t<td and t>td)
for i=1:length(t)
    if t(i) <=td
        u(i) = u_st*(1-cos(omega*t(i)));
   else
    arg1=(1-cos(omega*td))*cos(omega*(t(i)-td));
    arg2=sin(omega*td)*sin(omega*(t(i)-td));
        u(i)= u_st*(arg1+arg2);

    end
end


%%%Response and DLF estimation
figure (1)
plot (t, u);
xlabel ('Time (sec)');
ylabel ('Displacement');
grid on

u_max =abs(max(u))      %u_max
D = u_max/u_st          %DLF factor
```

MATLAB yields the maximum response of 0.673 in. This value is comparable to the value using Fig. 4.4. DLF can be also calculated from MATLAB. The value using MATLAB yield the same value from the spectra chart. Using MATLAB, the response can be varied depending on t_d Fig. 4.7.

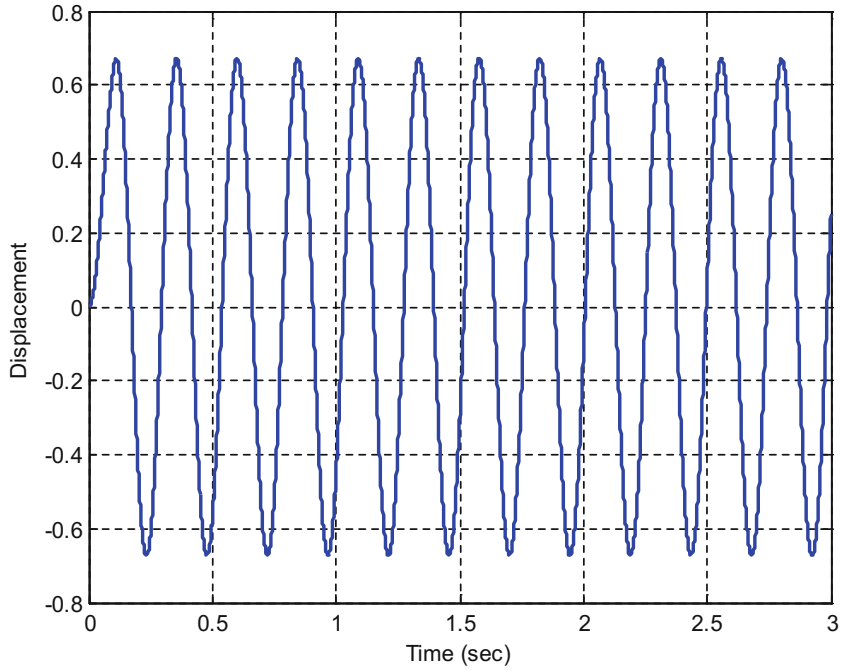

Fig. 4.7 Response of Illustrative Example 4.1

Illustrative Example 4.2
Plot Fig. 4.4 using MATLAB (Response Spectral Chart). The MATLAB code is presented below.

```
close all
clc

%%%%-GIVEN VALUES-%%%%
T = 0.2446;                    %Natural period
omega = 2*pi/T;                %Natural frequency
F_0=3000;                      %Maximum force
k=8544;                        %Stiffness

t = 0:0.01:3;                  %0 to 3 secs with the interval of 0.01 sec
td_T = 0:0.01:10;              %td/T ratio


%%%-ESTIMATION-%%%
td = td_T*T;                   %Duration of force
u_st = F_0/k;                  %Static displacement

%%%Estimate response spectral chart
for j =1:length(td)

for i=1:length(t)
    if t(i) <=td(j)
        u(i,j) = u_st*(1-cos(omega*t(i)));
   else
    arg1=(1-cos(omega*td(j)))*cos(omega*(t(i)-td(j)));
    arg2=sin(omega*td(j))*sin(omega*(t(i)-td(j)));
        u(i,j)= u_st*(arg1+arg2);
    end

end

end
```

```
%%%DLF estimation
u_max = max(u,[],1);          %u_max
D = u_max/u_st;               %DLF factor

figure (1)
semilogx(td_T, D);
xlabel ('t_d/T');
ylabel ('DLF');
grid on
```

4.2 Duhamel's Integral-Damped System

The response of a damped system expressed by the Duhamel's integral is obtained in a manner entirely equivalent to the undamped analysis except that the impulse $F(\tau)d\tau$ producing an initial velocity $dv = F(\tau)\frac{d\tau}{m}$ is substituted into the corresponding damped free-vibration Eq. (2.20). Setting $u_0 = 0, v_0 = F(\tau)\frac{d\tau}{m}$, and substituting t for $(t-\tau)$ in Eq. (2.20), we obtain the differential displacement $du(t)$ at a time t as

$$du(t) = e^{-\xi\omega(t-\tau)}\frac{F(\tau)\,d\tau}{m\omega_D}\sin\omega_D(t-\tau) \tag{4.14}$$

Summing or integrating these differential response terms over the entire loading interval results in

$$u(t) = \frac{1}{m\omega_D}\int_0^t F(\tau)e^{-\xi\omega(t-\tau)}\sin\omega_D(t-\tau)\,d\tau \tag{4.15}$$

which is the response for a damped system in terms of the Duhamel's integral. For numerical evaluation Eq. (4.15), we proceed as in the undamped case (See Problem 4.2).

4.3 Response by Direct Integration

The differential equation of motion for a one degree-of-freedom system represented by the damped simple oscillator, shown in Fig. 4.8a, is obtained by establishing the dynamic equilibrium of the forces in the free body diagram, Fig. 4.8b:

$$m\ddot{u} + c\dot{u} + ku = F(t) \tag{4.16}$$

in which the function F(t) represents the force applied to the mass of the oscillator.

When the structure, modeled by the simple oscillator, is excited by a motion at its support, as is shown in Fig. 4.9a, the equation of motion obtained using the free body diagram in Fig. 4.9b is

$$m\ddot{u} + c\left(\dot{u} - \dot{u}_s\right) + k(u - u_s) = 0 \tag{4.17}$$

In this case, it is convenient to express the displacement u_r of the mass relative to the displacement u_s of the support, namely

$$u_r = u - u_s \tag{4.18}$$

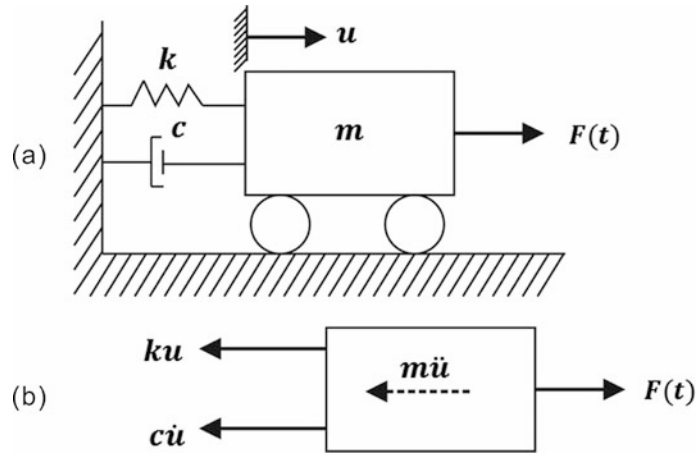

Fig. 4.8 (**a**) Damped simple oscillator excited by the force $F(t)$. (**b**) Free body diagram

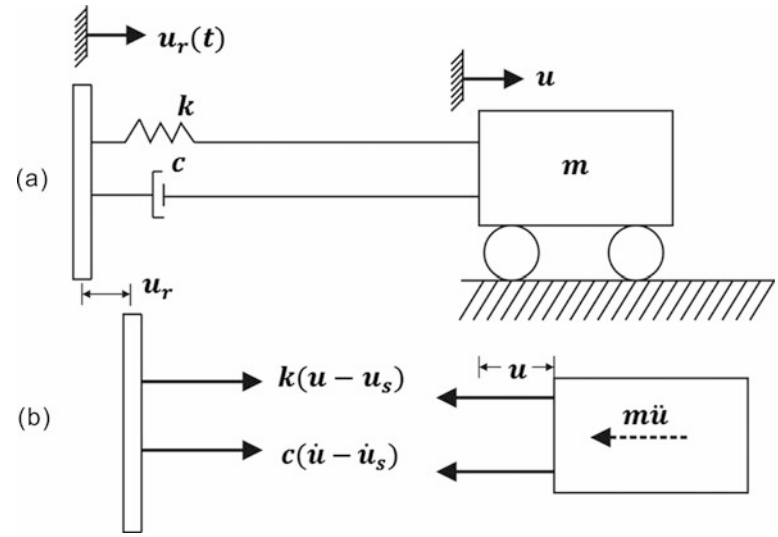

Fig. 4.9 (**a**) Damped Simple oscillator excited by the displacement $u_s(t)$ at its support. (**b**) Free body diagram

The substitution of u_r and its derivative from Eq.(4.18) into Eq. (4.17) results in

$$m\ddot{u}_r + c\dot{u}_r + ku_r = -mu_s(t) \tag{4.19}$$

Comparison of Eqs. (4.16 and 4.19) reveals that both equations are mathematically equivalent if the right-hand side of Eq. (4.19) is interpreted as the effective force

$$F_{eff}(t) = -m\ddot{u}_s(t) \tag{4.20}$$

Equation (4.19) may then be written as

$$m\ddot{u}_r + c\dot{u}_r + ku_r = F_{eff}(t) \tag{4.21}$$

Consequently, the solution of the second order differential Eq. (4.16) or Eq. (4.21) gives the response in terms of the absolute motion u for the case in which the mass is excited by a force, or in terms of the relative motion $u_r = u - u_s$, for the structure excited at the base.

4.4 Solution of the Equation of Motion

The method of solution for the differential equation of motion presented in this Section is exact for an excitation function described by linear segments. The process of solution requires for convenience that the excitation function be calculated at equal time intervals Δt. This result is accomplished by a linear interpolation between points defining the excitation. Thus, the time duration of the excitation, including a suitable extension of time after cessation of the excitation, is divided into N equal time intervals of duration Δt, For each interval Δt, the response is calculated by considering the initial conditions at the beginning of that time interval and the linear excitation during the interval. The initial conditions are, in this case, the displacement and velocity at the end of the preceding time interval. Assuming that the excitation function $F(t)$ is approximated by a piecewise linear function as shown in Fig. 4.10, we may express this function by

$$F(t) = \left(1 - \frac{t - t_i}{\Delta t}\right)F_i + \left(\frac{t - t_i}{\Delta t}\right)F_{i+1}, \qquad t_i \le t \le t_{i+1} \tag{4.22}$$

in which $t_i = i \cdot \Delta t$ for equal intervals of duration Δt and $i = 1,2,3,\ldots,N$. The differential equation of motion, Eq. (4.16), is then given by

$$m\ddot{u} + c\dot{u} + ku = \left(1 - \frac{t - t_i}{\Delta t}\right)F_i + \left(\frac{t - t_i}{\Delta t}\right)F_{i+1}, \qquad \mathrm{t_i} \le t \le t_{i+1} \tag{4.23}$$

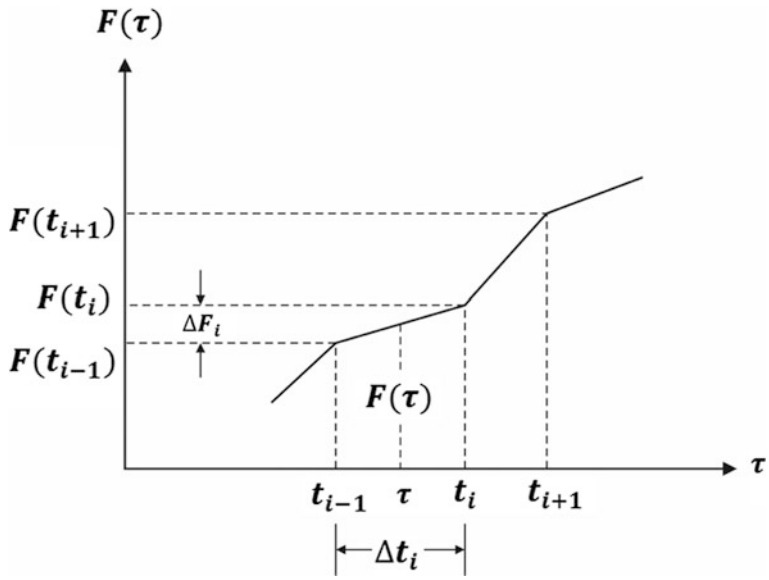

Fig. 4.10 Segmental linear loading function

The solution of Eq. (4.23) may be expressed as the sum of the complementary solution u_c for which the second member of Eq. (4.23) is set equal to zero, and the particular solution u_p, that is

$$u = u_c + u_p \tag{4.24}$$

The complementary solution is given in general by Eq. (2.15), which for the interval $t_i \le t \le t_i + \Delta t$ is

$$u_c = e^{\xi\omega(t-t_i)}[C_i \cos \omega_D(t-t_i) + D_i \sin \omega_D(t-t_i)] \tag{4.25}$$

On the other hand, the particular solution of Eq. (4.23) takes the form

$$u_p = B_i + A_i(t-t_i) \tag{4.26}$$

which upon its substitution into Eq. (4.23) gives

$$cA_i + k[B_i + A_i(t-t_i)] = \left(1 - \frac{t-t_i}{\Delta t}\right)F_i + \left(\frac{t-t_i}{\Delta t}\right)F_{i+1}$$

where A_i and B_i are constants for the interval $t_i \le t \le t_i, + \Delta t$ and where we use the notation $F_i = F(t_i)$ and $F_{i\,+\,1} = F(t + \Delta t)$. Establishing the identity of terms between the left-hand and right-hand sides, that is, between the constant terms and the terms with a factor (t - t_i) and then solving the resulting equations, we obtain

$$\begin{aligned} A_i &= \frac{F_{i+1} - F_i}{k\,\Delta t} \\ B_i &= \frac{F_i - cA_i}{k} \end{aligned} \tag{4.27}$$

The substitution into Eq. (4.24) of the complementary solution u_c from Eq. (4.25) and of the particular solution u_p from Eq. (4.26) gives the total solution as

$$u = e^{-\xi\omega(t-t_i)}[C_i \cos \omega_D(t-t_i) + D_i \sin \omega_D(t-t_i)] + B_i + A_i(t-t_i) \tag{4.28}$$

The velocity $\dot{u}$ is then given by the derivative of Eq. (4.28) as

$$\dot{u} = e^{\xi\omega(t-t_i)}[(\omega_D D_i - \xi\omega C_i)\cos \omega_D(t-t_i) - (\omega_D C_i + \xi\omega D_i)\sin \omega_D(t-t_i)] + A_i \tag{4.29}$$

The constants of integration C_i and D_i are obtained from Eqs. (4.28 and 4.29) introducing the initial conditions for the displacement u_i and for the velocity $\dot{u}_i$, at the beginning of the interval Δt, that is, at time t_i. Thus, introducing into Eqs. (4.28 and 4.29) these initial conditions and solving for the constants C_i and D_i in the resulting relations yields

$$\begin{aligned} C_i &= u_i - B_i \\ D_i &= \frac{\dot{u}_i - A_i - \xi\omega C_i}{\omega_D} \end{aligned} \tag{4.30}$$

The evaluation of Eqs. (4.28 and 4.29) at time $t_i + \Delta t$ results in the displacement $u_{i\,+\,1}$ and the velocity $\dot{u}_{i+1}$ at time $t_{i\,+\,1}$. Namely,

$$u_{i+1} = e^{-\xi\omega\Delta t}[C_i \cos \omega_D \Delta t + D_i \sin \omega_D \Delta t] + B_i + A_i \Delta t \tag{4.31}$$

and

$$\dot{u}_{i+1} = e^{-\xi\omega\Delta t}\left[D_i(\omega_D \cos \omega_D \Delta t - \xi\omega \sin \omega_D \Delta t) - C_i\left(\xi\omega \cos \omega_D \Delta t + \omega_D \sin \omega_D \Delta t\right)\right] - A_i \tag{4.32}$$

Finally, the acceleration at time $t_{i\,+\,1} = t_i + \Delta t$ is directly obtained after substituting $u_{i\,+\,1}$ and $\dot{u}_{i+1}$ from Eqs. (4.31 and 4.32) into the differential Eq. (4.16) and letting $t_{i\,+\,1} = t_i + \Delta t$. Specifically,

$$\ddot{u}_{i+1} = \frac{1}{m}\left(F_{i+1} - c\dot{u}_{i+1} - ku_{i+1}\right) \tag{4.33}$$

The substitution of the coefficient A_i, B_i, C_i and D_i from Eqs. (4.27 and 4.30), together with $\varepsilon = \frac{2\xi k}{\omega}$ into Eqs. (4.31 and 4.32), results in the following formulas to calculate the displacement, velocity and acceleration at the time step $t_{i+1} = t_i + \Delta t$:

$$u_{i+1} = A' u_i + B' \dot{u}_i + C' F_i + D_i' F_{i+1} \tag{4.34}$$

$$\dot{u}_{i+1} = A'' u_i + B'' \dot{u}_i + C'' F_i + D'' F_{i+1} \quad \dot{u}_{i+1} = A'' u_i + B'' \dot{u}_i + C'' F_i + D'' F_{i+1} \tag{4.35}$$

$$\ddot{u}_{i+1} = -\omega^2 u_{i+1} - 2\xi\omega\ddot{u}_{i+1} + \frac{F_{i+1}}{m} \tag{4.36}$$

Equations (4.34, 4.35 and 4.36) are recurrence formulas to calculate, respectively, the displacement, velocity, and acceleration at the next time step $t_{i+1} = t_i + \Delta t$ from the previously calculated values for these quantities at the preceding time step t_i. Because these recurrence formulas are exact, the only restriction in selecting the length of the time step, Δt, is that it allows a close approximation to the excitation function and that equally spaced time intervals do not miss the peaks of this function. This numerical procedure is highly efficient because the coefficients in Eqs. (4.34, 4.35 and 4.36) need to be calculated only once. The final expressions to calculate the coefficients $A', B', \ldots D''$ are given in Box 4.1.

Illustrative Example 4.3

Determine the dynamic response of a tower subjected to a blast loading. The idealization of the structure and the blast loading are shown, respectively, in Fig. 4.11a, b. Assume damping equal to 20% of the critical damping.

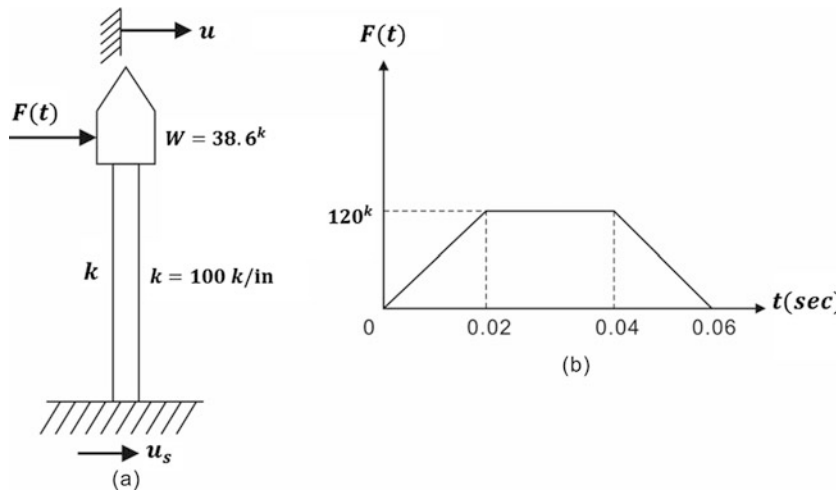

Fig. 4.11 (**a**) Idealized structure (**b**) idealized loading for Illustrative Example 4.4

Box 4.1 Coefficients in Eqs. (4.34, 4.35 and 4.36).

$$A' = e^{-\xi\omega\Delta t}\left(\frac{\xi\omega}{\omega_D}\sin\omega_D\Delta t + \cos\omega_D\Delta t\right)$$

$$B' = e^{-\xi\omega\Delta t}\left(\frac{1}{\omega_D}\sin\omega_D\Delta t\right)$$

$$C' = \frac{1}{k}\left\{e^{-\xi\omega\Delta t}\left[\left(\frac{1-2\xi^2}{\omega_D\Delta t} - \frac{\xi\omega}{\omega_D}\right)\sin\omega_D\Delta t - \left(1+\frac{2\xi}{\omega\Delta t}\right)\cos\omega_D\Delta t\right] + \frac{2\xi}{\omega\Delta t}\right\}$$

$$D' = \frac{1}{k}\left\{e^{-\xi\omega\Delta t}\left[\left(\frac{2\xi-1}{\omega_D\Delta t}\sin\omega_D\Delta t + \frac{2\xi}{\omega\Delta t}\cos\omega_D\Delta t\right)\right] + \left(1-\frac{2\xi}{\omega\Delta t}\right)\right\}$$

$$A'' = -e^{-\xi\omega\Delta t}\left(\frac{\omega^2}{\omega_D}\sin\omega_D\Delta t\right)$$

$$B'' = e^{-\xi\omega\Delta t}(\cos\omega_D\Delta t) - \frac{\xi\omega}{\omega_D}\sin\omega_D\Delta t$$

$$C'' = \frac{1}{k}\left\{-e^{-\xi\omega\Delta t}\left[\left(\frac{\omega^2}{\omega_D} + \frac{\omega\xi}{\Delta t\omega_D}\right)\sin\omega_D\Delta t + \frac{1}{\Delta t}\cos\omega_D\Delta t\right] - \frac{1}{\Delta t}\right\}$$

$$D'' = \frac{1}{k\Delta t}\left\{-e^{-\xi\omega\Delta t}\left(\frac{\omega\xi}{\omega_D}\sin\omega_D\Delta t + \cos\omega_D\Delta t\right) + 1\right\}$$

Solution:
Since the loading is given as a segmental linear function, the response obtained using the direct method will be exact. The necessary calculations are presented in a convenient tabular format in Table 4.1. For this system, the natural frequency is

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{100,000}{100}} = 31.623 \text{ rad/ sec}$$

and damped natural frequency

$$\omega_D = \omega\sqrt{1-\xi^2} = 31.623\sqrt{1-0.2^2} = 30.984 \text{ rad/ sec}$$

Hence, the natural period is

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{31.623} = 0.20 \text{ sec}$$

Recommended practice is to select $\Delta t \leq \frac{T}{10}$. Specifically, we select $\Delta t = 0.02$ s. We then calculate the coefficients of Eqs. (4.34 and 4.35):

$$A' = 0.82180 \quad B' = 0.16517 \quad C' = 1.16755 \times 10^{-6} \quad D' = 6.1439 \times 10^{-7}$$

$$A'' = -16.5170 \quad B'' = 0.61286 \quad C'' = 7.60730 \times 10^{-5} \quad D'' = 8.9097 \times 10^{-5}$$

with initial conditions $u_0 = 0$ and $v_0 = 0$, we obtain from Eqs. (4.34, 4.35, and 4.36).

$$u_1 = 6.1439 \times 10^{-7} \times 1.2 \times 10^5 = 0.074 \text{ in}$$

$$\dot{u}_1 = 8.9124 \times 10^{-5} \times 1.2 \times 10^5 = 10.692 \text{ in/ sec}$$

$$\ddot{u}_1 = 31.623^2 \times 0.074 - 2 \times 0.2 \times 31.623 \times 10.692 + 1.2 \times 10^5/100 = 990.754 \text{ in/ sec}^2$$

Thus, completing the first cycle of calculations in the direct method of solution. Introducing the calculated values u_1, $\dot{u}_1$, and $\ddot{u}_1$, into the recurrence formulas Eqs. (4.34, 4.35, and 4.36), we obtain the response at time $t_2 = 0.04$ s. The continuation of this process results in the response of this system as shown in Table 4.1 up to 0.10 s.

Table 4.1 Calculation of the response for Illustrative Example 4.4

T(s)	u_i in.	$\dot{u}_i$ in./sec	$\ddot{u}_i$ in./sec^2	$F(\tau)$
0.000	0	0	0	0
0.020	0.074	10,692	990.754	120,000
0.040	0.451	25.155	430.768	120,000
0.060	0.926	17.096	−1142.511	0
0.080	1.044	−4.821	−982.581	0
0.100	0.778	−20.191	−522.555	0

Illustrative Example 4.4

Consider the tower shown in Fig. 4.11, but now subjected to a constant impulsive acceleration of magnitude $\ddot{u}_s = 0.5\ g$ during 0.5 s applied at the foundation of the tower. Determine the response of the tower in terms of the displacement and velocity of the mass relative to the motion of the foundation. Also, determine the maximum acceleration of the mass.

Solution:

The following data are obtained from Fig. 4.12:

Mass: $m = 100$ (lb · sec^2/in)

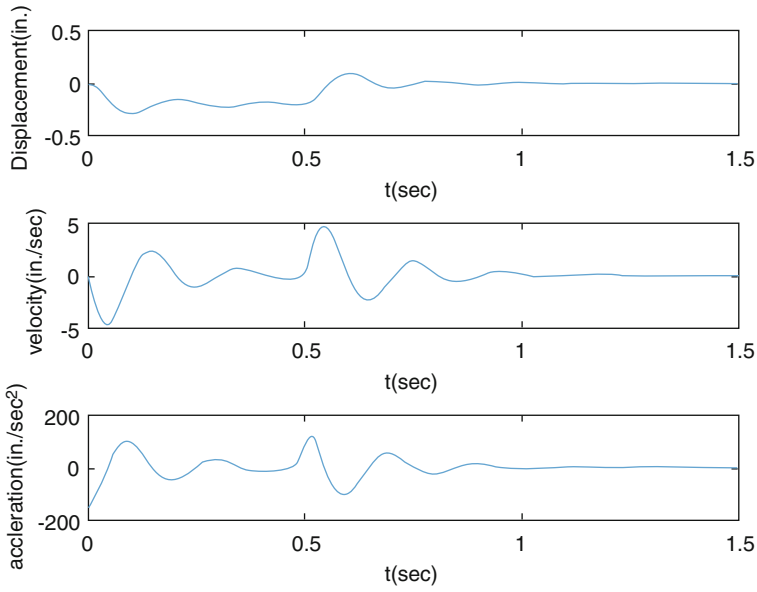

Fig. 4.12 Displacement, velocity, and acceleration (relative) for Illustrative Example 4.4

Spring constant: $k = 100{,}000$ (lb/in)
Damping coefficient: $c = 1265$ (lb · sec/in)
Acceleration of gravity: $g = 386$ (in/sec^2)
Select time step: $\Delta t = 0.002$ s
Excitation function:

Time (s)	Support acceleration (g)
0	0.5
0.5	0.5

Using ODE45 function in the MATLAB, two files can be used to create plots and determine maximum displacement, velocity, and acceleration.

Matlab file: Ex4_4.m

```
close all
clear
clc

%%%%-GIVEN VALUES-%%%%
deltat = 0.02;                %Time step
tspan=0:deltat:1.5;           %0 to 1.5 secs with the interval of 0.02 sec
IC = [0 0]';                  %Initial conditions (u0=0, v0=0)

%%%Plot the displacement using ODE45
%%%-ESTIMATION-%%%
[t, u] = ode45(@SDOF2, tspan, IC);


%Display responses(displacement,velocity,acceleration)
umax= max(abs(u(:,1)))

%Velocity
udot = u(:,2);
udotmax = max(udot)

%Acceleration
udotdot = gradient(u(:,2), deltat);
udotdotmax = max(udotdot)

figure

subplot(3,1,1);
plot(t, u(:,1));
%Create x&y labels
xlabel ('t(sec)');
ylabel ('Displacement(in.)');

subplot(3,1,2);
plot(t, u(:,2));

%Create x&y labels
xlabel ('t(sec)');
ylabel ('Velocity(in./sec)');

subplot(3,1,3);
plot(t, udotdot);

%Create x&y labels
xlabel ('t(sec)');
ylabel ('Accleration(in./sec^2)');
```

Matlab file: SDOF2.m

```
function  u = SDOF2(t, u)

%%%%-GIVEN VALUES-%%%%

        m=100;                  %Mass (lb.sec^2/in.)
        k =100000;              %Stiffness (lb/in.)
        omega = sqrt(k/m);      %Natural Frequency
        c=1265;                 %Damping coefficient (lb.sec/in.)
        g =386;                 %Acceleration of gravity (in./sec^2)
        c_cr=2*m*omega;         %Critical damping coefficient
        xi = c/c_cr;            %Damping ratio

%%%Define the forcing function
        if t<=0.5
               F = -0.5*g*m/m;
            else
               F =0;
        end

%%%%-ESTIMATION-%%%%
u = [u(2); -omega*omega*u(1)-2*xi*omega*u(2)+F];
```

Maximum displacement and velocity are 0.2945 in. and velocity is 4.748 in./sec, respectively. The maximum acceleration is 120.07 in./sec^2. For excitation at the support: Displacement, velocity, and acceleration are relative to the support. The absolute acceleration is equal to be 120.07 + 0.5 g in./sec^2 using the relationship of $\ddot{u} = \ddot{u}_r + \ddot{u}_s$.

Illustrative Example 4.5.
A structural system modeled in Fig. 4.13a by the simple oscillator with 10% ($\xi = 0.10$) of critical damping is subjected to the impulsive load as shown in Fig. 4.13b. Polt the response (Displacement, velocity, and acceleration) using ODE45 method in MATLAB.

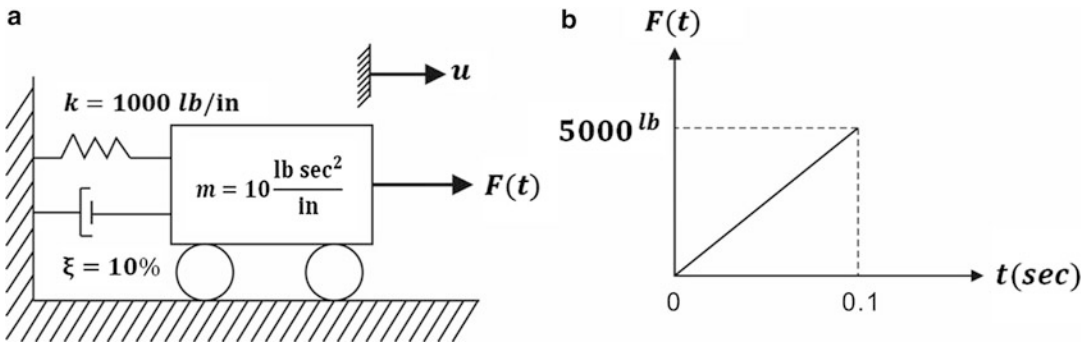

Fig. 4.13 Illustrative of Example 4.5 (a) Mathematical model, (b) Load function

Solution:
The following data are obtained from Fig. 4.14:

Mass: $m = 10$ lb $\cdot$ sec^2/in
Spring constant: $k = 10{,}000$ lb/in
Damping coefficient: $c = \xi c_{cr} = 2\xi\sqrt{km} = 63.25$ lb $\cdot$ sec /in
Natural period: $T = 2\pi\sqrt{m/k} = 0.20$ sec

Select time step for integration: $\Delta t = 0.02$ s
Excitation function: $F(t) = 50,000t\text{lb}, \quad 0 \leq t \leq 0.1\,\text{sec}\,.$

$$= 0 \qquad\qquad t > 0.1\,\text{sec}\,.$$

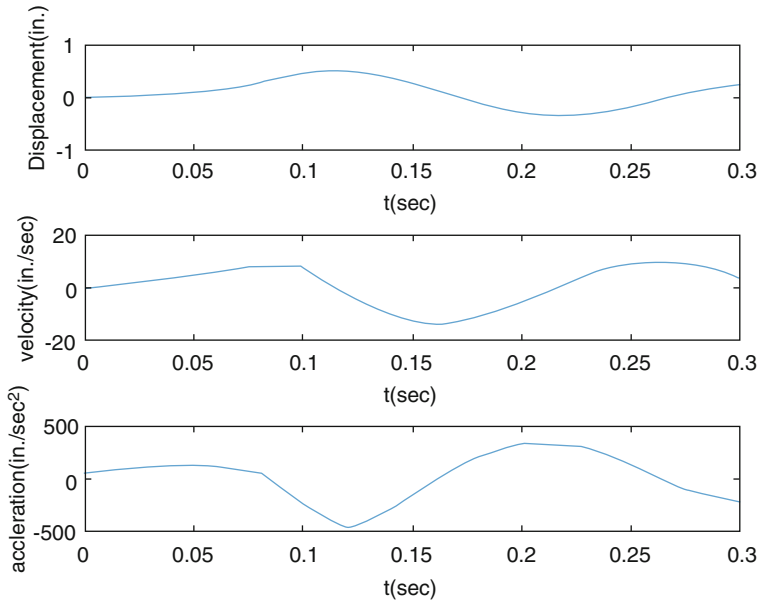

Fig. 4.14 Displacement, velocity, and acceleration for Illustrative Example 4.5 ($\Delta t = 0.02$ s)

Maximum time: $t_{\max} = 0.2$ s

4.5 Summary

In this chapter, we have shown that the differential equation of motion for a single-degree-of-freedom linear system can be solved for any forcing function $F(\tau)$ in terms of Duhamel's integral which for the undamped system is given by

$$u(t) = \frac{1}{m\omega}\int_0^t F(\tau)\sin\omega(t-\tau)\,d\tau \qquad (4.3)\text{ repeated}$$

and for a damped system by

$$u(t) = \frac{1}{m\omega_D}\int_0^t F(\tau)e^{-\xi\omega(t-\tau)}\sin\omega_D(t-\tau)\,d\tau \qquad (4.15)\text{ repeated}$$

where

$\omega = \sqrt{\frac{k}{m}}$ is the (undamped) natural frequency
$\omega_D = \omega\sqrt{1-\xi^2}$ is the damped frequency
$\xi = \frac{c}{c_{c1}}$ is the damping ratio

The solution of the differential equation of motion may also be obtained by application of the Direct Method. In this method, it is assumed that the forcing function is given by a segmental linear function between defining points. Based on this assumption, the solution obtained is exact. The response is calculated at each time increment for the conditions existent at the end of the preceding time interval (initial conditions for the new time interval) and the action of the excitation applied during the time interval, which is assumed to be linear.

4.6 Analytical Problems

Problem 4.1
Develop a numerical method to evaluate Duhamel's Integral which gives the response of an undamped elastic one-degree-of-freedom structure modeled by the simple oscillation shown in

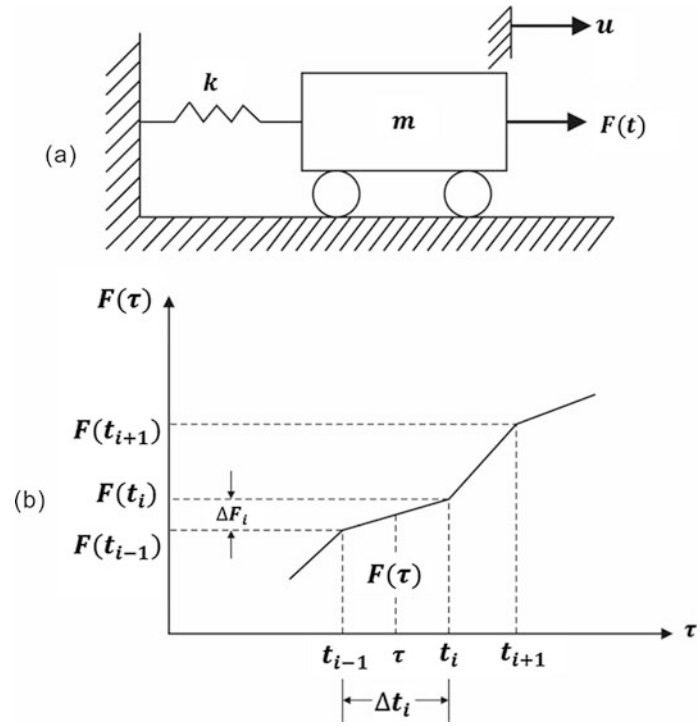

Fig. P4.1 (**a**) Structure modeled by the undamped simple oscillator. (**b**) Segmental linear forcing function

Fig. P4.1a. Assume that the forcing function $F(t)$ may be represented by a segmental linear function as shown in Fig. P4.1b.

Solution:
Assuming zero initial conditions, we obtain Duhamel's integral from Eq. (4.4) as

$$u(t) = \frac{1}{m\omega}\int_0^t F(\tau)\sin\omega(t-\tau)\,d\tau$$

Then, using the trigonometric identity

$$\sin\omega(t-\tau) = \sin\omega t\cos\omega\tau - \cos\omega t\sin\omega\tau$$

we obtain

$$u(t) = \sin\omega t\frac{1}{m\omega}\int_0^t F(\tau)\cos\omega\tau - \cos\omega t\frac{1}{m\omega}\int_0^t F(\tau)\sin\omega\tau\,d\tau$$

or

$$u(t) = \{A(t)\sin\omega t - B(t)\cos\omega t\}/m\omega \tag{a}$$

where

$$A(t) = \int_0^t F(\tau)\cos\omega\tau\,d\tau \tag{b}$$

$$B(t) = \int_0^t F(\tau)\sin\omega\tau\,d\tau \tag{c}$$

The calculation of Duhamel's integral thus requires the numerical evaluation of the integrals $A(t)$ and $B(t)$. It is more convenient to express the integrations in Eqs. (b and c) in incremental form, namely

$$A(t_i) = A(t_{i-1}) + \int_{t_{i-1}}^{t_i} F(\tau)\cos\omega\tau\,d\tau \tag{d}$$

$$B(t_i) = B(t_{i-1}) + \int_{t_{i-1}}^{t_i} F(\tau)\sin\omega\tau\,d\tau \tag{e}$$

where $A(t_i)$ and $B(t_i)$ represent the values of the integrals in Eq. (a) at time t_i. Assuming that the forcing function $F(\tau)$is approximated by a piecewise linear function as shown in Fig. P4.1b, we may write

$$F(\tau) = F(t_{i-1}) + \frac{\Delta F_i}{\Delta t_i}(\tau - t_{i-1}), \qquad \mathrm{t_{i-1}} \le \tau \le t_i \tag{f}$$

where

$$\Delta F_i = F(t_i) - F(t_{i-1})$$

and

$$\Delta t_i = t_i - t_{i-1}$$

The substitution of Eq. (4.20) into Eq. (4.18) and integration yield

$$A(t_i) = A(t_{i-1}) + \left(F(t_{i-1}) - t_{i-1}\frac{\Delta F_i}{\Delta t_i}\right)(\sin\omega t_i - \sin\omega t_{i-1})/\omega + \frac{\Delta F_i}{\omega^2 \Delta t_i}\{\sin\omega t_i - \sin\omega t_{i-1} + \omega(t_i\cos\omega t_i - \omega t_{i-1}\cos\omega t_{i-1})\} \tag{4.21}$$

Analogously from Eq. (4.19),

$$B(t_i) = B(t_{i-1}) + \left(F(t_{i-1}) - t_{i-1}\frac{\Delta F_i}{\Delta t_i}\right)(\cos\omega t_{i-1} - \cos\omega t_i)/\omega + \frac{\Delta F_i}{\omega^2 \Delta t_i}\{\sin\omega t_i - \sin\omega t_{i-1} - \omega(t_i\cos\omega t_i - t_{i-1}\cos\omega t_{i-1})\} \tag{4.22}$$

Equations (4.21 and 4.22) are recurrent formulas for the evaluation of the integrals in Eq. (4.15) at any time $t = t_i$.

Problem 4.2
Develop a numerical method to evaluate Duhamel's integral including damping in the system.

Solution:
The response of a damped single-degree-of-freedom system in terms of Duhamel's integral is given by Eq.(4.15) as

$$u(t) = \frac{1}{m\omega_D}\int_0^t F(\tau)e^{-\xi\omega(t-\tau)}\sin\omega_D(t-\tau)\,d\tau \quad \text{(repeated)} \tag{4.15}$$

For numerical evaluation, we proceed as in the undamped case and obtain from Eq. (4.15)

$$u(t) = \{A_D(t)\sin\omega_D t - B_D(t)\cos\omega_D t\}\frac{e^{-\xi\omega t}}{m\omega_D} \tag{g}$$

where

$$A_D(t_i) = A_D(t_{i-1}) + \int_{t_{i-1}}^{t_i} F(\tau)e^{\xi\omega\tau}\cos\omega_D\tau\,d\tau \tag{h}$$

$$B_D(t_i) = B_D(t_{i-1}) + \int_{t_{i-1}}^{t_i} F(\tau)e^{-\xi\omega\tau}\sin\omega_D\tau\,d\tau \tag{i}$$

For a linear piecewise loading function, $F(\tau)$ given by Eq. (f) of Problem 4.1, is substituted into Eqs. (h and i) which require the evaluation of the following integrals:

$$I_1 = \int_{t_{i-1}}^{t_i} e^{\xi\omega\tau}\cos\omega_D\tau\,d\tau = \frac{e^{\xi\omega\tau}}{(\xi\omega)^2 + \omega_D^2}\left(\xi\omega\cos\omega_D\tau + \omega_D\sin\omega_d\tau\right)\Big|_{t_{i-1}}^{t_i} \tag{j}$$

$$I_2 = \int_{t_{i-1}}^{t_i} e^{\xi\omega\tau}\sin\omega_D\tau\,d\tau = \frac{e^{\xi\omega\tau}}{(\xi\omega)^2 + \omega_D^2}\left(\xi\omega\sin\omega_D\tau - \omega_D\cos\omega_d\tau\right)\Big|_{t_{i-1}}^{t_i} \tag{k}$$

$$I_3 = \int_{t_{i-1}}^{t_i} \tau e^{\xi\omega\tau} \sin \omega_D \tau \, d\tau = \left(\tau - \frac{\xi\omega}{(\xi\omega)^2 + \omega_D^2} \right) I_2' + \frac{\omega_D}{(\xi\omega)^2 + \omega_D^2} I_1' \Big|_{t_{i-1}}^{t_1} \tag{l}$$

$$I_4 = \int_{t_{i-1}}^{t_i} \tau e^{\xi\omega\tau} \cos \omega_D \tau \, d\tau = \left(\tau - \frac{\xi\omega}{(\xi\omega)^2 + \omega_D^2} \right) I_1' - \frac{\omega_D}{(\xi\omega)^2 + \omega_D^2} I_2' \Big|_{t_{i-1}}^{t_1} \tag{m}$$

where I_1' and I_2' are the integrals indicated in Eqs. (j and k) before their evaluation at the limits. In terms of these integrals, $A_D(t_i)$ and $B_D(t_i)$ may be evaluated after substituting Eq. (f) of Problem 4.1 into Eqs. (h and i) as

$$A_D(t_i) = A_D(t_{i-1}) + \left(F(t_{i-1}) - t_{i-1} \frac{\Delta F_i}{\Delta t_i} \right) I_1 + \frac{\Delta F_i}{\Delta t_i} I_4 \tag{n}$$

$$B_D(t_i) = B_D(t_{i-1}) + \left(F(t_{i-1}) - t_{i-1} \frac{\Delta F_i}{\Delta t_i} \right) I_2 + \frac{\Delta F_i}{\Delta t_i} I_3 \tag{o}$$

Finally, the substitution of Eqs. (n and o) into Eq. (g) gives the displacement at time t_i as

$$u_i(t_i) = \frac{e^{-\xi\omega t_i}}{m\omega_D} \{ A_D(t_i) \sin \omega_D t_i - B_D(t_i) \cos \omega_D t_i \} \tag{p}$$

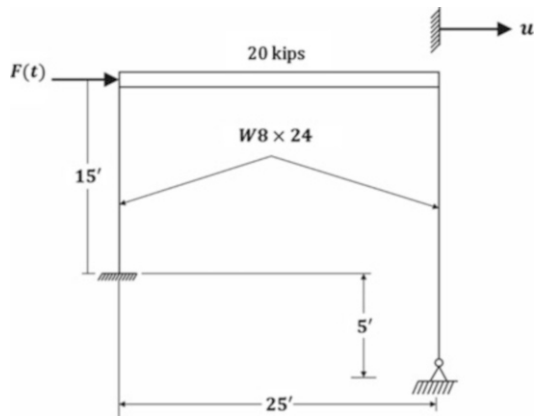

Fig. P4.3

4.7 Problems

Problem 4.3

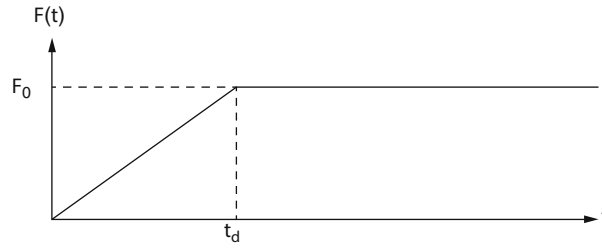

Fig. P4.5

The steel frame shown in Fig. P4.3 is subjected to a horizontal force F(t) applied at the girder level. The force decreases linearly from 5 kip at time $t = 0$ to zero at $t = 0.6$ s. Determine: (a) the horizontal deflection at $t = 0.5$ s and (b) the maximum horizontal deflection. Assume the columns massless and the girder rigid. Neglect damping.

Problem 4.4
Repeat Problem 4.3 for 10% of critical damping.

Problem 4.5
For the load-time function in Fig. P4.5, derive the expression for the dynamic load factor for the undamped simple oscillator as a function of t, ω, and t_d.

Problem 4.6
The frame shown in Fig. P4.3 is subjected to a sudden acceleration of 0.5 g applied to its foundation. Determine the maximum shear force in the columns. Neglect damping.

Problem 4.7
Repeat Problem 4.6 for 10% of critical damping.

Problem 4.8
Use Duhamel's integral to obtain the response of a damped simple oscillator of stiffness k, mass m, and damping ratio ξ, subjected to a suddenly applied force of magnitude F_0. Assume initial displacement and initial velocity equal zero.

Problem 4.9

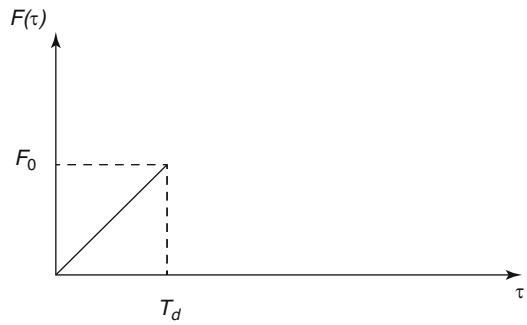

Fig. P4.11

Establish the equation of motion for the system in Problem 4.8 and solve it by superposition of complementary and [particular solutions with initial conditions for displacement and velocity equal to zero.

Problem 4.10

A trailer being pulled by a truck moving at constant speed v is idealized as a mass m connected to the truck by a spring of stiffness k. Determine the governing equation and its solution if the truck starts from rest.

Problem 4.11

Determine the response of an undamped system to a ramp force (Fig. P4.11) of maximum magnitude F_0 and duration t_d starting with zero initial conditions of displacement and velocity.

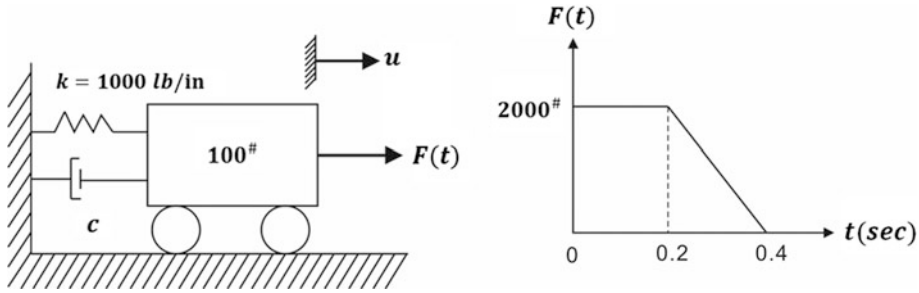

Fig. P4.14

Problem 4.12

Determine the maximum displacement at the top of the columns and maximum bending stress in the frame of Fig. P4.3 assuming that the columns are pinned at the base. Discuss the effect of base fixity.

Problem 4.13

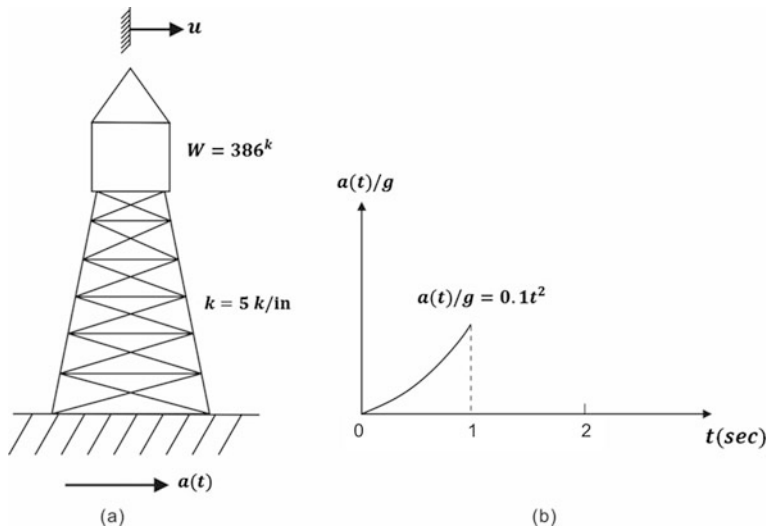

Fig. P4.16

Determine the maximum response (displacement and bending stress) for the frame of Illustrative Example 4.1 subjected to a triangular load of initial force $F_0 = 6000$ lb linearly decreasing to zero at time $t_d = 0.1$ s.

Problem 4.14.
For the dynamic system shown in Fig. P4.14, determine and plot the displacement as a function of time for the interval $0 \leq t \leq 0.5$ s. Neglect damping.

Problem 4.15
Repeat problem 4.14 for 10% critical damping.

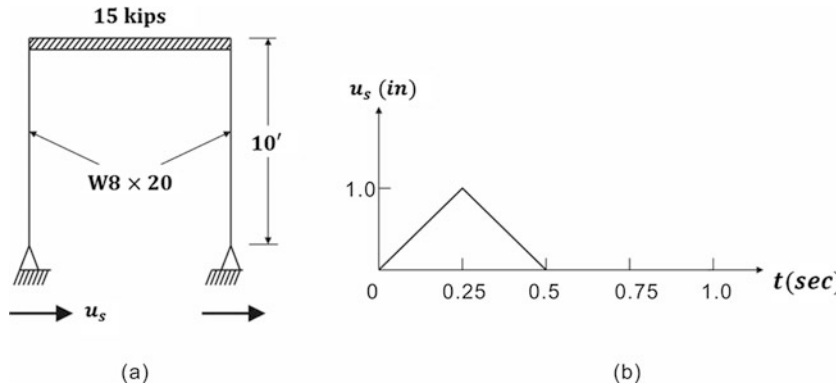

Fig. P4.19

Problem 4.16
The tower of Fig. P4.16a is subjected to horizontal ground acceleration $a(t)$ shown in Fig. P4.16b. Determine the relative displacement at the top of the tower at time $t = 1.0$ s. Neglect damping.

Problem 4.17
Repeat Problem 4.16 for 20% of critical damping.

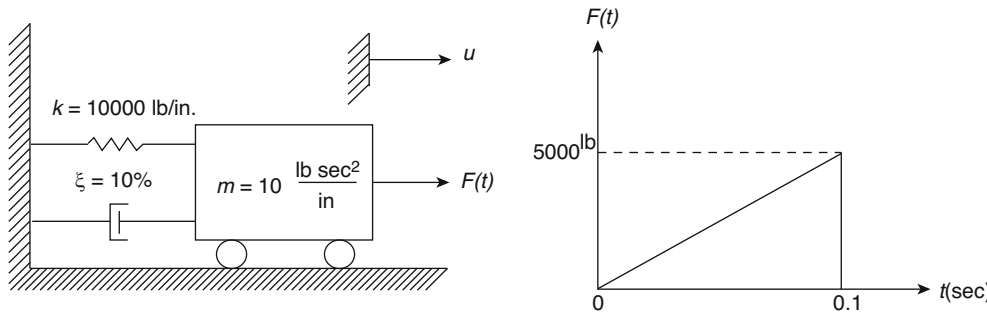

Fig. P4.21

Problem 4.18
Determine for the tower of Problem 4.17, the maximum displacement at the top of the tower relative to the ground displacement.

Problem 4.19

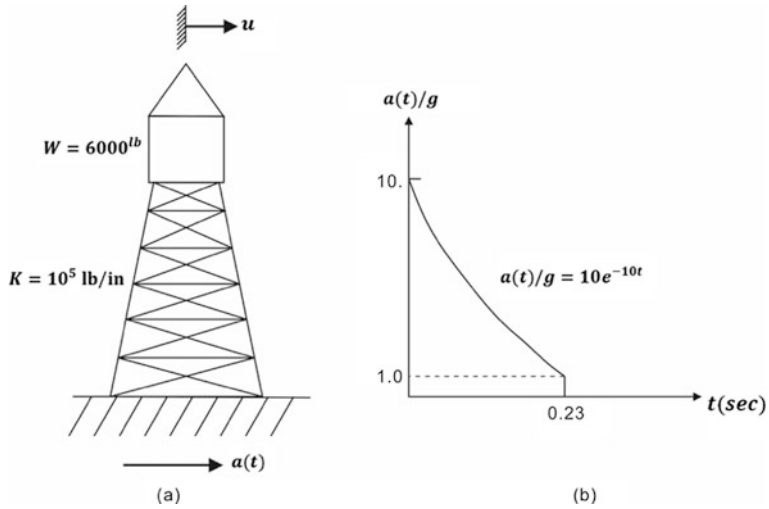

Fig. P4.22

The frame of Fig. P4.19a is subjected to horizontal support motion shown in Fig. P4.19b. Determine the maximum absolute deflection of top of the frame. Assume no damping.

Problem 4.20
Repeat Problem 4.19 for 10% of critical damping.

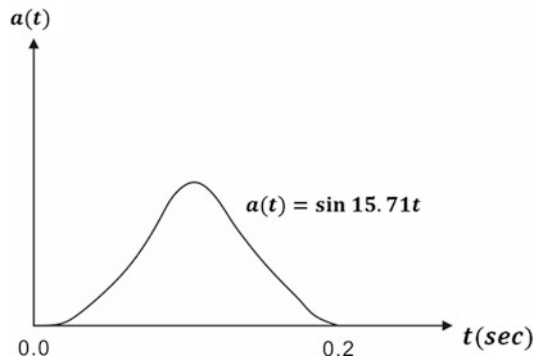

Fig. P4.24

Problem 4.21
A structural system modeled by the simple oscillator with 10% ($\xi = 0.10$) of critical damping is subjected to the impulsive load as shown in Fig. P4.21. Determine the response.

Problem 4.22
A water tower modeled as shown in Fig. P4.22a is subjected to ground shock given by the function

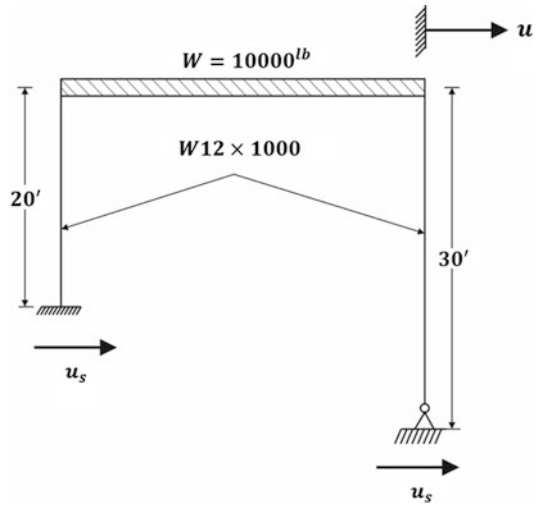

Fig. P4.25

depicted in Fig. P4.22b. Determine: (a) the maximum displacement at the top of the tower and (b) the maximum shear force at the base of the tower. Neglect damping. Use time step for integration $\Delta t = 0.005$ s.

Problem 4.23
Repeat Problem 4.22 for 20% of critical damping.

Problem 4.24
Determine the maximum response of the tower of Problem 4.22 when subjected to the impulsive ground acceleration depicted in Fig. P4.24.

Problem 4.25
The steel frame in Fig. P4.25 is subjected to the ground motion produced by a passing train in its vicinity. The ground motion is idealized as a harmonic acceleration of the foundation of the frame with amplitude 0.1 g at frequency 10 cps. Determine the maximum response in terms of the relative displacement of the girder of the frame and the motion of the foundation. Assume 10% of the critical damping.

Problem 4.26

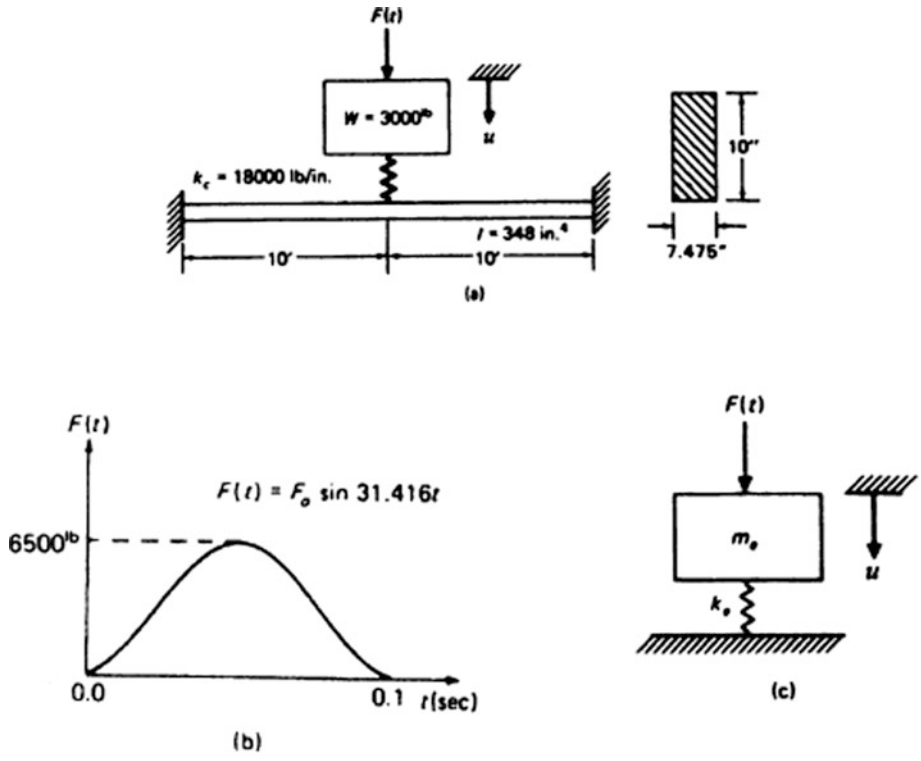

Fig. P4.27

Determine the maximum stresses in the columns of the frame in Problem 4.25 using the maximum response in terms of relative motion. Also check that the same results may be obtained using the response in terms of the maximum absolute acceleration.

Problem 4.27

A machine having a weight $W = 3000$ lb is mounted through coil springs to a steel beam of rectangular cross-section as shown in Fig. P4.27a. Due to malfunctioning, the machine produces a shock force represented in Fig. P4.27b. Neglecting the mass of the beam and damping in the system, determine the maximum displacement of the machine.

Problem 4.28

For Problem 4.27 determine: (a) the maximum tensile and compressive stresses in the beam and (b) the maximum force experienced by the coil springs during the shock.

Response Spectra

5

In this chapter, we introduce the concept of response spectrum, which in recent years has gained wide acceptance in structural dynamic practice, particularly in earthquake engineering design. Stated brief, the response spectrum is a plot of the maximum response (maximum displacement, velocity, acceleration, or any other quantity of interest) to a specified load function for all possible single-degree-of-freedom systems. The abscissa of the spectrum is the natural frequency (or period) of the system, and the ordinate the maximum response. A plot of this type is shown in Fig. 5.1, in which a one-story building is subject to a ground displacement indicated by the function $u_s(t)$. The response spectral curve shown in Fig. 5.1a gives, for any single-degree-of-freedom system, the maximum displacement of the response from an available spectral chart, for a specified excitation, we need only to know the natural frequency of the system.

5.1 Construction of Response Spectrum

To illustrate the construction of a response spectral chart, consider in Fig. 5.2a the undamped oscillator subject to one-half period of the sinusoidal exciting force shown in Fig. 5.2b.

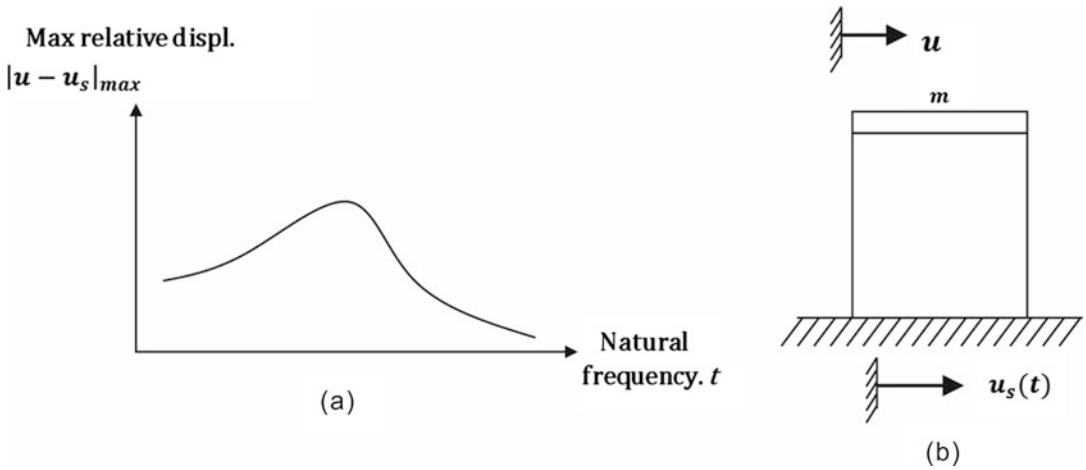

Fig. 5.1 (**a**) Typical response spectrum. (**b**) Single-degree-of-freedom system subject to ground excitation

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M. Paz, Y. H. Kim, *Structural Dynamics*, https://doi.org/10.1007/978-3-319-94743-3_5

The system is assumed to be initially at rest. The duration of the sinusoidal impulse is denoted by t_d. The differential equation of motion is obtained by equating to zero the sum of the forces in the corresponding free body diagram shown in Fig. 5.2c, that is,

$$m\ddot{u} + ku = F(t) \tag{5.1}$$

in which

$$F(t) = \begin{cases} F_0 \sin \varpi t & \text{for } 0 \leq t \leq t_d \\ 0 & \text{for } t > t_d \end{cases} \tag{5.2}$$

and

$$\varpi = \frac{\pi}{t_d} \tag{5.3}$$

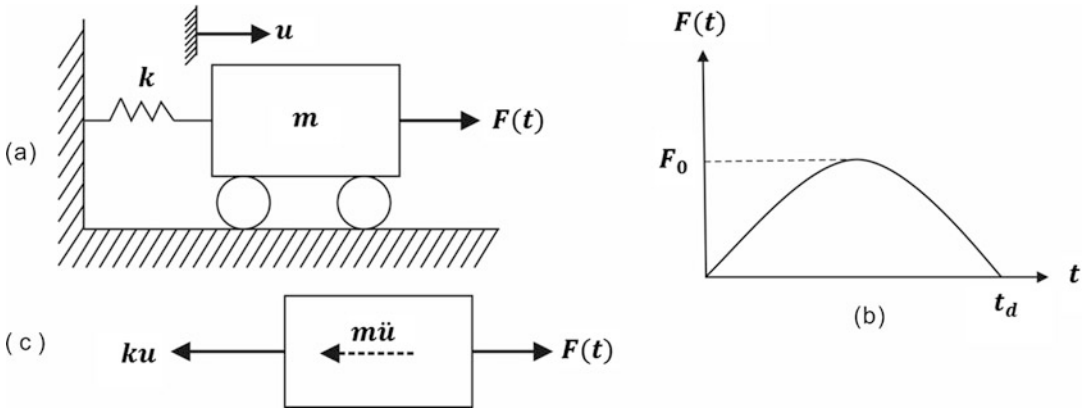

Fig. 5.2 (**a**) Undamped simple oscillator subjected to load $F(t)$. (**b**) Loading function $F(t) = F_0 \sin \varpi t$ $(0 \leq t \leq t_d)$, (**c**) Free body diagram

The solution of Eq. (5.1) may be found by any of the methods studied in the preceding Chap. 4 such as the use of Duhamel's integral or the direct method. However, for this example, owing to the simplicity of the exciting force, we can obtain the solution of Eq. (5.1) by the general method of integration of a linear differential equation, that is, the superposition of the complementary solution u_c and the particular solution $u_{p;}$

$$u = u_c + u_p \tag{5.4}$$

The complementary solution of Eq. (5.1) (right-hand side equals zero) is given by Eq. (1.17) as

$$u_c = A \cos \omega t + B \sin \omega t \tag{5.5}$$

in which $\omega = \sqrt{k/m}$ is the natural frequency. The particular solution for the time interval $0 \leq t \leq t_d$ is suggested by the right-hand side of Eq. (5.1) to be of the form

$$u_p = C \sin \varpi t \tag{5.6}$$

The substitution of Eq. (5.6) into Eq. (5.1) and solution of the resulting identity gives

$$C = \frac{F_0}{k - m\varpi^2} \tag{5.7}$$

Combining (5.4) through (5.7), we obtain the response for $0 \leq t \leq t_d$ *as*

$$u = A\cos\omega t + B\sin\omega t + \frac{F_0 \sin \varpi t}{k - m\varpi^2} \tag{5.8}$$

Introducing the initial conditions $u\,(0) = 0$ and $\dot{u}\,(0) = 0$ into Eq. (5.8) and calculating the constants of integration A and B, we obtain

$$u = \frac{F_0/k}{1 - (\varpi/\omega)^2}\left[\sin \varpi t - (\varpi/\omega)\sin\omega t\right] \tag{5.9}$$

It is convenient to introduce the following notation:

$$u_{st} = \frac{F_0}{k}, \quad \varpi = \frac{\pi}{t_d}, \quad \omega = \frac{2\pi}{T}$$

Then Eq. (5.9) becomes

$$\frac{u}{u_{st}} = \frac{1}{1 - \left(\frac{T}{2t_d}\right)^2}\left[\sin\pi\frac{t}{t_d} - \frac{T}{2t_d}\sin 2\pi\frac{t}{T}\right] \quad \text{for } 0 \leq t \leq t_d \tag{5.10a}$$

After a time t_d the external force becomes zero and the system is then in free vibration. Therefore, the response for $t > t_d$ is of the form given by Eq. (5.5) with the constants of integration determined from known values of displacement and velocity calculated from Eq. (5.10a) at time $t = t_d$. The expression obtained for the response is then given by

$$\frac{u}{u_{st}} = \frac{T/t_d}{\left(\frac{T}{2t_d}\right)^2 - 1}\cos\pi\frac{t}{t_d}\sin 2\pi\left(\frac{t}{T} - \frac{t_d}{2T}\right) \quad \text{for } t \geq t_d \tag{5.10b}$$

It may be seen from Eq. (5.10a) that the response in terms of u/u_{st}, is a function of the ratio of the pulse duration to the natural period of system (t_d/T) and of time expressed as t/T. Hence for any fixed value of the parameter t_d/T, we can obtain the maximum response from Eq. (5.10b). The plot in Fig. 5.3 of these maximum values as a function of t_d/T is the response spectrum for the half-sinusoidal force duration considered in this case. It can be seen from the response spectrum in Fig. 5.3 that the maximum value of the response (amplification factor) $u/u_{st} = 1.76$ occurs for this particular pulse when $t_d/T = 0.8$.

Owing to the simplicity of the input force, it was possible in this case to obtain a closed solution and to plot the response spectrum in terms of dimensionless ratios, thus making this plot valid for any impulsive force described by one-half of the sine cycle. However, in general, for an

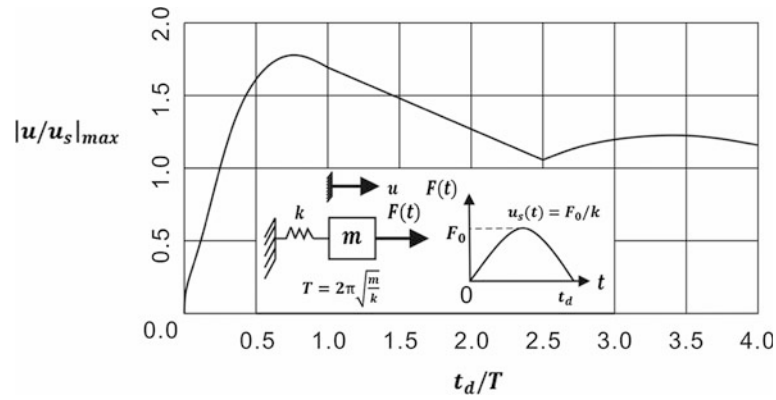

Fig. 5.3 Response spectrum for half-sinusoidal force of duration t_d

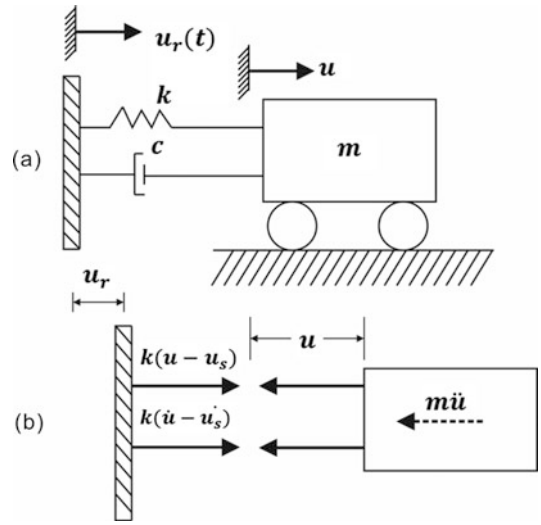

Fig. 5.4 (**a**) Damped simple oscillator subjected to support excitation. (**b**) Free body diagram.

5.2 Response Spectrum for Support Excitation

An important problem in structural dynamic is the analysis of a system subjected to excitation applied to the base or foundation of the structure. An example of such input excitation of the base acting on a damped oscillator which serves to model certain structures is shown in Fig. 5.4. The excitation in this case is given as an acceleration function which is represented in Fig. 5.5. The equation of motion that is obtained by equation to zero the sum of the forces in the corresponding free body diagram in Fig. 5.4b is

$$m\ddot{u} + c\left(\dot{u} - \dot{u}_s\right) + k(u - u_s) = 0 \tag{5.11}$$

or, with the usual substitution $\omega = \sqrt{k/m}$ and $\xi = c/c_{cr}$ $\left(c_{cr} = 2\sqrt{km}\right)$,

$$\ddot{u} + 2\xi\omega\dot{u} + \omega^2 u = \omega^2 u_s(t) + 2\xi\omega\dot{u}_s(t) \tag{5.12}$$

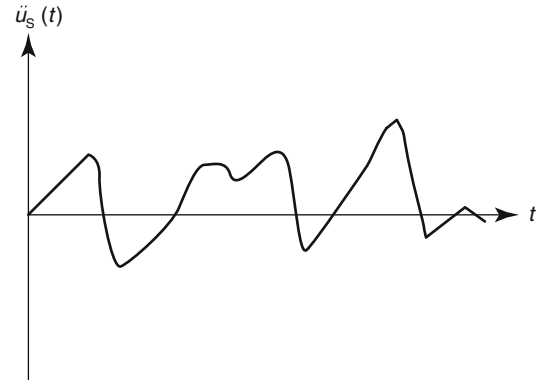

Fig. 5.5 Acceleration function exciting the support of the oscillator in Fig. 5.4

Equation (5.12) is the differential equation of motion for the damped oscillator in terms of its absolute motion. A more useful formulation of this problem is to express Eq. (5.12) in terms of the relative motion of the mass with respect to the motion of the support, that is, in terms of the spring deformation. The relative displacement u_r is then defined as

$$u_r = u - u_s \tag{5.13}$$

Substitution into Eq. (5.12) yields

$$\ddot{u}_r + 2\xi\omega\dot{u}_r + \omega^2 u_r = -\ddot{u}_s(t) \tag{5.14}$$

The formulation of the equation of motion in Eq. (5.14) as a function of the relative motion between the mass and the support is particularly important since in design it is the deformation or stress in the "spring element" that is required. Besides, the input motion at the base is usually specified by means of an acceleration function (e.g., earthquake accelerograph record); thus Eq. (5.14) containing in the right-hand side the acceleration of the excitation is a more convenient form than Eq. (5.12) which in the right–hand side has the support displacement and the velocity.

The solution of the differential equation, Eq. (5.14), may be obtained by any of the methods presented in previous chapters for the solution of one-degree-of-freedom systems. In particular, the solution is readily expressed using Duhamel's integral as

$$u(t) = -\frac{1}{\omega}\int_0^t \ddot{u}_s(\tau)e^{-\xi\omega(t-\tau)}\sin\omega(t-\tau)d\tau \tag{5.15}$$

5.3 Tripartite Response Spectra

It is possible to plot in a single chart using logarithmic scales the maximum response in terms of the acceleration, the relative displacement, and a third quantity known as the relative pseudovelocity. The pseudovelocity is not exactly the same as the actual velocity, but it is closely related and provides for a convenient substitute for the true velocity. These three quantities the maximum absolute acceleration, the maximum relative displacement, and the maximum relative pseudovelocity are known, respectively, as the spectral acceleration, spectral displacement, and spectral velocity.

It is significant that the spectral displacement S_D, that is, the maximum relative displacement, is proportional to the spectral acceleration S_a the maximum absolute acceleration. To demonstrate this fact, consider the equation of motion, Eq. (5.11), which, after using Eq. (5.13), becomes for the damped system

$$m\ddot{u} + c\dot{u}_r + ku_r = 0 \tag{5.16}$$

and for the undamped system

$$m\ddot{u} + ku_r = 0 \tag{5.17}$$

We observe from Eq. (5.17) that the absolute acceleration is at all times proportional to the relative displacement. In particular, at maximum values, the spectral acceleration is proportional to the spectral displacement, that is, form Eq. (5.17)

$$S_a = -\omega^2 S_D \tag{5.18}$$

where $\omega = \sqrt{k/m}$ is the natural frequency of the system, $S_a = \ddot{u}_{\max}$, and $S_D = u_{r,\max}$.

When damping is considered in the system, it may be rationalized that the maximum relative displacement occurs when the relative velocity is zero ($\dot{u} = 0$). Hence we again obtain Eq. (5.18) relating spectral acceleration and spectral displacement. However, for a damped system, the spectral acceleration S_a is not exactly equal to the maximum acceleration, although in general, it provides a good approximation. Equation (5.18) is by mere coincidence the same as the relationship between acceleration and displacement for a simple harmonic motion. The fictitious velocity associated with the apparent harmonic motion is the pseudovelocity and, its maximum value S_v is defined as the spectral velocity, that is

$$S_v = \omega S_D = \frac{S_a}{\omega} \tag{5.19}$$

Dynamic response spectra for a single-degree-of-freedom elastic systems have been computed for a number of input motions. A typical example of response spectrum for a single-degree-of freedom system subjected to support motion is shown in Fig. 5.6. This plot is the response for the input motion given by the recorded ground acceleration of the North-south component of the 1940 El Centro earthquake. The acceleration record of this earthquake has been used extensively in earthquake engineering investigations. A plot of the acceleration record for this earthquake is show in Fig. 5.7. Until the time of the San Fernando, California earthquake of 1971, the El Centro record was one of the few records available for long and strong earthquake motions. In Fig. 5.8, the same type of data that were used to obtain the displacement response spectrum in Fig. 5.6 are plotted in terms of the spectral velocity, for several values of the damping coefficient, with the difference that the abscissa as well as the ordinate are in these cases plotted on a logarithmic scale. In this type of plot, because of Eqs. (5.18 and 5.19), it is possible to draw diagonal scales for the displacement sloping 135 ° with the abscissa, and for the acceleration 45 °, so that we can from a single plot read values of spectral acceleration, spectral velocity, and spectral displacements.

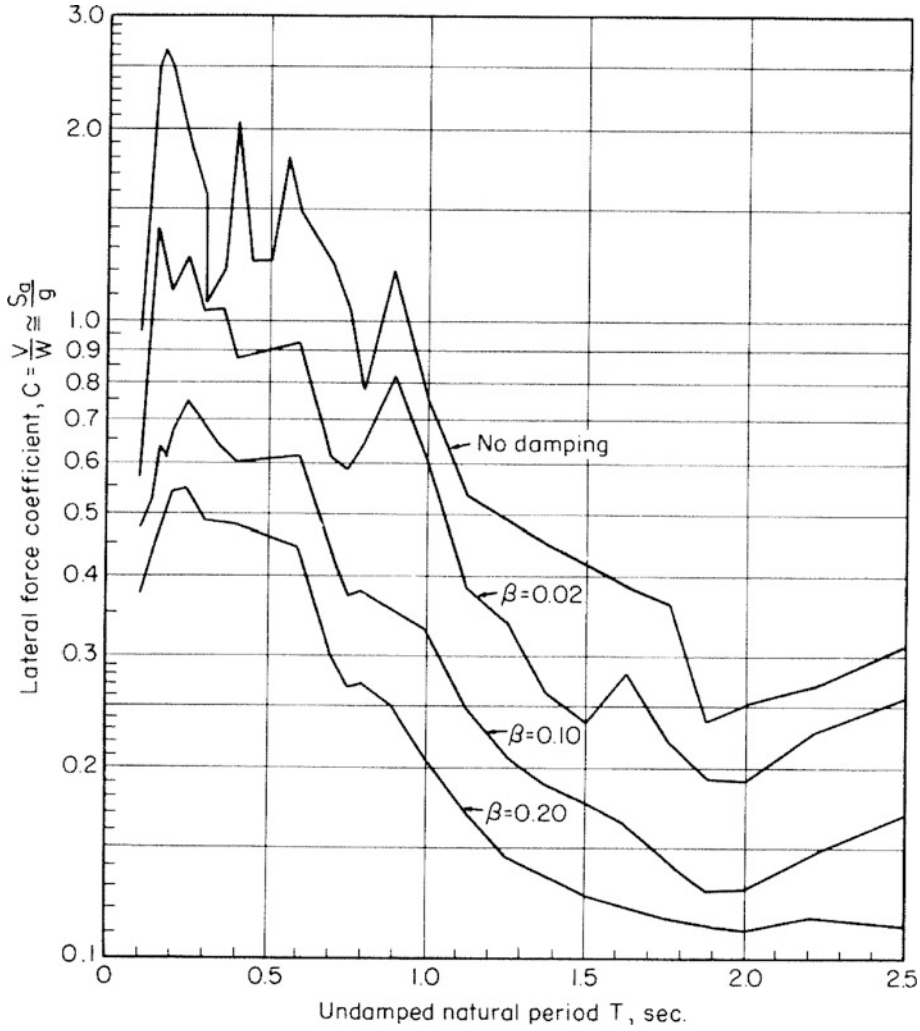

Fig. 5.6 Acceleration spectra for elastic system (from *Design of Multistory Reinforced Building for Earthquake Motions* by J. A. Blum, N. M. Newmark, and L. H. Corning, Portland Cement Association 1961)

To demonstrate the construction of a tripartite diagram such as the one of Fig. 5.8, we write Eq. (5.19) in terms of the natural frequency f in cycles per second (cps) and take the logarithm of the terms, so that

$$\begin{aligned} S_v &= \omega S_D = 2\pi f S_D \\ \log S_v &= \log f + \log(2\pi S_D) \end{aligned} \tag{5.20}$$

For constant values of S_D, Eq. (5.20) is the equation of a straight line of $\log S_v$ versus $\log f$ with a slope of 45 °. Analogously, from Eq. (5.19)

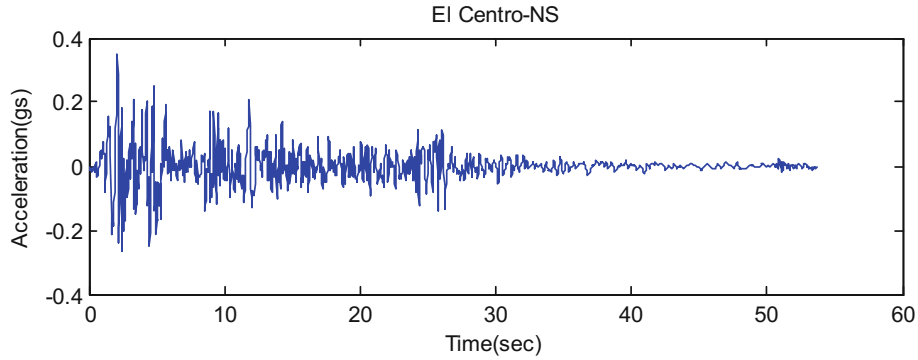

Fig. 5.7 Ground acceleration record for El Centro, California earthquake of May 18, 1940 north–south component

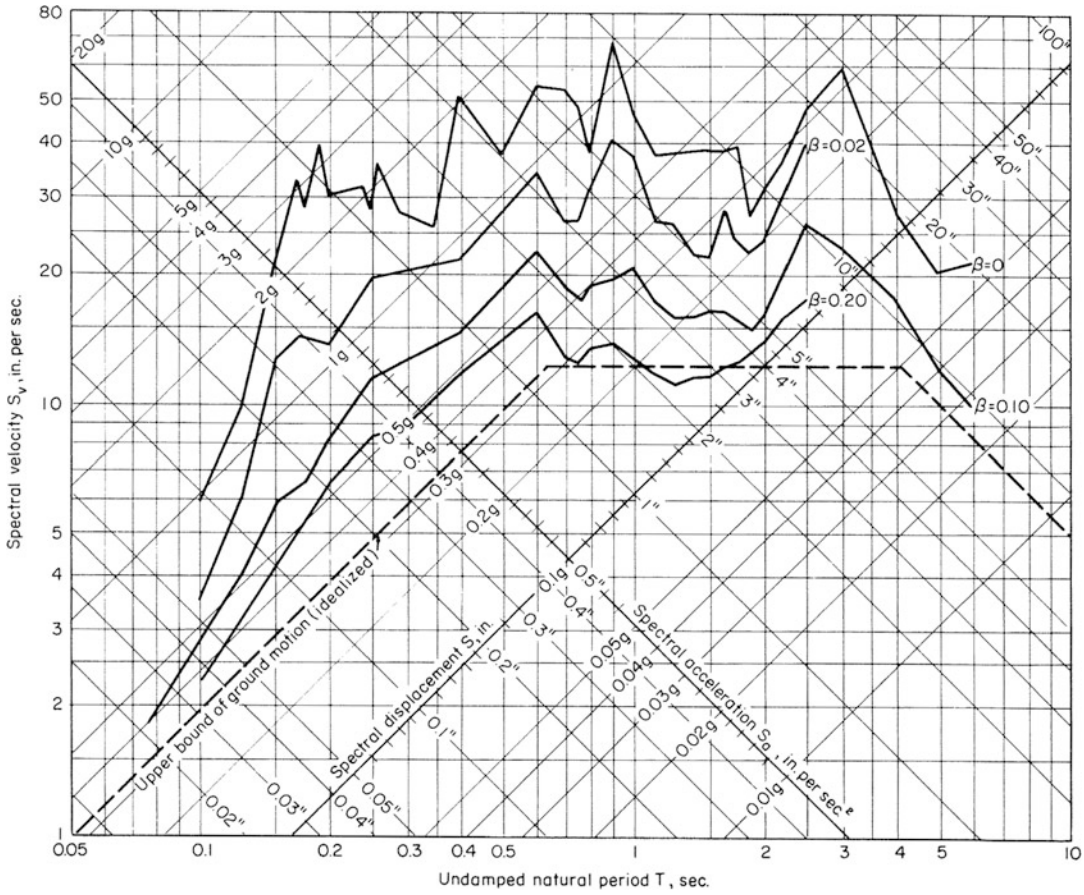

Fig. 5.8 Response spectra for elastic system for the 1940 El Centro earthquake (from Blume et al. 1961)

$$S_v = \frac{S_a}{\omega} = \frac{S_a}{2\pi f}$$
$$\log S_v = -\log f + \log \frac{S_a}{2\pi} \tag{5.21}$$

For a constant value of S_a, Eq. (5.21) is the equation of a straight line of log S_v versus log f with a slope of 135 °.

5.4 Response Spectra for Elastic Design

In general, response spectral charts are prepared by calculating the response to a specified excitation of single-degree-of-freedom systems with various amounts of damping. Numerical integration with short time intervals are applied to calculate the response of the system. The step-by-step process is continued until the total earthquake record has been completed. The greatest value of the function of interest is recorded and becomes the maximum response of the system to that excitation. Changing the parameters of the system to change the natural frequency, the process is continued and a new maximum response is recorded. This process is continued until all frequencies of interest have been covered and the results plotted. Since no two earthquakes are alike, this process must be repeated for all earthquakes of interest.

As already stated, until the San Fernando, California earthquake of 1971, there were few recorded strong earthquake motions because there were few accelerometers emplaced to measure them. The El Centro, California earthquake of 1940 was the most severe earthquake recorded and was used as the basis for much analytical work. Since that year, however, many other strong earthquakes have been recorded. Maximum values of ground motion of about 0.32 g for the El Centro earthquake to values of more than 0.5 g for other earthquakes have been recorded. It can be expected that even larger values will be recorded as more instruments are placed closer to the epicenters of earthquakes.

Earthquakes consist of a series of essentially random ground motions. Usually the north-south, east-west, and vertical components of the ground acceleration are measured. Currently, no accurate method is available to predict the particular motion that a site can be expected to experience in future earthquakes. Thus it is reasonable to use a design response spectrum which incorporates the spectra for several earthquakes and which represents a kind of "average" response spectrum for design. Such a design response spectrum is shown in Fig. 5.9 normalized for a maximum ground acceleration of

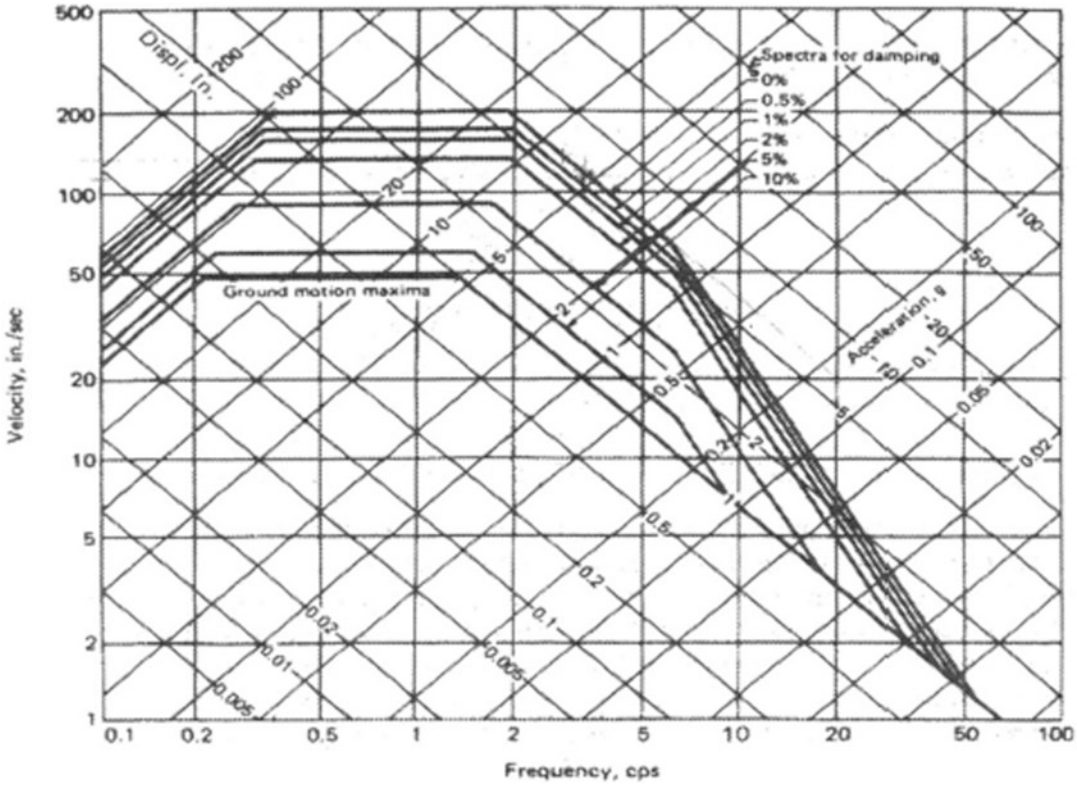

Fig. 5.9 Basic design spectra normalized to peak ground acceleration of 1.0 *g*. (From Newark and Hall 1973)

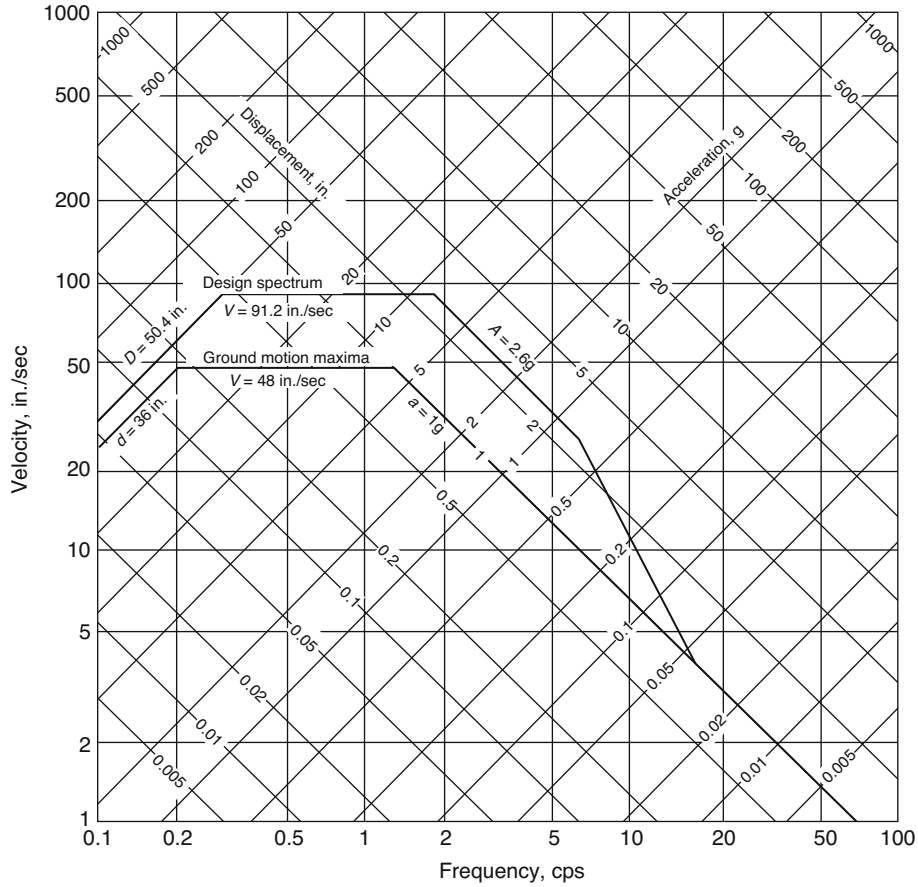

Fig. 5.10 Elastic design spectrum normalized to peak ground acceleration of 1.0 *g* for 5% damping. (Newmark and Hall 1973)

1.0 *g*. This figure shows the design maximum ground motion and a series of response spectral plots corresponding to various values of the damping ratio in the system.

Details for the construction of the basic spectrum for design purposes are given by Newmark and Hall (1973), who have shown that smooth response spectra of idealized ground motion may be obtained by amplifying the ground motion by factors depending on the damping in the system . In general, for any given site, estimates might be made of the expected maximum ground acceleration, maximum ground velocity, and maximum ground displacement. The lines representing these maximum values are drawn on a tripartite logarithmic paper of which Fig. 5.10 is an example. The lines in this figure are shown for a maximum ground acceleration of 1.0 *g*, a velocity of 48 in/sec, and a displacement of 36 in. These values correspond to motions that are more intense than those generally expected in seismic design. They are, however, of proportional magnitudes that are generally acceptable in practice.

Table 5.1 Design spectrum amplification factors (Newmark and Hall, 1973)

Amplification factors			
Percent damping	Displacement	Velocity	Acceleration
0	2.5	4.0	6.4
0.5	2.2	3.6	5.8
1	2.0	3.2	5.2
2	1.8	2.8	4.3
5	1.4	1.9	2.6
7	1.2	1.5	1.9
10	1.1	1.3	1.5
20	1.0	1.1	1.2

These maximum values normalized for a ground acceleration of the ground of 1 g are simply scaled down for other than 1 g maximum acceleration of the ground. Recommended amplification factors to obtain the response spectra from maximum values of the ground motion are given in Table 5.1. For each value of the damping coefficient, the amplified displacement lines are drawn at the left, the amplified velocities at the top, and the amplified acceleration at the right of the chart. At a frequency of approximately 6 cps (Fig. 5.9), the amplified acceleration region line intersects a line sloping down toward the maximum ground acceleration value at a frequency of about 30 cps for a system with 2% damping. The lines corresponding to other values of damping are drawn parallel to the 2% damping line as shown in Fig. 5.9.

The amplification factors in Table 5.1 were developed on the basis of earthquake records available at the time. As new records of more recent earthquakes become available, these amplification factors have been recalculated. Table 5.2 shows the results of a statistical study based on a selection of 10 strong motion earthquakes. The table gives recommended amplification factors as well as the corresponding standard deviation values obtained in the study. The relatively large values shown in Table 5.2 for the standard deviation of the amplification factors provides further evidence on the uncertainties surrounding earthquake prediction for structural analysis. The response spectra for designs presented in Fig. 5.9 has been constructed using the amplification factors shown in Table 5.1.

Table 5.2 Spectral amplification factors and standard deviation values[a]

	Displacement		Velocity		Acceleration	
Percent damping	Factor	Standard deviation	Factor	Standard deviation	Factor	Standard deviation
2	1.691	0.828	2.032	0.853	3.075	0.738
5	1.465	0.630	1.552	0.605	2.281	0.502
10	1.234	0.481	1.201	0.432	1.784	0.321

[a]Newmark, N. M., and Riddell, R., Inelastic Spectra for Seismic Design: Seventh World Earthquake Conference, Istanbul, Turkey. Vol. 4, pp. 129–136, 1980

Illustrative Example 5.1

A structure modeled as a single-degree-of-freedom system has a natural period, $T = 1$ sec. Use the response spectral method to determine the maximum absolute acceleration, the maximum relative displacement, and the maximum relative pseudovelocity for: (a) a foundation motion equal to the N-S component of the El Centro earthquake of 1940, and (b) the design spectra earthquake with a maximum ground acceleration equal to 0.32 g. Assume 10% of the critical damping.

Solution:

(a) From the response spectra in Fig. 5.8 with $f = 1/T = 1.0$ cps, corresponding to the curve labeled $\xi = 0.10$, we read on the three scales the following values:

$$S_D = 3.3 \text{ in.}$$

$$S_v = 18.5 \text{ in/ sec}$$

$$S_a = 0.30g$$

(b) From the basic design spectra in Fig. 5.9 with frequency $f = 1$ cps and 10% critical damping, we obtain after scaling for 0.32 g maximum ground acceleration, the following results:

$$S_D = 9.5 \times 0.32 = 3.04 \text{ in}$$

$$S_v = 60 \times 0.32 = 19.2 \text{ in/ sec}$$

$$S_a = 0.95 \times 0.32 = 0.304\ g$$

5.5 Influence of Local Soil Conditions

Before the San Fernando earthquake of 1971, earthquake accelerograms were limited in number, and the majority had been recorded on alluvium. Therefore, it is only natural that the design spectra based on those data, such as those suggested by Housner (1959) and Newmark-Hall (1973), mainly represent alluvial sites. Since 1971, the wealth of information obtained from earthquakes worldwide and from subsequent studies have shown the very significant effect that the local site conditions have on spectral shapes.

An example of a conservative design spectrum is shown in Fig. 5.11. This figure shows four spectral acceleration curves representing the average of normalized spectral values corresponding to several sets of earthquake records registered on four types of soils. The dashed line through the points A, B, C, and D defines a possible conservative design spectrum for rock and stiff soil sites. Normalized design spectral shapes, as those includes in the recent edition of the Uniform Building Code (ICBO, 1994) (Fig. 5.12) are based on such simplifications. The UBC spectral shapes become

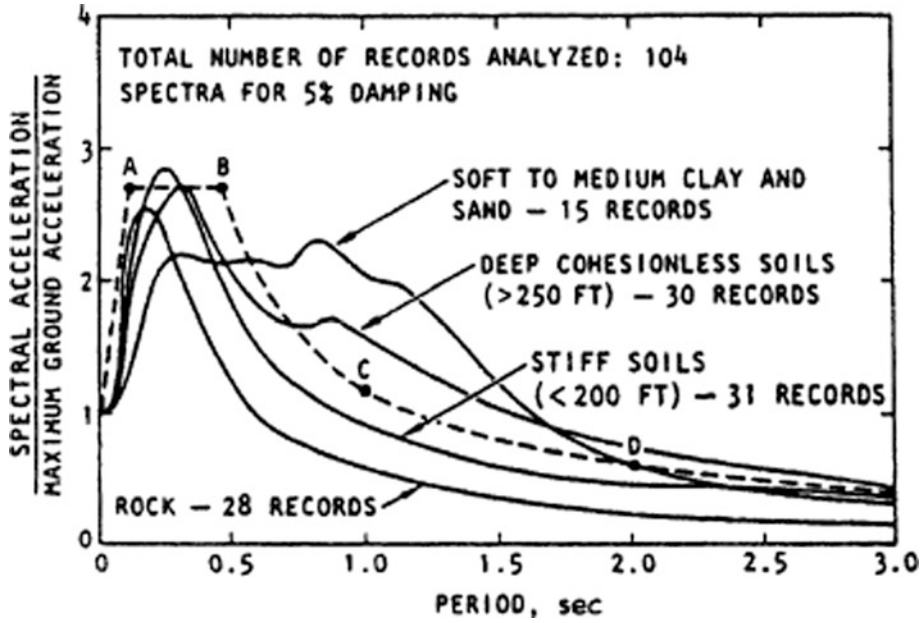

Fig. 5.11 Average acceleration spectra for different soil conditions (After Seed et al. 1976; from Seed and Idriss 1982)

trilinear, when drawn on a tripartite logarithmic chart, similar in shape to a Newmark-Hall spectrum. UBC spectral shapes (ICBO, 1994) can be expressed[1] by the following rather simple formulas:

Soil Type I (Rock and Stiff Soils):

$$\begin{aligned} S_A &= 1 + 10T \quad \text{for} \quad 0 < T \leq 0.15 \ \text{sec} \\ S_A &= 2.5 \quad\quad\ \text{for} \quad 0.15 < T \leq 0.39 \ \text{sec} \\ S_A &= 0.975/T \quad \text{for} \quad T > 0.39 \ \text{sec} \end{aligned} \tag{5.22}$$

Soil Type II (Deep Cohesionless or Stiff Clay Soils);

$$\begin{aligned} S_A &= 1 + 10T \quad \text{for} \quad 0 < T \leq 0.15 \ \text{sec} \\ \mathrm{S_A} &= 2.5 \quad\quad\ \text{for} \quad 0.15 < T \leq 0.585 \ \text{sec} \\ \mathrm{S_A} &= 1.463/T \quad \text{for} \quad T > 0.585 \ \text{sec} \end{aligned} \tag{5.23}$$

Soil Type III (Soft to Medium Clays and Sands):

$$\begin{aligned} S_A &= 1 + 75T \quad \text{for} \quad 0 < T \leq 0.2 \ \text{sec} \\ S_A &= 2.5 \quad\quad\ \text{for} \quad 0.2 < T \leq 0.915 \ \text{sec} \\ S_A &= 2.288/T \quad \text{for} \quad T > 0.915 \ \text{sec} \end{aligned} \tag{5.24}$$

where S_A in the spectral acceleration for 5% damping normalized to a peak ground acceleration of one g, and T is the fundamental period of the building. It should be noted that values obtained from the UBC spectral chart of Fig. 5.12, or alternatively, calculated with Eqs. (5.22, 5.23, and 5.24) are too conservative. In actual design practice, these values are scaled down by the structural factor R (Chap. 24) having values between 4 and 12, depending on the type of structural resisting system.

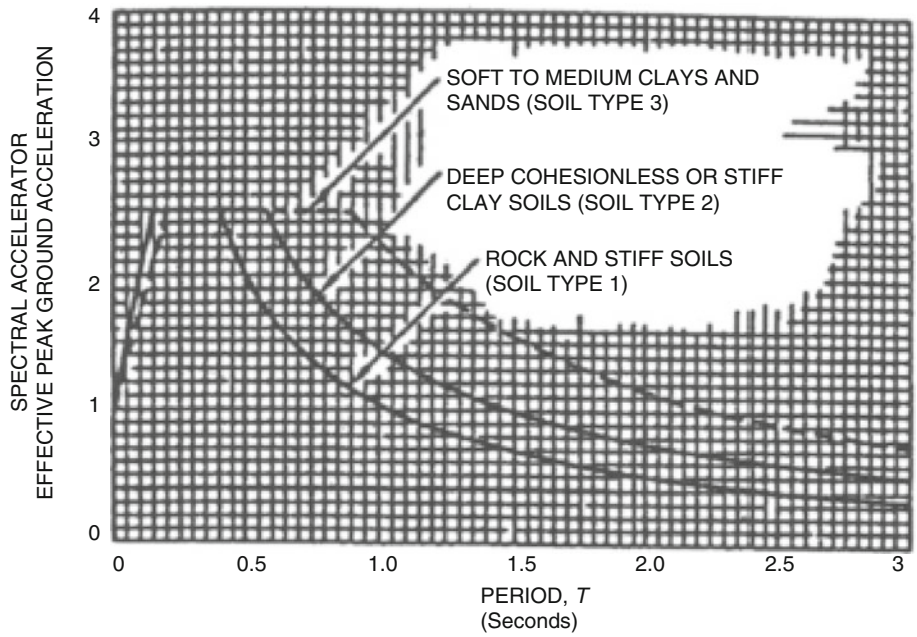

Fig. 5.12 Normalized design spectra shapes contained in uniform building code. (ICBO 1994)

[1] From Recommended Lateral Force Requirement and Tentative Commentary SEAOL-90, P. 36-C.

5.6 Response Spectra for Inelastic Systems

For certain types of extreme events such as nuclear blast explosions or very strong motion earthquakes, it is sometimes necessary to design structures to withstand strains beyond the elastic limit For example, in seismic design for an earthquake of moderate intensity, it is reasonable to assume elastic behavior for a well-designed and well-constructed structure. However, for very strong motions, this is not a realistic assumption even for a well-designed structure. Although structures can be designed to resist severe earthquakes, it is not feasible economically to design buildings to elastically withstand earthquakes of the greatest foreseeable intensity. In order to design structures for strain levels beyond the linear range, the response spectrum has been extended to include the inelastic range (Newmark and Hall 1973). Generally, the elastoplastic relation between force and displacement is assumed in structural dynamics. Such a force-displacement relationship is shown in Fig. 5.13b. Because of the assumption of elastoplastic behavior, if the force is removed prior to the occurrence of yielding, the material will return along its loading line to the origin. However, when yielding occurs at displacement u_t, the restoring force remains constant at a magnitude R_t. If the displacement is not reversed, the displacement may reach a maximum value u_{max}. If, however, the displacement is reversed, the elastic recovery follows along a line parallel to the initial line and the recovery proceeds elastically until a negative yield value R_c is reached in the opposite direction.

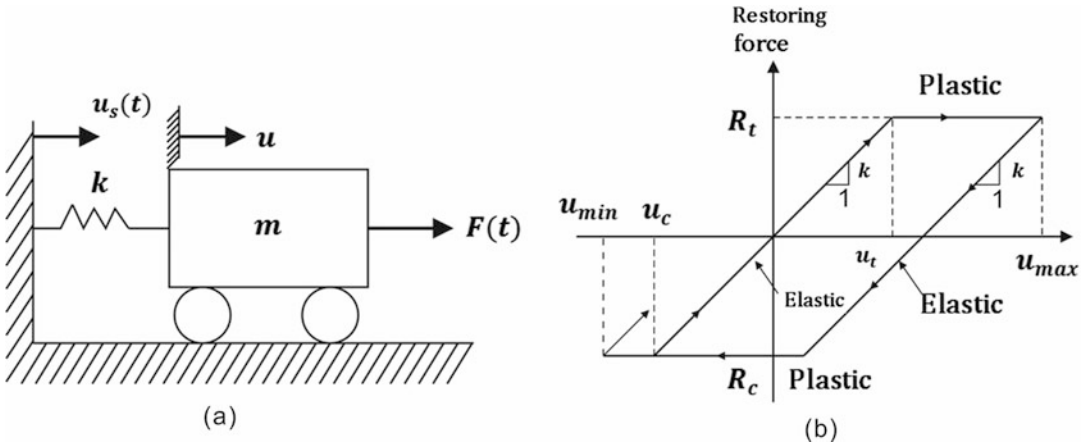

Fig. 5.13 Force-displacement relationship for an elastoplastic single-degree-of-freedom system

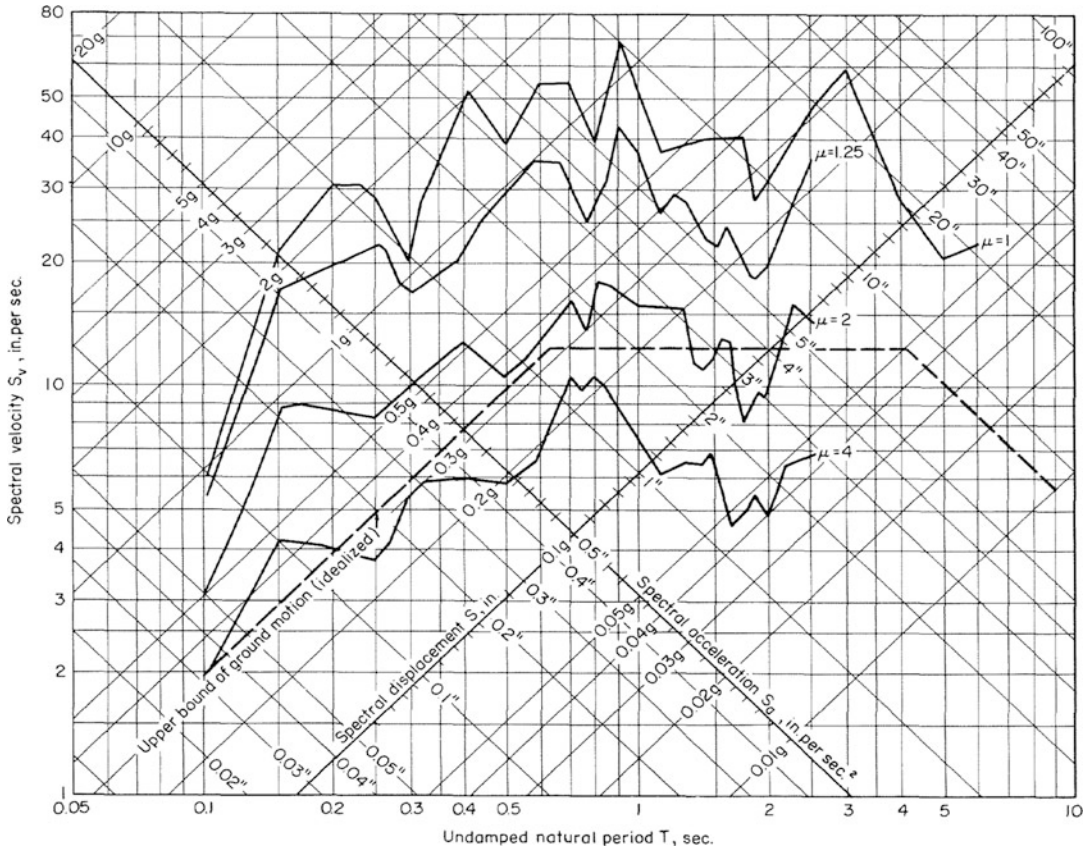

Fig. 5.14 Response spectra for undamped elastoplastic system for the 1940 El Centro earthquake. (From Blume et al. 1961)

The preparation of response spectra for such an inelastic system is more complex than that for elastic systems. However, response spectra have been prepared for several different earthquake motions. These spectra are usually plotted as a series of curves corresponding to definite values of the ductility ratio μ. The ductility ratio μ is defined as the ratio of the maximum displacement u_{max} of the structure in the inelastic range to the displacement corresponding to the yield point u_y, that is

$$\mu = \frac{u_{\max}}{u_y} \tag{5.25}$$

The response spectra for an undamped single-degree-of-freedom system subjected to a support motion equal to the El Centro 1940 earthquake is shown in Fig. 5.14 for several values of the ductility ratio. The tripartite logarithmic scales used to plot these spectra give simultaneously for any single-degree-of-freedom system of natural period T and specified ductility ratio μ, the spectral values of displacement, velocity, and acceleration. Similarly, in Fig. 5.15 are shown the response spectra for an elastoplastic system with 10% of critical damping. The spectral velocity and the spectral acceleration are read directly from the plots in Figs. 5.14 and 5.15, whereas the values obtained for the spectral displacement must be multiplied by the ductility ratio in order to obtain the correct value for the spectral displacement.

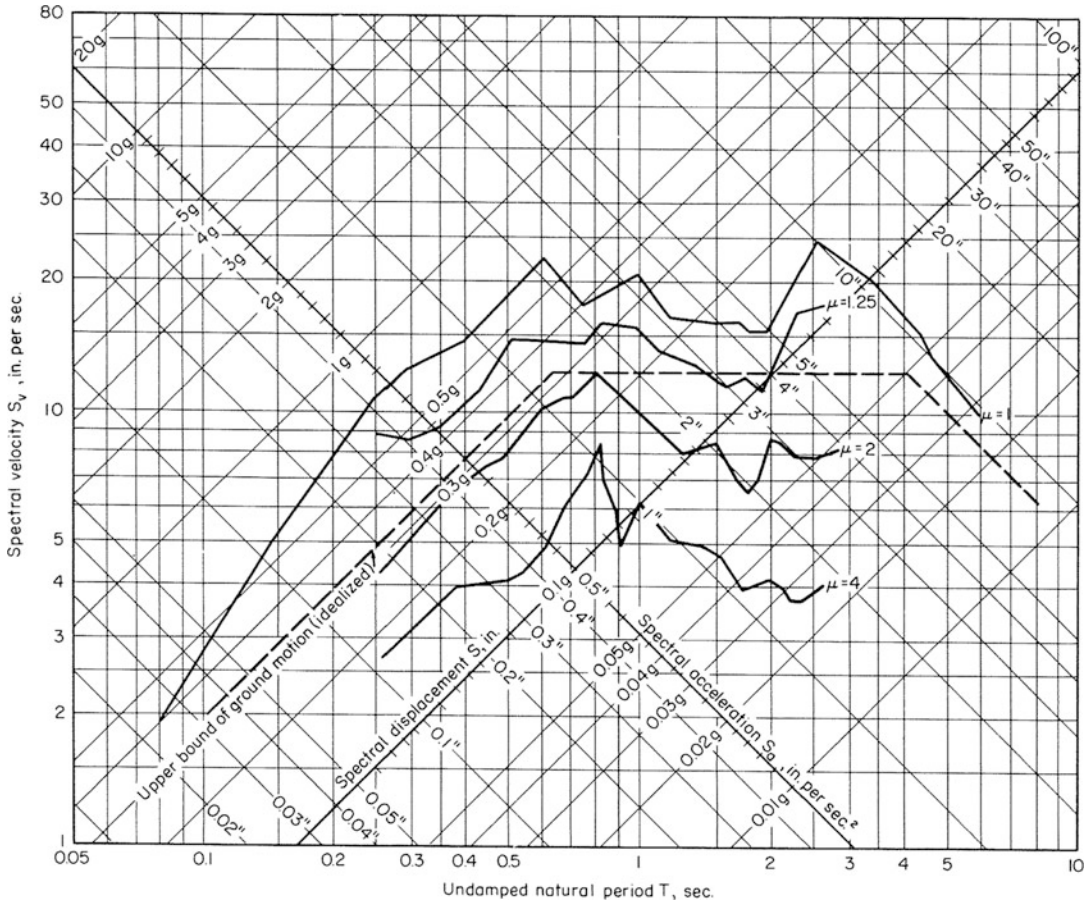

Fig. 5.15 Response spectra for elastoplastic system with 10% critical damping for the 1940 El Centro earthquake. (From Blume et al. 1961)

The concept of ductility ratio has been associated mainly with steel structures. These structures have a load-deflection curve that is often approximated as the elastoplastic curve shown in Fig. 5.13b. For other types of structures, such as reinforced concrete structures or masonry shear walls, still, conveniently, the load-deflection curve is modeled as the elastoplastic curve. Although for steel structures, ductility factors as high as 6 are often used in collapse-level earthquake design, lower values for the ductility ratio are applicable to reinforced concrete or masonry shear walls. The selection of ductility values for seismic design must also be based on the design objectives and the loading criteria as well as the risk level acceptable for the structure as it relates to its use.

For reinforced concrete structures or masonry walls, a ductility factor of 1.0–1.5 seems appropriate for earthquake design where the objective is to limit damage. In other words, the objective of limit damage requires that structural members should be designed to undergo little if any yielding. When the design objective is to prevent collapse of the structure, ductilities of 2–3 are appropriate in this case.

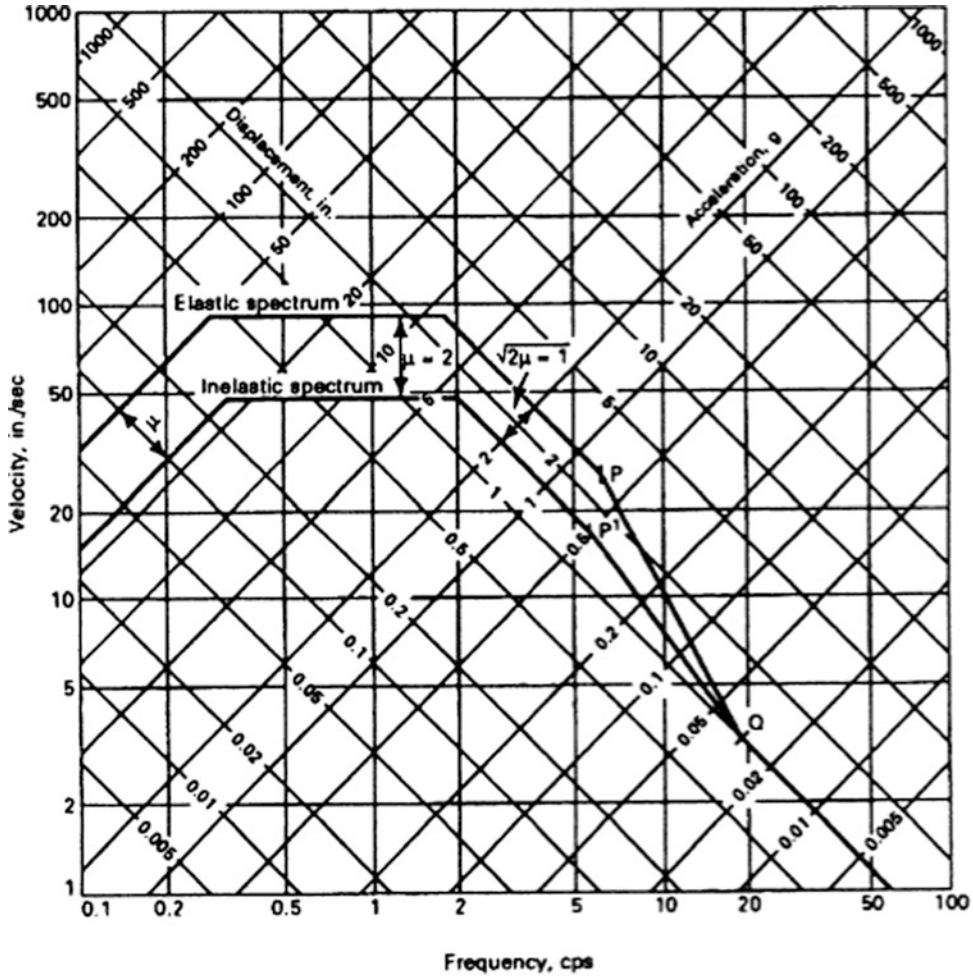

Fig. 5.16 Inelastic design spectrum normalized to peak ground acceleration of 1.0 *g* for 5% damping and ductility ratio $\mu = 2.0$. (NOTE: Chart gives directly S_v and S_a; however, displacement values obtained from the chart should be multiplied by μ to obtain S_D)

5.7 Response Spectra for Inelastic Design

In the preceding section of this chapter, we discussed the procedure for calculating the seismic response spectra for elastic design. Figure 5.9 shows the elastic design spectra for several values of the damping ratio. The same procedure of constructing of a basic response spectrum that consolidates the "average" effect of several earthquake records may also be applied to design in the inelastic range. The spectra for elastoplastic systems have the same appearance as the spectra for elastic systems, but the curves are displaced downward by an amount that is related to the ductility ratio μ. Figure 5.16 shows the construction of a typical design spectrum currently recommended (Newmark and Hall 1973) for use when inelastic action is anticipated.

The elastic spectrum for design from Fig. 5.9 corresponding to the desired damping ratio is copied in a tripartite logarithmic paper as shown in Fig. 5.16 for the spectra corresponding to 5% damping. Then lines reduced by the specified ductility ratio are drawn parallel to elastic spectral lines in the displacement region (the left region) and in the velocity region (the central region). However, in the acceleration region (the right region) the recommended reducing factor is $\sqrt{2\mu - 1}$. This last line is extended up to a frequency of about 6 cps (point P' in Fig. 5.16). Then the inelastic design spectrum is

completed by drawing a line from this last point P' to point Q, where the descending line from point P of the elastic spectrum intersects the line of constant acceleration as shown in Fig. 5.16.

The development of the reduction factors in the displacement and velocity regions is explained with the aid of Fig. 5.17a. This figure shows the force-displacement curves for elastic and for elastoplastic behavior. At equal maximum displacement u_{max} for the two curves, we obtain from Fig. 5.17a the following relationship:

$$\mu = \frac{u_{\max}}{u_y} = \frac{F_E}{F_y}$$

or

$$F_y = \frac{F_E}{\mu} \tag{5.26}$$

where F_E is the force corresponding to the maximum displacement u_{max} in the elastic curve and F_y is the force at the yield condition Eq. (5.26) then shows that the force and consequently the acceleration in the elastoplastic system is equal to the corresponding value in the elastic system reduced by the ductility ratio. Therefore, the spectral acceleration S_a for elastoplastic behavior is related to the elastic spectral acceleration S_{aE} by

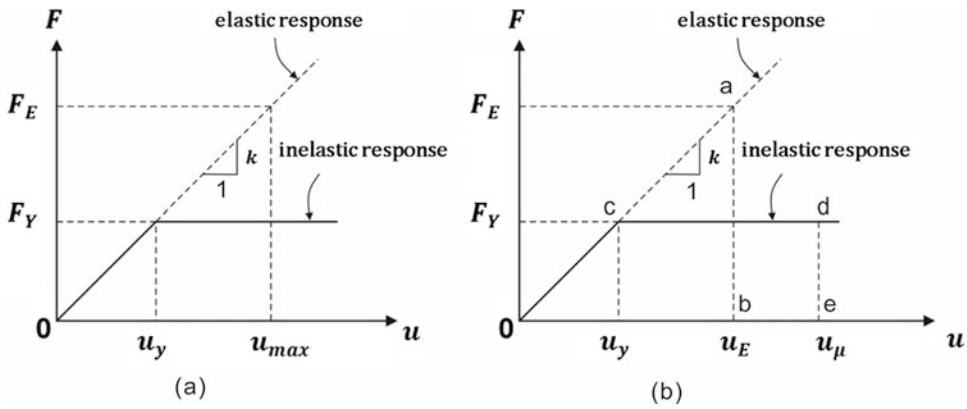

Fig. 5.17 Elastic and inelastic force-displacement curves. (**a**) Displacement and velocity regions. (**b**) Acceleration region

$$S_a = \frac{S_{aE}}{\mu} \tag{5.27}$$

However, in the acceleration region of the response spectrum, a ductility reduction factor μ does not result in close agreement with experimental data as does the recommended reduction factor $\sqrt{2\mu - 1}$. This factor may be rationally obtained by establishing the equivalence of the energy between the elastic and the inelastic system. In reference to Fig. 5.17b this equivalence is established by equating the area under elastic curve "*Oab*" with the area under the inelastic curve "*Ocde*." Namely,

$$\frac{F_E u_E}{2} = \frac{F_y u_y}{2} + F_y\left(u_\mu - u_y\right) \tag{5.28}$$

where F_E and F_y are the elastic and inelastic forces corresponding respectively to the maximum elastic displacement u_E and to the maximum inelastic displacement u_μ and where u_y is the yield displacement.

The substitution of $u_E = F_E/k$, $u_y = F_y/k$, and $u_\mu = \mu\, u_y$ into Eq. 5.28 gives

$$\frac{F_E^2}{2k} = \frac{F_y^2}{2k}(2\mu - 1)$$

or

$$\frac{F_E}{F_y} = \sqrt{2\mu - 1}$$

Thus,

$$F_y = \frac{F_E}{\sqrt{2\mu - 1}}$$

and, consequently

$$S_a = \frac{S_{aE}}{\sqrt{2\mu - 1}}$$

Thus, the inelastic spectral acceleration in the acceleration region is obtained by reducing the elastic spectral by the factor $\sqrt{2\mu - 1}$.

The inelastic response spectrum thus constructed and shown in Fig. 5.16 gives directly the values for the spectral acceleration S_a and spectral velocity S_v. However, the values read from this chart in a displacement scale must be multiplied by the ductility ratio μ to obtain the spectral displacement S_D. Inelastic response spectral charts for design developed by the procedure explained are presented in

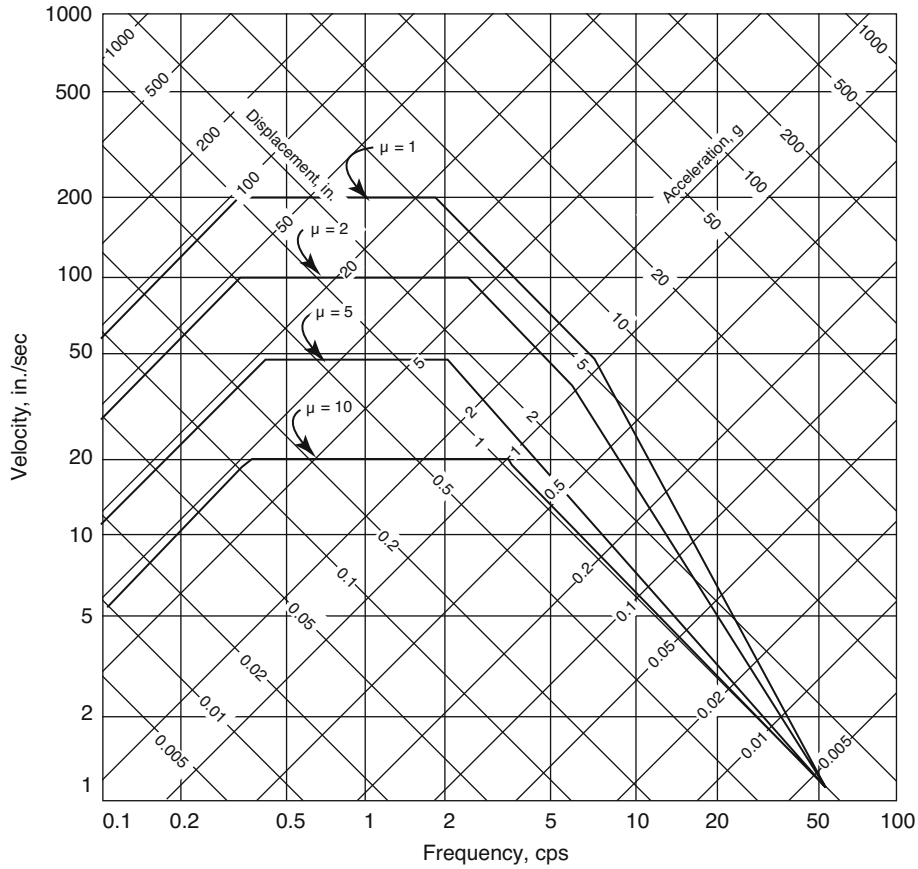

Fig. 5.18 Undamped inelastic design spectra normalized to peak ground acceleration of 1.0 g. (Spectral values S_a for acceleration and S_v for velocity are obtained directly from the graph; however, values of S_D for spectral displacement should be amplified by the ductility ratio μ)

Figs. 5.18, 5.19, and 5.20 correspondingly to damping factors $\xi = 0$, 5%, and 10% and for ductility ratios $\mu = 1$, 2, 5, and 10.

Illustrative Example 5.2

Calculate the response of the single-degree-of-freedom system of Example 5.1, assuming that the structure is designed to withstand seismic motions with an elastoplastic behavior having a ductility ratio $\mu. = 4.0$. Assume damping equal to 10% of the critical damping. (a) Use the response spectra of the El Centro earthquake and (b) Use the design response spectra.

Solution:

(a) From the response spectrum corresponding to 10% of the critical damping (Fig. 5.15), we read for $T = 1$ sec and the curve labeled $\mu = 4.0$

$$S_D = 1.0 \times 4.0 = 4.0 \text{ in}, \quad S_v = 6.2 \text{ in/ sec} \quad \text{and} \quad S_a = 0.1\ g.$$

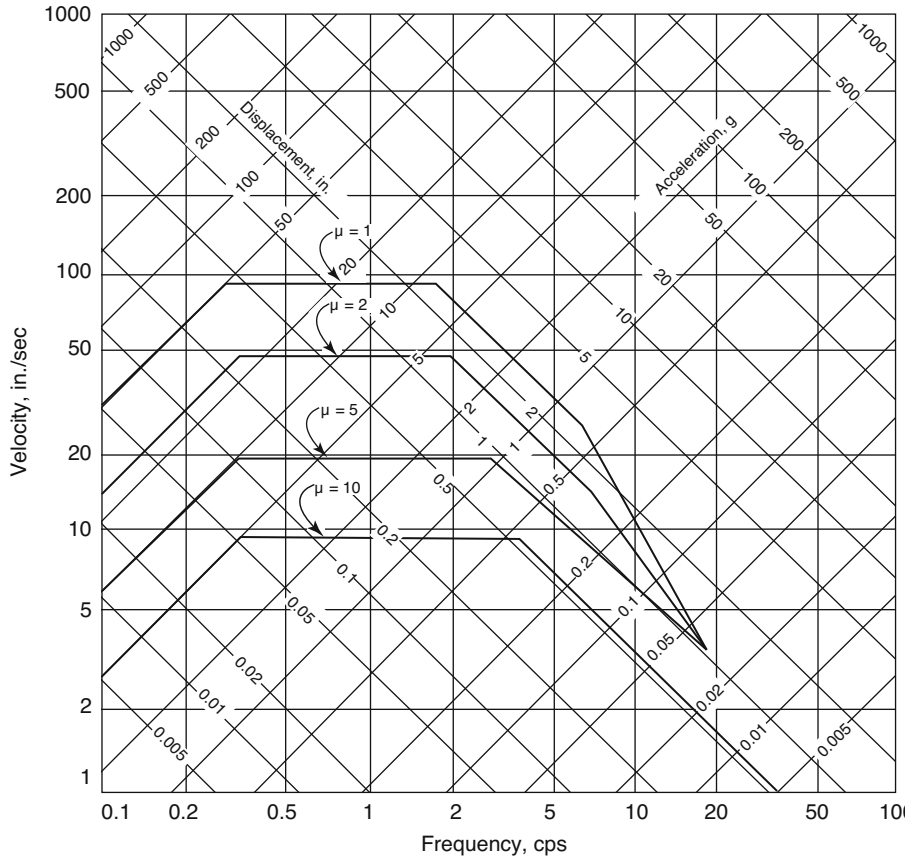

Fig. 5.19 Inelastic design spectra normalized to peak ground acceleration of 1.0 g for 5% damping. (Spectral values S_a for acceleration and S_v for velocity are directly obtained from the graph. However, values of S_D for spectral displacement should be amplified by the ductility ratio μ)

The factor 4.0 is required in the calculation of S_D since as previously noted the spectra plotted in Fig. 5.15 are correct for acceleration and for pseudovelocity, but for displacements it is necessary to amplify the values read from the chart by the ductility ratio.

(b) Using the inelastic design spectra with 10% damping for design in Fig. 5.20, corresponding to 1 cps, we obtain the following maximum values for the response:

$$S_D = 3.0 \times 0.32 \times 4.0 = 3.84 \text{ in}$$
$$S_v = 15.6 \times 0.32 = 5.00 \text{ in/ sec}$$
$$S_a = 0.3 \times 0.32 = 0.096\ g$$

As it can be seen, these spectral values based on the design spectrum are somewhat different than those obtained from the response spectrum of the El Centro earthquake of 1940. Also, if we compare these results for the elastoplastic behavior with the results in Example 5.1 for the elastic structure, we observe that the maximum relative displacement has essentially the same magnitude whereas the acceleration and the relative pseudovelocity are appreciably less. This observation is in general true for any structure when inelastic response is compared with the response based on elastic behavior.

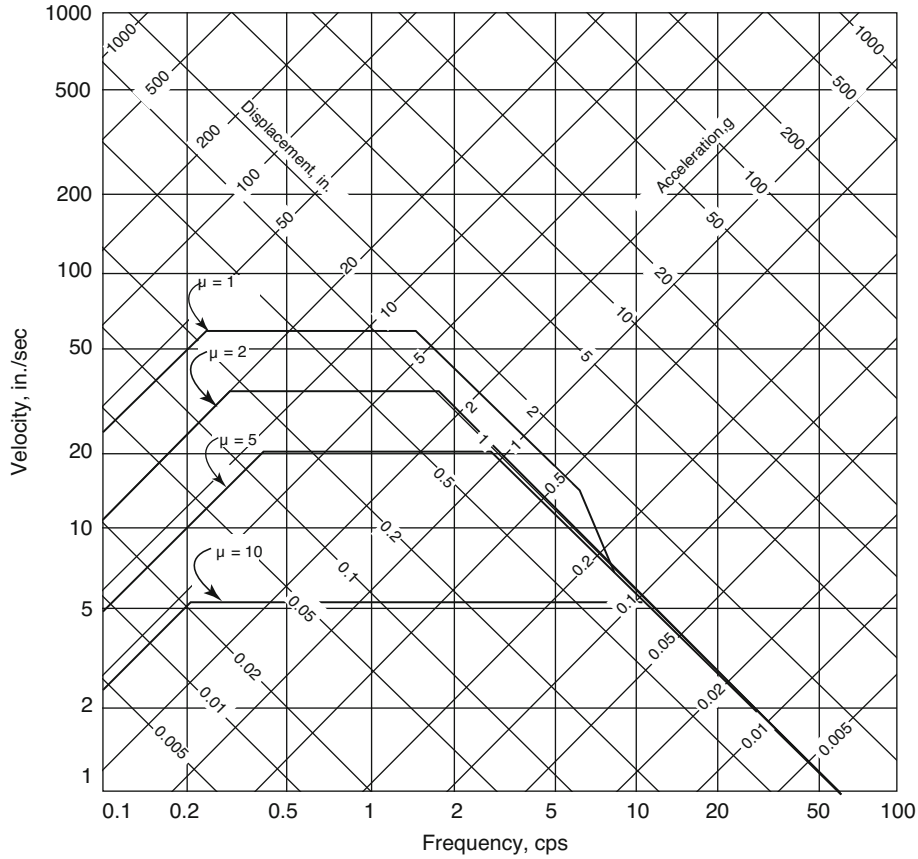

Fig. 5.20 Inelastic design spectra normalized to peak ground acceleration of 1.0 g' for 10% damping. (Spectral values S_a for acceleration and S_v for velocity are directly obtained from the graph; however, values of S_D for spectral displacement should be amplified by the ductility ratio μ)

5.8 Seismic Response Spectra Using MATLAB

The computer program described in this chapter serves to calculate elastic response spectra in terms of spectral displacement (maximum relative displacement), spectral velocity (maximum relative pseudovelocity), and spectral acceleration (maximum absolute acceleration) for any prescribed

time-acceleration-seismic excitation. The response is calculated in the specified range of frequencies or period using the Newmark beta's method in Chap. 6.

```
clear
close all
clc

%%%%-GIVEN VALUES-%%%%
load elcentro_NS.dat.txt     %elcentro_NS.dat_Table5_3.txt % ;
t = elcentro_NS_dat(:,1);    %elcentro_NS_dat_Table5_3(:,1);
Ag = elcentro_NS_dat(:,2);   %elcentro_NS_dat_Table5_3(:,2);
g = 386;                     %Acceleration of gravity
Dt = t(2)-t(1);              %Time interval
m=1;                         %Mass (unitless)

T(1,1)=0.00;                 %T=0
tt= length(t);               %Number of row @ t

xi = 0.1;                    %Damping ratio
endp =25;                    %Last value of T in x axes

%%%-ESTIMATION-%%%

for j=1:round(endp/Dt)

omega(j,1)=2*pi/T(j);
k =omega(j)^2*m;             %Stiffness
c=2*m*omega(j)*xi;           %Damping coefficient

%%%%Linear acceleration method (Newmark beta method[Ch.6])
gamma =1/2;                  %Parameter gamma
beta = 1/6;                  %Parameter beta

u(1)=0;                      %initial condition; Displ.
v(1)=0;                      %initial condition; Velocity
a(1)=0;                      %initial condition; Acceleration

kbar = k +gamma*c/(beta*Dt)+m/(beta*Dt*Dt);     %Eq.6.45
A = m/(beta*Dt)+gamma*c/beta;                   %A in DFbar = DF + A*v0+B*a0 (Eq. 6.46)
B = m/(2*beta)+Dt*c*((0.5*gamma/beta)-1);       %B in DFbar = DF + A*v0+B*a0 (Eq. 6.46)


u0=u;                        %setting initial values for loop(u)
v0=v;                        %setting initial values for loop(v)
a0=a;                        %setting initial values for loop(a)

%%%% Loop over each time step

for i = 1:(tt-1)
DF =-g.*(Ag(i+1)-Ag(i));

    [t,u,v,a] = Newmark(t, A, B, DF, Dt, kbar, u0, v0, a0, gamma, beta);

u_t1(:,i) = u(:,1)';
v_t1(:,i) = v(:,1)';
a_t1(:,i) = a(:,1)';

%%%Setting initial values for next loop (displacement, velopcity,
%%%acceleration and time)
u0 = u;
v0 = v;
a0 = a;
t=t;
end

%%%Estimation maximum values at the given period
umax = max(abs(u_t1));       %Maximum displacement
vmax = max(abs(v_t1));       %Maximum velocity
amax = max(abs(a_t1));       %Maximum acceleration
```

```
%%%Estimation
Sd(j,1)=umax;                          %Spectral displacement
Spv(j,1)=Sd(j)*omega(j);               %Spectral velocity
Spa(j,1)=Sd(j)*(omega(j))^2/g*g;       %Spectral acceleration
T(j+1,1)=T(j,1)+Dt;
end

Sd(1:2,1)=0; Spv(1:2,1)=0;Spa(1:2,1)=max(abs(Ag))/g*g;
Ag(end)=[];
T(end)=[];

f = 1./T;

figure(1)
subplot(3,1,1),
plot (T,Spa');
title('(a) Spectral acceleration (\xi=0.1)')
%Create xlabel
xlabel ('Period(sec)');
%Create ylabel
ylabel ('S_a');

subplot(3,1,2),
plot (T, Spv');
title('(b) Spectral velocity (\xi=0.1)')
%Create xlabel
xlabel ('Period(sec)');
%Create ylabel
ylabel ('S_v');

subplot(3,1,3),
plot (T, Sd');
title('(c) Spectral displacement (\xi=0.1)')
%Create xlabel
xlabel ('Period(sec)');
%Create ylabel
ylabel ('S_d');
```

The function of Newmark beta method in MATLAB is presented below.

```
function  [t,u,v,a] = Newmark(t, A, B, DF, Dt, kbar, u0, v0, a0, gamma, beta)

    DFbar = DF + A*v0+B*a0;                                              %Eq.6.46
    Du = DFbar/kbar;                                                     %Eq.6.44
    Dudot = gamma*Du/(beta*Dt)-gamma*v0/beta+ Dt*a0*(1-0.5*gamma/beta); %Eq.6.43
    Dudotdot = Du/(beta*Dt*Dt)-v0/(beta*Dt)-a0/(2*beta);                 %Eq.6.42

    u=u0+Du;
    v=v0+Dudot;
    a=a0+Dudotdot;
    t=t+Dt;
```

Illustrative Example 5.3

Use MATLAB program to develop the response spectra for elastic systems subjected to the first 10 sec of the 1940 El Centro earthquake. Assume 10% damping.

The digitized values corresponding to the accelerations recorded for the first 10 sec of the El Centro earthquake are given in Table 5.3.

Table 5.3 Digitized Values of the Acceleration Recorded for the First 10 sec of the N-S component for the El Centro Earthquake of 1940

Time (sec)	Acc. (acc. *g*')	Time (sec)	Acc. (acc. *g*')	Time (sec)	Acc. (acc. *g*')	Time (sec)	Acc. (acc. *g*')
0.0000	0.0108	0.0420	0.0020	0.0970	0.0159	0.1610	−0.0001
0.2210	0.0189	0.2630	0.0001	0.2910	0.0059	0.3320	−0.0012
0.3740	0.0200	0.4290	−0.0237	0.4710	0.0076	0.5810	0.0425
0.6230	0.0094	0.6650	0.0138	0.7200	−0.0088	0.7400	−0.0256
0.7890	−0.0387	0.8290	−0.0568	0.8720	−0.0232	0.9020	−0.0343
0.9410	−0.0402	0.9610	−0.0603	0.9970	−0.0789	1.0660	−0.0666
1.0760	−0.0381	1.0940	−0.0429	1.1680	0.0897	1.3150	−0.1696
1.3840	−0.0828	1.4120	−0.0828	1.4400	−0.0945	1.4810	−0.0885
1.5090	−0.1080	1.5370	−0.1280	1.6280	0.1144	1.7030	0.2355
1.8550	0.1428	1.8800	0.1777	1.9240	−0.2610	2.0070	−0.3194
2.2150	0.2952	2.2700	0.2634	2.3200	−0.2984	2.3950	0.0054
2.4500	0.2865	2.5190	−0.0469	2.5750	0.1516	2.6520	0.2077
2.7080	0.1087	2.7690	−0.0325	2.8930	0.1033	2.9760	−0.0803
3.0680	0.0520	3.1290	−0.1547	3.2120	0.0065	3.2530	−0.2060
3.3860	0.1927	3.4190	−0.0937	3.5300	0.1708	3.5990	−0.0359
3.6680	0.0365	3.7380	−0.0736	3.8350	0.0311	3.9040	−0.1833
4.0140	0.0227	4.0560	−0.0435	4.1060	0.0216	4.2220	−0.1972
4.3140	−0.1762	4.4160	0.1460	4.4710	−0.0047	4.6180	0.2572
4.6650	−0.2045	4.7560	0.0608	4.8310	−0.2733	4.9700	0.1779
5.0390	0.0301	5.1080	0.2183	5.1990	0.0267	5.2330	0.1252
5.3020	0.1290	5.3300	0.1089	5.3430	−0.0239	5.4540	0.1723
5.5100	−0.1021	5.6060	0.0141	5.6900	−0.1949	5.7730	−0.2420
5.8000	−0.0050	5.8090	−0.0275	5.8690	−0.0573	5.8830	−0.0327
5.9250	0.0216	5.9800	0.0108	6.0130	0.0235	6.0850	−0.0665
6.1320	0.0014	6.1740	0.0493	6.1880	0.0149	6.1980	−0.0200
6.2290	−0.0381	6.2790	0.0207	6.3260	−0.0058	6.3680	−0.0603
6.3820	−0.0162	6.4090	0.0200	6.4590	−0.1760	6.4780	−0.0033
6.5200	0.0043	6.5340	−0.0040	6.5620	−0.0099	6.5750	−0.0017
6.6030	−0.0170	6.6450	0.0373	6.6860	0.0457	6.7140	0.0385
6.7280	0.0009	6.7490	−0.0288	6.7690	0.0016	6.8110	0.0113
6.8520	0.0022	6.9080	0.0092	6.9910	−0.0996	7.0740	0.0360
7.1210	0.0078	7.1430	−0.0277	7.1490	0.0026	7.1710	0.0272
7.2260	0.0576	7.2950	−0.0492	7.3700	0.0297	7.4060	0.0109
7.4250	0.0186	7.4610	−0.2530	7.5250	−0.0347	7.5720	0.0036
7.6000	−0.0628	7.6410	−0.0280	7.6690	−0.0196	7.6910	0.0068
7.7520	−0.0054	7.7940	−0.0603	7.8350	−0.0357	7.8770	−0.0716
7.9600	−0.0140	7.9870	−0.0056	8.0010	0.0222	8.0700	0.0468
8.1260	0.0260	8.1660	−0.0335	8.1950	−0.0128	8.2230	0.0661
8.2780	0.0305	8.3340	0.0246	8.4030	0.0347	8.4580	−0.0369
8.5330	−0.0344	8.5960	−0.0104	8.6380	−0.0260	8.7350	0.1534
8.8180	−0.0028	8.8600	0.0233	8.8820	−0.0261	8.9150	−0.0022
8.9560	−0.1849	9.0530	0.1260	9.0950	0.0320	9.1230	0.0955
9.1500	0.1246	9.2530	−0.0328	9.2890	−0.0451	9.4270	0.1301
9.4410	−0.1657	9.5100	0.0419	9.6350	−0.0936	9.7040	0.0816
9.8150	−0.0881	9.8980	0.0064	9.9390	−0.0006	9.9950	0.0586
10.0200	−0.0713	10.0500	−0.0448	10.0800	−0.0221	10.1000	0.0093
10.1500	0.0024	10.1900	0.0510				

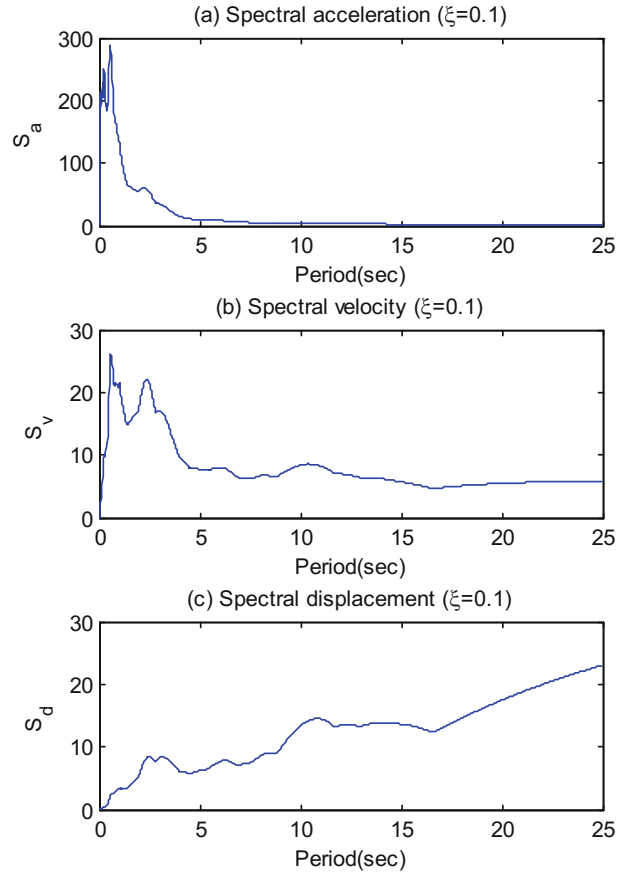

Fig. 5.21 Spectrum chart (**a**) Sa, (**b**) Sv, (**c**) Sd

5.9 Summary

Response spectra are plots that give the maximum response for a single-degree-of-freedom system subjected to a specified excitation. The construction of these plots requires the solution of single-degree-of-freedom systems for a sequence of values of the natural frequency and of the damping ratio in the range of interest. Every solution provides only one point (the maximum value) of the response spectrum. In solving the single-degree-of-freedom systems, use is made of Duhamel's integral or of the direct method (Chap. 4) for elastic systems and of the step-by-step linear acceleration method for inelastic behavior (Chap. 6). Since a large number of systems must be analyzed to fully plot each response spectrum, the task is lengthy and time consuming even with the use of a computer. However, once these curves are constructed and are available for the excitation of interest, the analysis for the design of structures subjected to dynamic loading is reduced to a simple calculation of the natural frequency of the system and the use of the response spectrum.

In the chapters of Part II dealing with structures that are modeled as systems with many degrees of freedom, it will be shown that the dynamic analysis of a system with n degrees of freedom can be transformed to the problem of solving n systems in which each one is a single-degree-of-freedom system. Consequently, this transformation extends the usefulness of response spectra for single-degree-of-freedom systems to the solution of systems of any number of degrees of freedom.

The reader should thus realize the full importance of a thorough understanding and mastery of the concepts and methods of solutions for single-degree-of-freedom systems, since these same methods are also applicable to systems of many degrees of freedom after the problem has been transformed to independent single-degree-of-freedom systems.

5.10 Problems

Problem 5.1
The steel frame shown in Fig. P5.1 is subjected to horizontal force at the girder level of (1000 sin $10t$ lb) for a time duration of half a cycle of the forcing sine function. Use the appropriate response spectral chart to obtain the maximum displacement. Neglect damping.

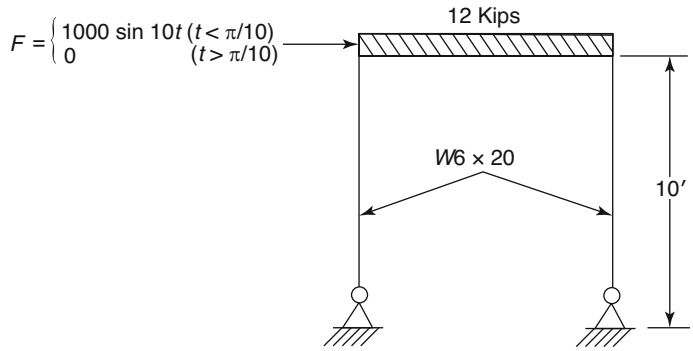

Fig. P5.1

Problem 5.2
Determine the maximum stresses in the columns of the frame of Problem 5.1.

Problem 5.3
Consider the frame shown in Fig. P5.3a subjected to a foundation excitation produced by a half cycle of the function $a_g = 200 \sin 10t$ in/sec^2 as shown in Fig. P5.3b. Determine the maximum horizontal displacement of the girder relative to the motion of the foundation. Neglect damping.

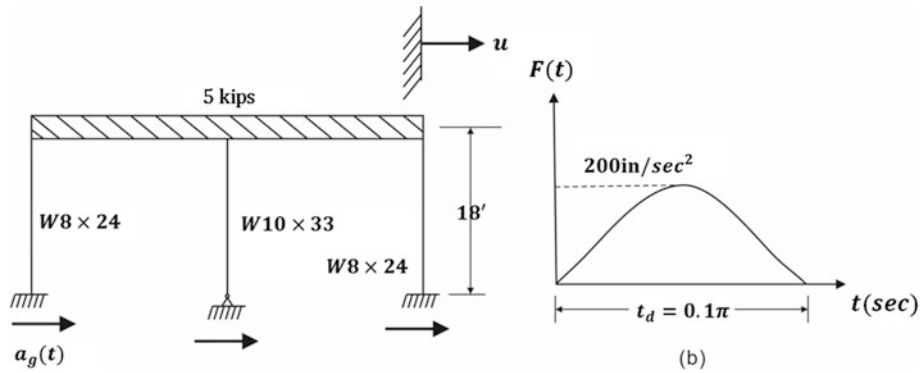

Fig. P5.3

Problem 5.4
Determine the maximum stress in the columns of the frame of Problem 5.3.

Problem 5.5
The frame shown in Fig. P5.1 is subjected to the excitation produced by the El Centro earthquake of 1940. Assume 10% damping and from the appropriate chart determine the spectral values for displacement, velocity, and acceleration. Assume elastic behavior.

Problem 5.6
Repeat Problem 5.5 using the basic design spectra given in Fig. 5.9 to determine the spectral values for acceleration, velocity, and displacement. (Scale down spectral values by factor 0.32, the estimated peak ground acceleration).

Problem 5.7
A structure modeled as the spring-mass system shown in Fig. P5.7 is assumed to be subjected to a support motion produced by the El Centro earthquake of 1940. Assuming elastic behavior and using the appropriate response spectral chart find the maximum relative displacement between the mass and the support. Also compute the maximum force acting on the spring. Neglect damping.

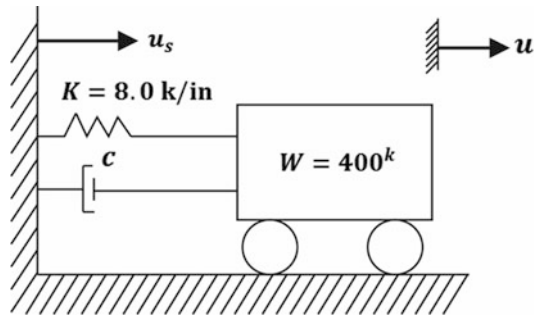

Fig. P5.7

Problem 5.8
Repeat Problem 5.7 assuming that the system has 10% of the critical damping.

Problem 5.9
Determine the force transmitted to the foundation for the system analyzed in Problem 5.8.

Problem 5.10
Consider the spring-mass system of Problem 5.7 and assume that the spring element follows, an elastoplastic behavior with a maximum value for the restoring force in tension or in compression is equal to half the value of the elastic maximum force in the spring calculated in Problem 5.7. Determine the spectral value for the displacement. Neglect damping. (Hint: Start by assuming $\mu = 2$, find spectral value S_D, calculate μ and and use this calculated value of μ to determine new spectral values, etc.)

Problem 5.11
Repeat Problem 5.10 for 10% damping.

Problem 5.12
A structure modeled as a single-degree-of-freedom system has a natural period $T = 0.5$ sec. Use the response spectral method to determine in the elastic range the maximum absolute acceleration, the maximum relative displacement, and the maximum pseudovelocity for: (a) a foundation motion equal to the El Centro earthquake of 1940 and (b) the design spectrum with a maximum ground acceleration equal to 0.3 g. Neglect damping.

Problem 5.13
Solve Problem 5.12 assuming elastoplastic behavior of the system with ductility ratio $\mu = 4$.

Problem 5.14
Use Program 6 to develop a table having the spectral values for displacements, velocities, and accelerations in the range of frequency from 0.10 cps to 1.0 cps. The excitation is a constant acceleration of magnitude 0.01 g applied for 10 sec. Neglect damping.

Problem 5.15
Use Program 6 to develop the response spectra for elastic systems subjected to the first 10 s of the 1940 El Centro earthquake. Neglect damping. The digitized values corresponding to the accelerations recorded for the first 10 sec of the El Centro earthquake are given in Table 5.3.

Problem 5.16
Repeat Problem 5.15 corresponding to a system with 10% of the critical damping.

Check for updates

Nonlinear Structural Response

6

In discussing the dynamic behavior of single-degree-of-freedom systems, we assumed that in the model representing the structure, the restoring force was proportional to the displacement. We also assumed the dissipation of energy through a viscous damping mechanism in which the damping force was proportional to the velocity. In addition, the mass in the model was always considered to be unchanging with time. As a consequence of these assumptions, the equation of motion for such a system resulted in a linear, second order ordinary differential equation with constant coefficients, namely,

$$m\ddot{u} + c\dot{u} + ku = F(t) \tag{6.1}$$

In the previous chapters it was illustrated that for particular forcing functions such as harmonic functions, it was relatively simple to solve this Eq. (6.1) and that a general solution always existed in terms of Duhamel's integral. Equation (6.1) thus represents the dynamic behavior of many structures modeled as a single-degree-of-freedom system. There are, however, physical situations for which this linear model does not adequately represent the dynamic characteristics of the structure. The analysis in such cases requires the introduction of a model in which the spring force or the damping force may not remain proportional, respectively, to the displacement or to the velocity. Consequently, the resulting equation of motion will no longer be linear and its mathematical solution, in general, will have a much greater complexity, often requiring a numerical procedure for its integration.

6.1 Nonlinear Single-Degree-of-Freedom Model

Figure 6.1a shows the model for a single-degree-of-freedom system and in Fig. 6.1b the corresponding free body diagram. The dynamic equilibrium in the system is established by equating to zero the sum of the inertial force $F_I(t)$, the damping force $F_D(t)$ the spring force $F_s(t)$, and the external force $F(t)$. Hence, at time t, the equilibrium of these forces is expressed as

$$F_I(t) + F_D(t) + F_s(t) = F(t) \tag{6.2}$$

Considering the case that in this equation, the mass is constant, $F_I(t) = m\ddot{u}$; that the damping force is proportional to the velocity with the damping coefficient also constant, $F_D(t) = c\dot{u}$; and the resisting force or spring force is a function of the displacement, $F_s(t) = F_s(u)$, we may then express Eq. (6.2) as

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M. Paz, Y. H. Kim, *Structural Dynamics*, https://doi.org/10.1007/978-3-319-94743-3_6

$$m\ddot{u} + c\dot{u} + F_s(u) = F(t) \tag{6.3}$$

Hence, at time t_i, using the notation $u_i = u(t_i)$, $\dot{u}_i = \dot{u}(t_i)$ and $\ddot{u}_i = \ddot{u}(t_i)$,

$$m\ddot{u}_i + c\dot{u}_i + F_s(u_i) = F(t_i) \tag{6.4}$$

and at shoet time later, $t_{i+1} = t_i + \Delta t$

$$m\ddot{u}_{i+1} + c\dot{u}_{i+1} + F_s(u_{i+1}) = F(t_{i+1}) \tag{6.5}$$

Subtracting Eq. (6.4) from Eq. (6.5) results in the difference equation of motion in terms of increments, namely

$$m\Delta\ddot{u}_i + c\Delta\dot{u}_i + \Delta F_s(u_i) = \Delta F_i \tag{6.6}$$

Furthermore, we assume that the incremental resisting or spring force is proportional to the incremental displacement, that is $\Delta F_s(u_i) = k_i \Delta u_i$,

$$m\Delta\ddot{u}_i + c\Delta\dot{u}_i + k_i\Delta u_i = \Delta F_i \tag{6.7}$$

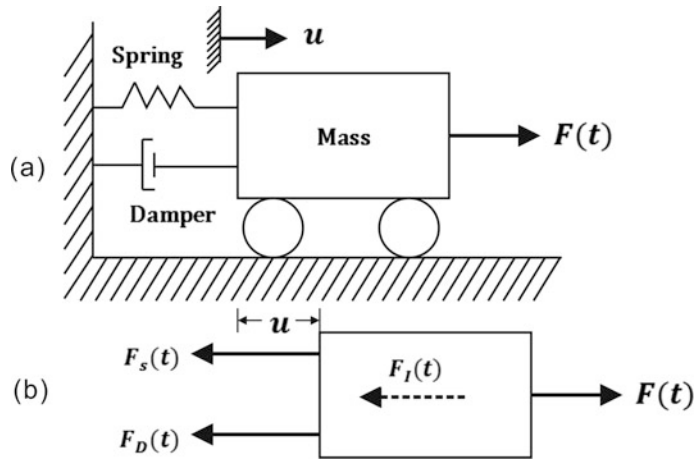

Fig. 6.1 (**a**) Model for a single-degree-of-freedom system. (**b**) Free body diagram showing the inertial force, the damping force, the spring force, and the external force

The coefficient k_i, is defined as the current evaluation for the resisting force per unit displacement (stiffness coefficient) which may be taken as the slope of the tangent of the force-displacement function, at the initiation of the time step Δt for the interval Δt or as the slope of the secant line as shown in Fig. 6.2 for the plot of the resisting force $F_s(u)$. The value of the coefficient k_i, is calculated at a displacement corresponding to time t, and assumed to remain constant during the increment of time Δt. Since, in general, this coefficient does not remain constant during that time increment, Eq. (6.7) is an approximate equation.

The incremental displacement Δu_i incremental velocity $\Delta\dot{u}_i$ and incremental acceleration $\Delta\ddot{u}$ are given by

$$\Delta u_i = u(t_i + \Delta t) - u(t_i) \tag{6.8}$$

$$\Delta\dot{u}_i = \dot{u}(t_i + \Delta t) - \dot{u}(t_i) \tag{6.9}$$

$$\Delta\ddot{u}_i = \ddot{u}(t_i + \Delta t) - \ddot{u}(t_i) \tag{6.10}$$

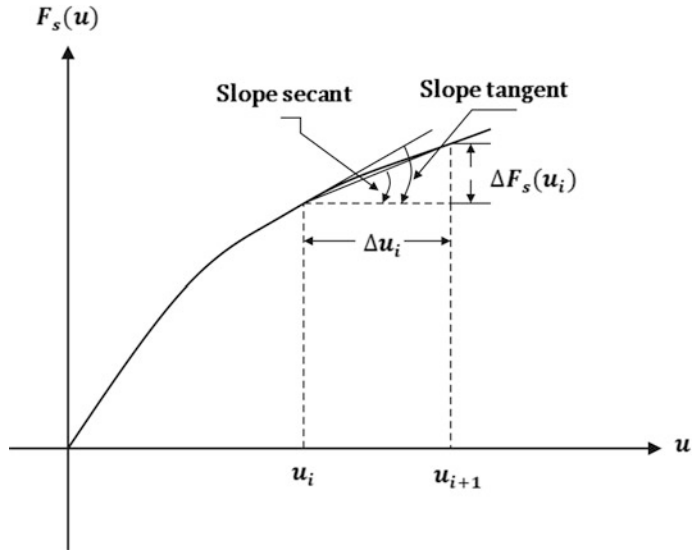

Fig. 6.2 Nonlinear stiffness function, spring force v.s. displacement

6.2 Integration of the Nonlinear Equation of Motion

Among the many methods available for the solution of the nonlinear equation of motion, probably one of the most effective is the step-by-step integration method. In this method, the response is evaluated at successive increments Δt of time, usually taken of equal lengths of time for computational convenience. At the beginning of each interval, the condition of dynamic equilibrium is established. Then, the response for a time increment Δt is evaluated approximately on the basis that the coefficients $k(u)$ and $c(\dot{u})$ remain constant during the interval Δt. The nonlinear characteristics of these coefficients are considered in the analysis by reevaluating these coefficients at the beginning of each time increment. The response is then obtained using the displacement and velocity calculated at the end of the time interval as the initial conditions for the next time step.

As we have said for each time interval, the stiffness coefficient $k(u)$ and the damping coefficient $c(\dot{u})$ are evaluated at the initiation of the interval but are assumed to remain constant until the next step; thus the nonlinear behavior of the system is approximated by a sequence of successively changing linear systems. It should also be obvious that the assumption of constant mass is unnecessary; it could just as well also be represented by a variable coefficient.

There are many procedures available for performing the step-by-step integration of Eq. (6.12). Two of the most popular methods are the constant acceleration method and the linear acceleration method. As the names of these methods imply, in the first method the acceleration is assumed to remain constant during the time interval Δt, while in the second method, the acceleration is assumed to vary linearly during the interval. As may be expected, the constant acceleration method is simpler but less accurate when compared with the linear acceleration method for the same value of the time increment. We shall present here in detail both methods.

6.3 Constant Acceleration Method

In the constant acceleration method, it is assumes that acceleration remains constant for the time step between times t_i and $t_{i+1} = t_i + \Delta t$ as shown in Fig. 6.3. The value of the constant acceleration during the interval Δt is taken as the average of the values of the acceleration $\ddot{u}_i$ at the initiation of the time step and $\ddot{u}_{i+1}$, the acceleration at the end of the time step. Thus, the acceleration $\ddot{u}(t)$ at any time t during the time interval Δt is given by

$$\ddot{u}(t) = \frac{1}{2}\left(\ddot{u}_i + \ddot{u}_{i+1}\right) \tag{6.11}$$

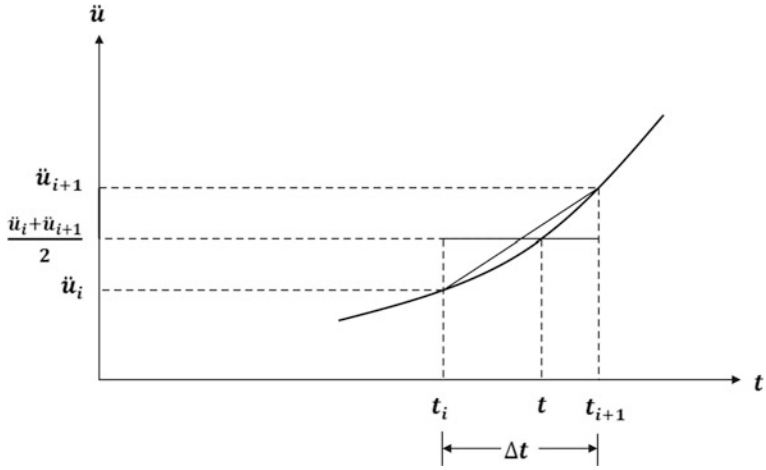

Fig. 6.3 Constant acceleration assumed during time interval Δt

Integrating this equation twice with respect to the time between the limits t_i and t results in

$$\dot{u}(t) = \dot{u}_i + \frac{1}{2}\left(\ddot{u}_i + \ddot{u}_{i+1}\right)(t - t_i) \tag{6.12}$$

and

$$u(t) = u_i + \dot{u}_i(t - t_i) + \frac{1}{4}\left(\ddot{u}_i + \ddot{u}_{i+1}\right)(t - t_i)^2 \tag{6.13}$$

The evaluation of Eqs. (6.12) and (6.13) at time $t_{i+1} = t_i + \Delta t$ gives

$$\Delta\dot{u}_i = \frac{\Delta t}{2}\left(\ddot{u}_i + \ddot{u}_{i+1}\right) \tag{6.14}$$

and

$$\Delta u_i = \dot{u}_i \Delta t + \frac{\Delta t^2}{2}\left(\ddot{u}_i + \ddot{u}_{i+1}\right) \tag{6.15}$$

where, Δu_i and, $\Delta\dot{u}_i$ are respectively the incremental displacement and incremental velocity defined by Eqs. (6.8) and (6.9).

To use the incremental displacement in the analysis, Eq. (6.15) is solved for $\ddot{u}_{i+1}$ and substituted into Eq. (6.14) to obtain:

$$\ddot{u}_{i+1} = \frac{4}{\Delta t^2}\Delta u_i - \frac{4}{\Delta t}\dot{u}_i - \ddot{u}_i \tag{6.16}$$

and

$$\Delta\dot{u}_i = \frac{2}{\Delta t}\Delta u_i - 2\dot{u}_i \tag{6.17}$$

Now subtracting $\ddot{u}_i$ on both sides of Eq. (6.16) results in

$$\Delta\ddot{u}_i = \frac{4}{\Delta t^2}\Delta u - \frac{4}{\Delta t}\dot{u}_i - 2\ddot{u}_i \tag{6.18}$$

The substitution into Eq. (6.7) of $\Delta\dot{u}_i$ and $\Delta\ddot{u}_i$, respectively, from Eqs. (6.17) and (6.18) gives

$$m\left(\frac{4}{\Delta t^2}\Delta u_i - \frac{4}{\Delta t}\dot{u}_i - 2\ddot{u}_i\right) + c\left(\frac{2}{\Delta t}\Delta u - 2\dot{u}_i\right) + k_i\Delta u_i = \Delta F_i \tag{6.19}$$

Equation (6.19) is then solved for the incremental displacement Δu_i to obtainin

$$\Delta u_i = \frac{\Delta\bar{F}_i}{\bar{k}_i} \tag{6.20}$$

in which the effective stiffness $\bar{k}_i$ is

$$\bar{k}_i = \frac{4m}{\Delta t^2} + \frac{2c}{\Delta t} + k_i \tag{6.21}$$

and the effective incremental force $\Delta\bar{F}_i$ *is*

$$\Delta\bar{F}_i = \Delta F_i + \frac{4m}{\Delta t}\dot{u}_i + 2m\Delta\ddot{u}_i + 2c\dot{u}_i \tag{6.22}$$

The displacement $u_{i+1} = u\,(t_i + \Delta t)$ at time $t_{i+1} = t_i + \Delta t$ is obtained from Eq. (6.8) after solving for incremental displacement Δu_i in Eq. (6.20). The incremental velocity is calculated by Eq. (6.17) and the velocity at time $t_{i+1} = t_i + \Delta t$ from Eq. (6.9) as

$$\dot{u}_{i+1} = \dot{u}_i + \Delta\dot{u}_i \tag{6.23}$$

Finally, the acceleration $\ddot{u}_{i+1}$ at the end of the time step, $t_{i+1} = t_i + \Delta t$, is obtained directly from the differential equation of motion, Eq. (6.3), rather than using Eq. (6.16), Hence, from Eq. (6.3):

$$\ddot{u}_{i+1} = \frac{1}{m}\left[F(t_{i+1}) - c\dot{u}_{i+1} - F_s(u_{i+1})\right] \tag{6.24}$$

in which $F_s(u_{i+1})$ is the restoring force evaluated at time $t_{i+1} = t_i + \Delta t$

After the displacement, velocity, and acceleration have been determined at time $t_{i+1} = t_i + \Delta t$, the outlined procedure is repeated to calculate these quantities at the following time step $t_{i+2} = t_{i+1} + \Delta$t, and the process is continued to any desired final value of time.

6.4 Linear Acceleration Step-by-Step Method

In the linear acceleration method, it is assumed that the acceleration may be expressed by a linear function of time during the time interval Δt. Let t_i and $t_{i+1} = t_i + \Delta t$ be, respectively, the designation for the time at the beginning and at the end of the time interval Δt. In this type of analysis, the material properties of the system c_i and k_i may include any form of nonlinearity. Thus it is not necessary for the spring force to be only a function of displacement or for the damping force to be specified only as a function of velocity. The only restriction in the analysis is that we evaluate these coefficients at an instant of time t_i and then assume that they remain constant during the increment of time Δt. When the acceleration is assumed to be a linear function of time for the interval of time t_i or $t_{i+1} = t_i + \Delta t$ as depicted in Fig. 6.4, we may express the acceleration as

$$\ddot{u}(t) = \ddot{u}_i + \frac{\Delta \ddot{u}_i}{\Delta t}(t - t_i) \tag{6.25}$$

where $\Delta \ddot{u}_i$ is given by Eq. (6.10). Integrating Eq. (6.25) twice with respect to time between the limits t_i and t yields

$$\dot{u}(t) = \dot{u}_i + \ddot{u}_i(t - t_i) + \frac{1}{2}\frac{\Delta \ddot{u}_i}{\Delta t}(t - t_i)^2 \tag{6.26}$$

and

$$u(t) = u_i + \dot{u}_i(t - t_i) + \frac{1}{2}\ddot{u}_i(t - t_i)^2 + \frac{1}{6}\frac{\Delta \ddot{u}_i}{\Delta t}(t - t_i)^3 \tag{6.27}$$

The evaluation of Eqs. (6.26) and (6.27) at time $t = t_i + \Delta t$ gives

$$\Delta \dot{u}_i = \ddot{u}_i \Delta t + \frac{1}{2}\Delta \ddot{u}_i \Delta t \tag{6.28}$$

and

$$\Delta u_i = \dot{u}_i \Delta t + \frac{1}{2}\ddot{u}_i \Delta t^2 + \frac{1}{6}\Delta \ddot{u}_i + \Delta t^3 \tag{6.29}$$

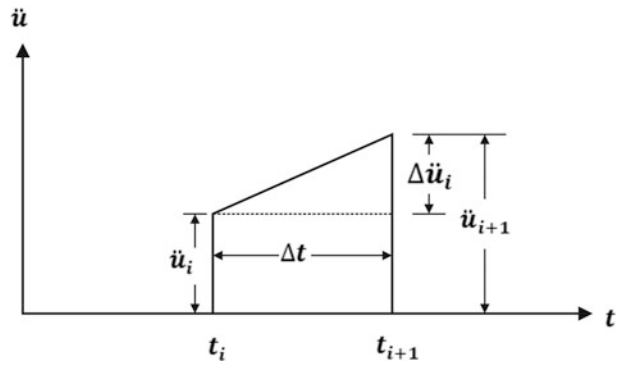

Fig. 6.4 Assumed linear variation of the acceleration during a time interval

where Δu_i and $\Delta \dot{u}_i$ are defined in Eqs. (6.8) and (6.9), respectively. Now, to use the incremental displacement Δu as the basic variable in the analysis, Eq. (6.29) is solved for the incremental acceleration $\Delta \ddot{u}_i$ and then substituted into Eq. (6.28) to obtain

$$\Delta \ddot{u}_i = \frac{6}{\Delta t^2}\Delta u_i - \frac{6}{\Delta t}\dot{u}_i - 3\ddot{u}_i \tag{6.30}$$

and

$$\Delta \dot{u}_i = \frac{3}{\Delta t}\Delta u_i - 3\dot{u}_i - \frac{\Delta t}{2}\ddot{u}_i \tag{6.31}$$

The substitution of Eqs. (6.30) and (6.31) into Eq. (6.7) leads to the following form of the equation of motion:

$$m\left\{\frac{6}{\Delta t^2}\Delta u_i - \frac{6}{\Delta t}\dot{u}_i - 3\ddot{u}_i\right\} + c_i\left\{\frac{3}{\Delta t}\Delta u_i - 3\dot{u}_i - \frac{\Delta t}{2}\ddot{u}_i\right\} + k_i\Delta u = \Delta F_i \tag{6.32}$$

Finally, transferring in Eq. (6.32) all the terms containing the unknown incremental displacement Δu_i to the left-hand side gives

$$\bar{k}_i \Delta u_i = \Delta \bar{F}_i \tag{6.33}$$

in which $\bar{k}_i$ is the effective spring constant, given by

$$\bar{k}_i = k_i + \frac{6m}{\Delta t^2} + \frac{3c_i}{\Delta t} \tag{6.34}$$

and $\Delta \bar{F}_i$ is the effective incremental force, expressed by

$$\Delta \bar{F}_i = \Delta F_i + m\left\{\frac{6}{\Delta t}\dot{u}_1 + 3\ddot{u}_i\right\} + c_i\left\{3\dot{u}_i + \frac{\Delta t}{2}\ddot{u}_i\right\} \tag{6.35}$$

It should be noted that Eq. (6.33) is equivalent to the static incremental equilibrium equation, and may be solved for the incremental displacement by simply dividing the effective incremental force $\Delta \bar{F}_i$ by the effective spring constant $\bar{k}_i$, that is,

$$\Delta u_i = \frac{\Delta \bar{F}_i}{\bar{k}_i} \tag{6.36}$$

To obtain the displacement $u_{i+1} = u\,(t_i + \Delta t)$ at time $t_{i+1} = t_i + \Delta t$, this value of Δu_i is substituted into Eq. (6.8) yielding

$$u_{i+1} = u_i + \Delta u_i \tag{6.37}$$

Then the incremental velocity $\Delta \dot{u}_i$ is obtained from Eq. (6.31) and the velocity at time $t_{i+1} = t_i\,\Delta + t$ from Eq. (6.9) as

$$\dot{u}_{i+1} = \dot{u}_i + \Delta \dot{u}_i \tag{6.38}$$

Finally, the acceleration $\dot{u}_{i+1}$ at the end of the time step is obtained directly from the differential equation of motion, Eq. (6.2), where the equation is written for time $t_{i+1} = t_i + \Delta t$. Hence, after setting $F_1 = m\ddot{u}_{i+1}$ in Eq. (6.2), it follows that

$$\ddot{u}_{i+1} = \frac{1}{m}\{F(t_{i+1}) - F_D(t_{i+1}) - F_s(t_{i+1})\} \tag{6.39}$$

where that damping force $F_D\ (t_{i\ +\ 1})$.and the spring force $F_S\ (t_{i\ +\ 1})$ are now evaluated at time $t_{i\ +\ 1} = t_i + \Delta t$.

After the displacement, velocity, and acceleration have been determined at time $t_{i\ +\ 1} = t_i + \Delta t$, the outlined procedure is repeated to calculate these quantities at the following time step $t_{i\ +\ 2} = t_{i\ +\ 1} + \Delta t$, and the process is continued to any desired final value of time. The reader should, however, realize that this numerical procedure involves two significant approximations: (1) the acceleration is assumed to vary linearly during the time increment Δt; and (2) the damping and stiffness properties of the system are evaluated at the initiation of each time increment and assumed to remain constant during the time interval. In general, these two assumptions introduce errors that are small if the time step is short. However, these errors generally might tend to accumulate from step to step. This accumulation of errors should be avoided by imposing a total dynamic equilibrium condition at each step in the analysis. This is accomplished by expressing the acceleration at each step using the differential equation of motion in which the displacement and velocity as well as the stiffness and damping forces are evaluated at that time step.

There still remains the problem of the selection of the proper time increment Δt. As in any numerical method, the accuracy of the step-by-step integration method depends upon the magnitude of the time increment selected. The following factors should be considered in the selection of Δt: (1) the natural period of the structure; (2) the rate of variation of the loading function; and (3) the complexity of the stiffness and damping functions.

In general, it has been found that sufficiently accurate results can be obtained if the time interval is taken to be no longer than one-tenth of the natural period of the structure. The second consideration is that the interval should be small enough to represent properly the variation of the load with respect to time. The third point that should be considered is any abrupt variation in the rate of change of the stiffness or damping function. For example, in the usual assumption of elastoplastic materials, the stiffness suddenly changes from linear elastic to a yielding plastic phase. In this case, to obtain the best accuracy, it would be desirable to select smaller time steps in the neighborhood of such drastic changes.

6.5 The Newmark: β Method

The Newmark-β Method includes, in its formulation, several time-step methods used for the solution of linear or nonlinear equations. It uses a numerical parameter designated as β. The method, as originally proposed by Newmark (1959), contained in addition to β, a second parameter γ. Particular numerical values for these parameters leads to well-known methods for the solution of the differential equation of motion, the constant acceleration method, and the linear acceleration method.

The Newmark equation can be written in incremental quantities for a constant time step Δt, as

$$\Delta \dot{u}_i = \ddot{u}_i \Delta t + \gamma \Delta \ddot{u}_i \Delta t \tag{6.40}$$

and

$$\Delta u_i = \dot{u}_i \Delta t + \frac{1}{2}\ddot{u}_i \Delta t^2 + \beta \Delta \ddot{u}_i \Delta t^2 \tag{6.41}$$

in which the incremental displacement Δu_i and incremental velocity $\Delta \dot{u}_i$ are defined, respectively, by Eqs. (6.8) and (6.9).

It has been found that for values of γ different than 1/2, the method introduced a superfluous damping in this system. For this reason, this parameter is generally set as $\gamma = 1/2$. The solution of Eq. (6.41) for $\Delta \ddot{u}_i$ and its subsequent substitution into Eq. (6.40) after setting $\gamma = 1/2$ yield

$$\Delta \ddot{u}_i = \frac{1}{\beta \Delta t^2} \Delta u_i - \frac{1}{\beta \Delta t} \dot{u}_i - \frac{1}{2\beta} \ddot{u}_i \tag{6.42}$$

and

$$\Delta \dot{u}_i = \frac{1}{2\beta \Delta t} \Delta u_i - \frac{1}{2\beta} \dot{u}_i + \left(1 - \frac{1}{4\beta}\right) \Delta t \cdot \ddot{u}_i \tag{6.43}$$

Now, the substitution of Eqs. (6.42) and (6.43) into the incremental equation of motion, Eq. (6.7), results in an equation to calculate the incremental displacement Δu_i, namely,

$$\bar{k}_i \, \Delta u_i = \Delta \bar{F}_i \tag{6.44}$$

where the effective stiffness $\bar{k}_i$, and the effective incremental force $\Delta \bar{F}_i$ are given respectively by

$$\bar{k}_i = k_i + \frac{m}{\beta \Delta t^2} + \frac{c_i}{2\beta \Delta t} \tag{6.45}$$

and

$$\Delta \bar{F}_i = \Delta F_i + \frac{m}{\beta \Delta t} \dot{u}_i + \frac{c_i}{2\beta} \ddot{u}_i + \frac{m}{2\beta} \ddot{u}_i - c_i \Delta t \left(1 - \frac{1}{4\beta}\right) \ddot{u}_i \tag{6.46}$$

In these equations, c_i and k_i are respectively the damping and stiffness coefficients evaluated at the initial time t_i of the time step $\Delta t = t_{i+1} - t_i$

In the implementation of the Newmark Beta Method, a numerical value for the parameter β is selected. Newmark suggested a value in the range $1/6 \leq \beta \leq 1/2$. For $\beta = 1/4$, the method corresponds to the constant acceleration method and for $\beta = 1/6$ to the linear acceleration method.

Illustrative Example 6.1: Response by Newmark-β Method Using MATLAB

There are two methods to obtain solution depending on the selection of gamma and beta in the equation.

Using the linear variation method of Newmark-β approach, solve the Problem 4.2

Solution:

From Fig. 4.11, *we have the following data:*

Mass	$m = W/g = (38.6 \times 1000)/386 = 100$ *(lb · sec2/in)*
Spring constant	$k = 100 \times 1000 = 100{,}000$ *(lb · /in)*
Damping coefficient	$c = 2\xi\sqrt{km} = 1265$ *(lb · sec/in)*
Natural period	$T = 2\pi\sqrt{m/k} = 0.20$ *sec*
Select time step for integration	$\Delta t = T/10 = 0.02$ *sec*

```
close all
clear all
clc

%%%%-GIVEN VALUES-%%%%

t=0:0.02:0.1;
Dt = t(2)-t(1);                  %Time interval

m=38.6/386;                      %Mass
k =100;                          %Stiffness
xi =0.2;                         %Damping ratio
omega = sqrt(k/m);               %Natural frequency
c=2*m*omega*xi;                  %Damping coefficient

%%%%Linear acceleration method (Newmark beta method[Ch.6])

gamma =1/2;                      %Parameter gamma
beta = 1/6;                      %Parameter beta
tt= length(t);

%%%%Define the Forcing Function, F, and using Matlab function diff, an
%%%%array containing changes in F during each time step

for i= 1:tt
    if t(i)<=0.02
        F(i) = 120*t(i)/0.02;
    elseif t(i) <=0.04
        F(i) =120;
    else
        F(i)=max(0, -120*(t(i)-0.06)/0.02);
    end
end
DF1 =diff(F)                                %Delta F

%%%Initial calculation

u(1)=0;                                     %Initial condition; Displ.
v(1)=0;                                     %Initial condition; Velocity
a(1)=(F(1)-c*v(1)-k*u(1))/m;
kbar = k +gamma*c/(beta*Dt)+m/(beta*Dt*Dt);      %Eq.6.45
A = m/(beta*Dt)+gamma*c/beta;                    %A in DFbar = DF + A*v0+B*a0 (Eq. 6.46)
B = m/(2*beta)+Dt*c*((0.5*gamma/beta)-1);        %B in DFbar = DF + A*v0+B*a0 (Eq. 6.46)

%%%Setting up for initial value of Loop over

u0=u;
v0=v;
a0=a;
t =t(1);

%%%%Iteration for each time step using Newmark beta method

fori = 1:(tt-1)
DF=DF1(i);

[t,u,v,a] = Newmark( t, A, B, DF, Dt, kbar, u0, v0, a0, gamma, beta);

ti(:,i)=t(:,1)';
u_t1(i,:) = u(1,:)';
v_t1(:,i) = v(1,:)';
a_t1(:,i) = a(1,:)';

u0 = u;
v0 = v;
a0 = a;
t = t;
end

%%%Maximum values

umax = max(u_t1)
vmax = max(v_t1)
amax = max(a_t1)

%%%Plot response

plot (ti, u_t1);
xlabel ('t(sec)');
ylabel ('Displacement(in.)');
```

The function of Newmark-β method in MATLAB is presented below (Fig. 6.5 and 6.6).

```
function  [t,u,v,a] = Newmark(t, A, B, DF, Dt, kbar, u0, v0, a0, gamma, beta)

    DFbar = DF + A*v0+B*a0;                                                    %Eq.6.46
    Du = DFbar/kbar;                                                           %Eq.6.44
    Dudot = gamma*Du/(beta*Dt)-gamma*v0/beta+ Dt*a0*(1-0.5*gamma/beta);        %Eq.6.43
    Dudotdot = Du/(beta*Dt*Dt)-v0/(beta*Dt)-a0/(2*beta);                       %Eq.6.42

    u=u0+Du;
    v=v0+Dudot;
    a=a0+Dudotdot;
    t=t+Dt;
```

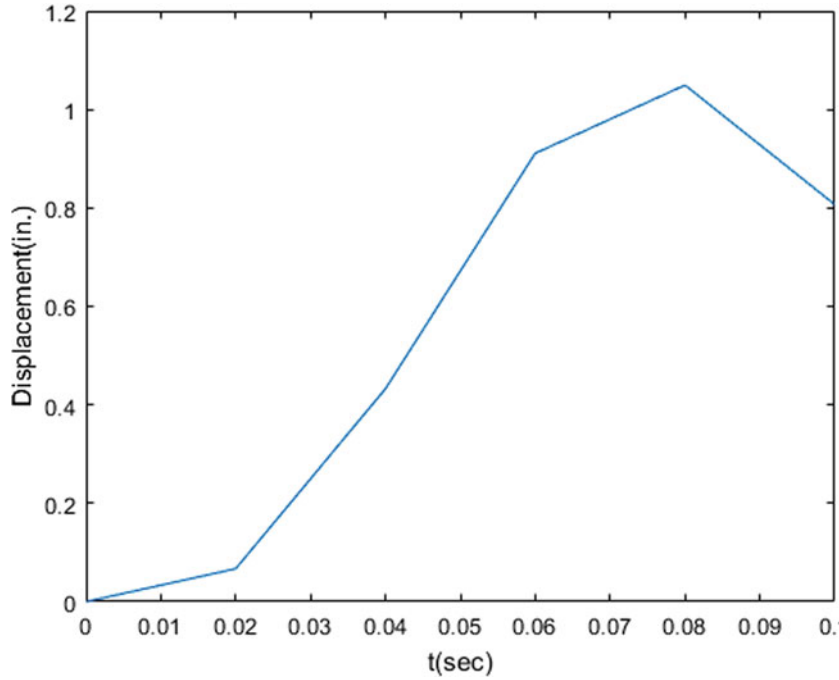

Fig. 6.5 Calculation of the response for Illustrative Example 6.1 ($\Delta t = 0.02$ sec)

The following table records the results provided by the MATLAB and their comparison with the Table 4.1 in Example 4.4:

Time (sec)	Direct integration (Table 4.1)	$\Delta t = 0.02$ sec		$\Delta t = 0.01$ sec		$\Delta t = 0.005$ sec	
		Displ.		Displ.		Displ.	
		u(t) (in.)	% error	u(t) (in.)	% error	u(t) (in.)	% error
0.000	0	0	0	0	0	0	0
0.020	0.074	0.067	−9.5	0.072	−2.7	0.073	−1.4
0.040	0.451	0.433	−4.0	0.446	−1.1	0.45	−0.2
0.060	0.926	0.911	−1.6	0.922	−0.4	0.925	−0.1
0.080	1.044	1.049	0.5	1.045	0.1	1.044	0.0
0.100	0.778	0.807	3.7	0.785	0.9	0.78	0.3

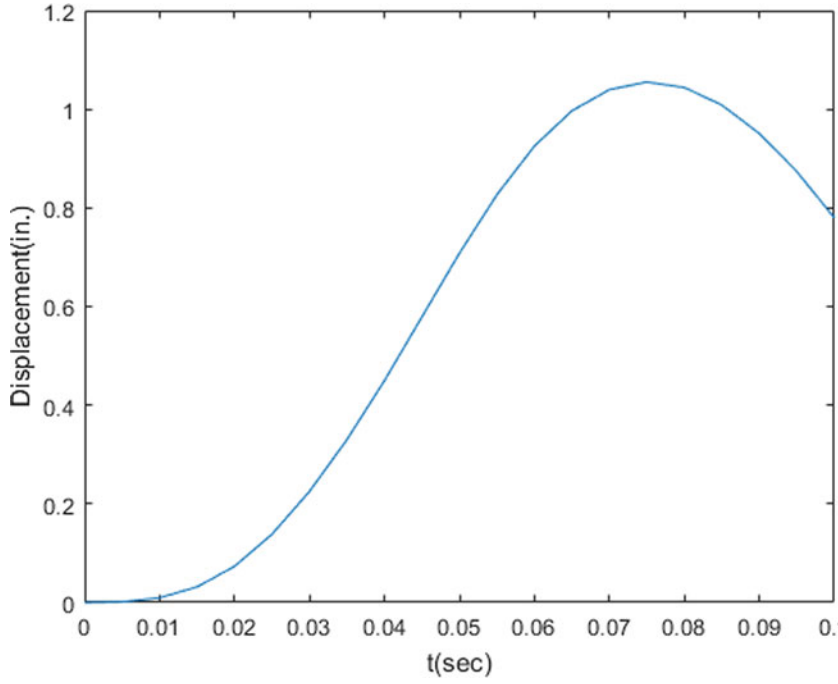

Fig. 6.6 Calculation of the response for Illustrative Example 6.1 ($\Delta t = 0.005$ sec)

Illustrative Example 6.2

(a) Determine the dynamic response of the tower shown in Fig. 6.7 *subjected to the sinusoidal force* $F(t) = F_0 \sin \varpi t$ *applied at its top for 0.30 sec. (b) Check results using the exact solution which in this case is available in closed form. Neglect damping.*

Solution:

(a) *The following data is obtained from* Fig. 6.7*:*

Mass	$m = w/g = (38.6 \times 1000)/386 = 100\ (lb \cdot sec2/in)$
Spring constant	$k = 100 \cdot 1000 = 100{,}000\ (lb/in)$
Natural frequency	$\omega = \sqrt{k/m} = 31.623\ rad/sec$
Natural period	$T = 2\pi/\omega = 0.20\ sec$
Select time step	$\Delta T = 0.01\ sec$

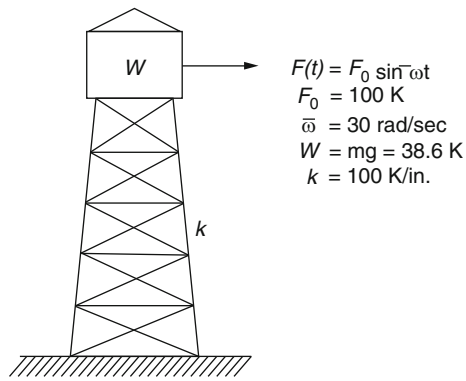

Fig. 6.7 Tower for Illustrative Example 6.2

(b) *The exact solution for the response of a simple oscillator to the sinusoidal force* $F_0 = sin\ \varpi\ t$, *with zero initial displacement and velocity, from* Eq. (3.8) *is*

$$u(t) = \frac{F_0}{k - m\overline{\omega}^2}\left(\sin \overline{\omega}t - \frac{\overline{\omega}}{\omega}\sin \omega t\right)$$

where ω *is the natural frequency in rad/sec,* ϖ *the forced frequency also in rad/sec, and* F_0 *the amplitude of the sinusoidal force.*

Substituting corresponding numerical values for this example yields

$$\begin{aligned} u(t) &= \frac{100,000}{100,000 - 100(30)^2}\left(\sin 30t - \frac{30}{31.623t}\sin 31.623t\right) \\ &= 10(\sin 30t - 0.94868 \sin 31.623t) \end{aligned}$$

The velocity and acceleration functions are then given by

$$\dot{u}\,(t) = 300\cos 30t - 300\cos 31.623t$$

and

$$\ddot{u}(t) = -9000\sin 30t + 948.7\sin 31.623t$$

The evaluation of the response at specific values of time results in the following table:

t (sec)	u(t) (in)	$\dot{u}$ (t) (in/sec)	$\ddot{u}$(t) (in/sec^2)
0.1	1.6076	2.9379	−1466.51
0.2	−3.1865	−11.6917	2907.53
0.3	4.7420	26.0822	−4298.04

Illustrative Example 6.3

Solve Example 4.5 *selecting the time step* Δt *equal to 0.02 and 0.005. Then compare the displacements at time* $t = 0.1, 0.2,$ *and 0.3 sec with the response obtained in Example* 4.5 *using the exact solution of the differential equation.*

Solution:

In solving the problem using MATLAB, the results from Newmark's method and ODE 45 are compared. The following table presented the summary of results:

		Newmark's method (Linear variation method)				ODE 45 method			
		$\Delta t = 0.02$ sec		$\Delta t = 0.005$ sec		$\Delta t = 0.02$ sec		$\Delta t = 0.005$ sec	
		Displ.		Displ.		Displ.		Displ.	
Time (sec)	Exact displ. (in.)	(in.)	% error	(in.)	% error	(in.)	% error	(in.)	% error
0.1	1.6076	1.547	−3.8	1.604	−0.2	1.608	0.0	1.608	0.0
0.2	−3.1865	−3.055	−4.1	−3.178	−0.3	−3.187	0.0	−3.188	0.0
0.3	4.742	4.487	−5.4	4.696	−1.0	4.71	−0.7	4.71	−0.7

Results shown in the above table are sufficiently close to corresponding values given by the MATLAB in part (a) of this problem (Fig. 6.8).

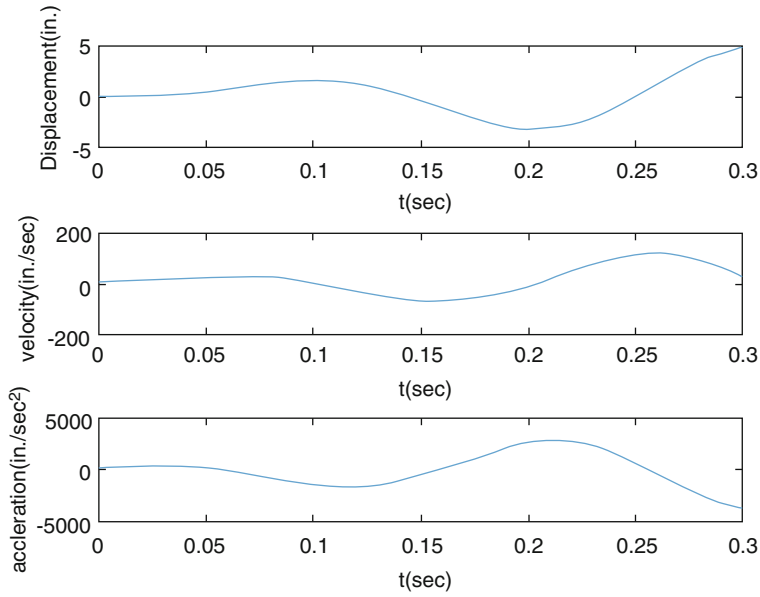

Fig. 6.8 Displacement, velocity, and acceleration for Illustrative Example 4.4 ($\Delta t = 0.02$ sec)

6.6 Elastoplastic Behavior

If any structure modeled as a single-degree-of-freedom system (spring-mass system) is allowed to yield plastically, then the restoring force exerted is likely to be of the form shown in Fig. 6.5a. There is a portion of the curve in which linear elastic behavior occurs, whereupon, for any further deformation, plastic yielding takes place. When the structure is unloaded, the behavior is again elastic until further reverse loading produces compressive plastic yielding. The structure may be subjected to cyclic loading and unloading in this manner. Energy is dissipated during each cycle by an amount that is proportional to the area under the curve (hysteresis loop) as indicated in Fig. 6.5a. This behavior is often simplified by assuming a definite yield point beyond which additional displacement takes place at a constant value for the restoring force without any further increase in the load. Such behavior is known as elastoplastic behavior; the corresponding force-displacement curve is shown in Fig. 6.5b.

For the structure modeled as a spring-mass system, expressions of the restoring force for a system with elastoplastic behavior are easily written (Fig. 6.9).

These expressions depend on the magnitude of the restoring force as well as upon whether the motion is such that the displacement is increasing $\left(\dot{u} > 0\right)$ or decreasing $\left(\dot{u} < 0\right)$. Referring to Fig. 6.5b in which a general elastoplastic cycle is represented, we assume that the initial conditions are zero ($u_0 = 0, \dot{u} = 0$) for the unloaded structure. Hence, initially, as the load is applied, the system behaves elastically along curve E_0. The displacement u_t at which plastic behavior in tension may be initiated, and the displacement u_c, at which plastic behavior in compression may be initiated, are calculated, respectively, from

$$u_t = R_t/k \tag{6.47}$$

and

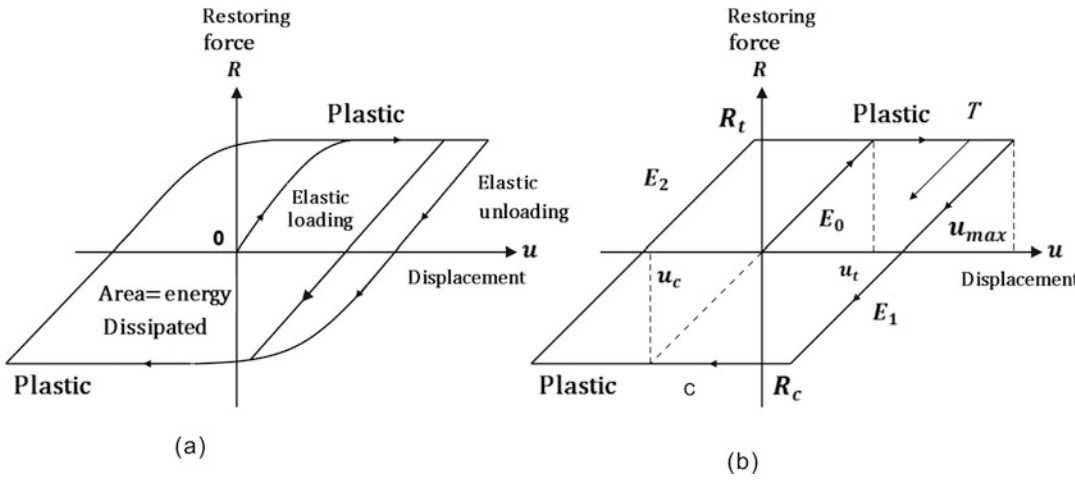

Fig. 6.9 Elastic-plastic structural models. (**a**) General plastic behavior. (**b**) Elastoplastic behavoir

$$u_c = R_c/k$$

where R_t and R_c are the respective values of the forces that produce yielding in tension and compression and k is the elastic stiffness of the structure. The system will remain on curve E_0 as long as the displacement u satisfies

$$u_c < u < u_t \tag{6.48}$$

If the displacement u increases to u_t the system begins to behave plastically in tension along curve T on Fig. 6.5b; it remains on curve T as long as the velocity $\dot{u} > 0$. When $\dot{u} < 0$, the system reverses to elastic behavior on a curve such as E_1 with new yielding points given by

$$u_t = u_{\max}$$

$$u_c = u_{\max} - (R_t - R_c)/k \tag{6.49}$$

in which u_{max} is the maximum displacement along curve T, which occurs when $\dot{u} = 0$.

Conversely, if u decreases to u_c the system begins a plastic behavior in compression along curve C and it remains on this curve as long as $\dot{u} < 0$. The system returns to an elastic behavior when the velocity again changes direction and $\dot{u} > 0$. In this case, the new yielding limits are given by

$$u_c = u_{\min}$$

$$u_t = u_{\min} + (R_t - R_c)/k \tag{6.50}$$

in which u_{min} is the minimum displacement along curve C, which occurs when $\dot{u} = 0$. The same condition given by Eq. (6.48) is valid for the system to remain operating along any elastic segment such as E_0, E_1, E_2,. . .as shown in Fig. 6.5b.

We are now interested in calculating the restoring force at each of the possible segments of the elastoplastic cycle. The restoring force on an elastic phase of the cycle (E_0. E_1. E_2,. . .) may be calculated as.

$$R = R_t - (u_t - u)k \tag{6.51}$$

on a plastic phase in tension as

$$R = R_t \tag{6.52}$$

and on the plastic compressive phase as

$$R = R_c \tag{6.53}$$

The algorithm for the step-by-step linear acceleration method of a single degree-of-freedom system assuming an elastoplastic behavior is outlined in the following section.

6.7 Algorithm for Step-by-Step Solution for Elastoplastic Single-Degree-of-Freedom System

Initialize and input data:

1. Input values for k, m, c, R_1, R_c, and a table giving the time t_i and magnitude of the excitation F_j.
2. Set $u_0 = 0$ and $\dot{u}_0 = 0$.
3. Calculate initial acceleration:

$$\ddot{u}_0 = \frac{F(t=0)}{m} \tag{6.54}$$

4. Select time step Δt and calculate constants:

$$a_1 = 3/\Delta t, a_2 = 6/\Delta t, a_3 = \Delta t/2, a_4 = 6/\Delta t^2$$

5. Calculate initial yield points:

$$u_t = R_t/k$$

$$u_c = R_c/k \tag{6.55}$$

For each time step:

1. Use the following code to establish the elastic or plastic state of the system:

$$\begin{aligned} &\text{KEY} = 0 \text{ (elastic behavior)} \\ &\text{KEY} = -1 \text{ (plastic behavior in compression)} \\ &\text{KEY} = 1 \text{ (plastic behavior in tension)} \end{aligned} \tag{6.56}$$

2. Calculate the displacement $\dot{u}$ and velocity u at the end of the time step and set the value of KEY according to the following conditions:

   (a) When the system is behaving elastically at the beginning of the time step and

$$\begin{aligned} u_c < u < u_t \quad & \text{KEY} = 0 \\ u > u_t \quad & \text{KEY} = 1 \\ u < u_c \quad & \text{KEY} = -1 \end{aligned}$$

(b) When the system is behaving plastically in tension at the beginning of the time step and

$$\begin{aligned} \dot{u} > 0 \ \text{KEY} = 1 \\ \dot{u} < 0 \ \text{KEY} = 0 \end{aligned}$$

(c) When the system is behaving plastically in compression at the beginning of the time step and

$$\begin{aligned} \dot{u} < 0 \ \text{KEY} = -1 \\ \dot{u} > 0 \ \text{KEY} = 0 \end{aligned}$$

3. Calculate the effective stiffness:

$$\bar{k}_i = k_p + a_4 m + a_1 c_i \tag{6.57}$$

where

$$\begin{aligned} k_p = k \quad & \text{forelasticbehavior}\,(\text{KEY} = 0) \\ k_p = 0 \quad & \text{forplasticbehavior}\,(\text{KEY} = 1\,\text{or} - 1) \end{aligned} \tag{6.58}$$

4. Calculate the incremental effective force:

$$\Delta\bar{F}_i = \Delta F_i + (a_2 m + 3c_i) + (3m + a_3 c_i)\ddot{u}_i \tag{6.59}$$

5. Solve for the incremental displacement:

$$\Delta u_i = \Delta\bar{F}_i / \bar{k}_i \tag{6.60}$$

6. Calculate the incremental velocity:

$$\Delta\dot{u}_i = a_1 \Delta u_i - 3\dot{u}_i - a_3 \ddot{u}_i \tag{6.61}$$

7. Calculate displacement and velocity at the end of time interval:

$$u_{i+1} = u_i + \Delta u_i \tag{6.62}$$

$$\dot{u}_{i+1} = \dot{u}_i + \Delta u_i \tag{6.63}$$

8. Calculate acceleration $\ddot{u}_{i+1}$ at the end of time interval using the dynamic equation of equilibrium:

$$\ddot{u}_{i+1} = \frac{1}{m}[F(t_{i+1}) - c_{i+1}\dot{u}_{i+1} - R] \tag{6.64}$$

in which

$$\begin{aligned} R = R_t - (u_i - u_{i+1})k \quad & \text{if KEY} = 0 \\ R = R_t \quad & \text{if KEY} = 1 \end{aligned} \tag{6.65}$$

or

$$R = R_c \qquad \text{if KEY} = -1$$

Illustrative Example 6.4

To illustrate the hand calculations in applying the step-by-step integration method described above, consider the single-degree-of-freedom system in Fig. 6.6 with elastoplastic behavior subjected to the loading history as shown. For this example, we assume that the damping coefficient remains constant ($\xi = 0.087$). Hence the only nonlinearities in the system arise from the changes in stiffness as yielding occurs (Fig. 6.10).

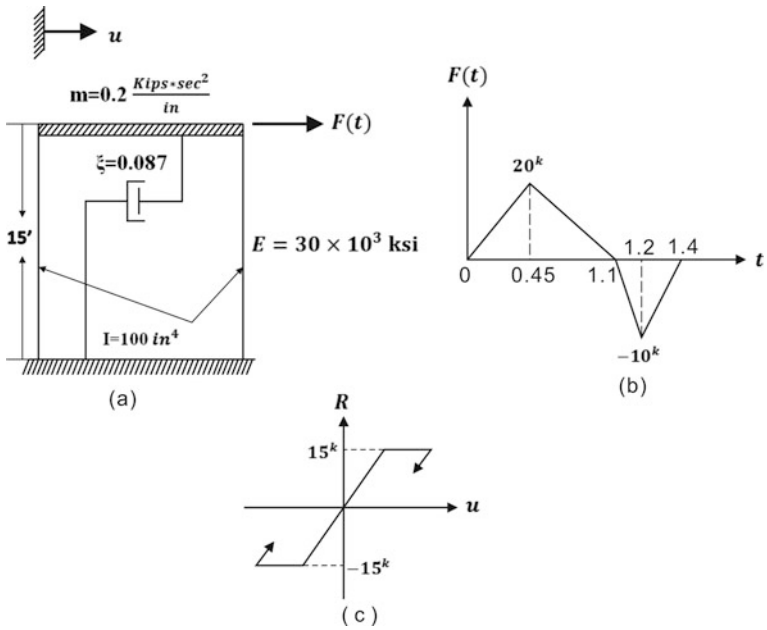

Fig. 6.10 Frame with elastoplastic behavior subjected to dynamic loading. (**a**) Frame. (**b**) Loading. (**c**) Elastoplastic behavior

Solution:

The stiffness of the system during elastic behavior is

$$k = \frac{12E(2I)}{L^3} = \frac{12x30x10^3x2x100}{(15x12)^3} = 12.35\ kip/in$$

and the damping coefficient

$$c = \xi c_{cr} = (0.087)(2)\sqrt{0.2 \times 12.35} = 0.274\ \text{kip} \cdot \text{sec}/\text{in}$$

Initial displacement and initial velocity are $u_0 = \dot{u}_0 = 0$.

The initial acceleration is

$$\ddot{u}_0 = \frac{F(0)}{k} = 0$$

Yield displacements are

$$u_t = \frac{R_t}{k} = \frac{15}{12.35} = 1.215 \text{ in}$$

and

$$u_c = -1.215 \text{ in}$$

The natural period is $T = 2\pi\sqrt{m/k} = 0.8$ (for the elastic system). For numerical convenience, we select $\Delta t = 0.1$ sec. The effective stiffness from Eq. (6.57) is

$$\bar{k} = k_p + \frac{6}{0.1^2}0.2 + \frac{3}{0.1}0.274$$

or

$$\bar{k} = k_p + 128.22$$

where

$$k_p = k = 12.35 \quad \text{(elasticbehavior)}$$
$$k_p = 0 \quad \text{(plasticbehavior)}$$

The effective incremental loading from Eq. (6.59) is

$$\Delta\bar{F}_i = \Delta F_i + \left(\frac{6}{\Delta t}m + 3c\right)\dot{u}_i + \left(3m + \frac{\Delta t}{2}\right)\ddot{u}_i$$
$$\Delta\bar{F}_i = \Delta F_i + 12.822\dot{u}_i + 0.613\ddot{u}_i$$

The velocity increment given by Eq. (6.61) becomes

$$\Delta\dot{u}_i = 30\Delta u_i - 3\dot{u}_i - 0.05u_i$$

The necessary calculations may be conveniently arranged as illustrated in Table 6.1. In this example with elastoplastic behavior, the response changes abruptly as the yielding starts and stops. To obtain better accuracy, it would be desirable to subdivide the time step in the neighborhood of the change of state; however, an iterative procedure would be required to establish the length of the subintervals. This refinement has not been used in the present analysis or in the computer program described in the next section. The stiffness computed at the initiation of the time step has been assumed to remain constant during the entire time increment. The reader is again cautioned that a significant error may arise during phase transitions unless the time step is selected relatively small.

Table 6.1 Nonlinear response – linear acceleration step-by-step method for Illustrative Example 6.4

t (sec)	F (Kip)	u (in)	KEY	$\dot{u}$ (in/sec)	R (Kip)	$\ddot{u}$ (in/sec^2)	k_p Kip/in	$\bar{k}$ Kip/in	ΔF (Kip)	$\Delta \bar{F}$ (Kip)	Δu (in)	$\Delta \dot{u}$ (in/sec)
0	0		0	0	0	0	12.35	140.57	4.444	4.444	0.0316	0.9485
0.1	4.444	0.0316	0	0.9485	0.390	18.972	12.35	140.57	4.444	28.249	0.2010	2.2359
0.2	8.888	0.2326	0	3.1844	2.871	25.723	12.35	140.57	4.444	61.050	0.4343	2.193
0.3	13.333	0.6669	0	5.3760	8.233	18.134	12.35	140 57	4.444	84.510	0.6012	1.000
0.4	17.777	1.2681	1	6.3768	15.00	5.152	0	128.22	0.685	85.609	0.6677	0.6422
0.5	18.462	1.9358	1	7.0190	15.00	7.691	0	128.22	−3.077	91.641	0.7147	−0.0001
0.6	15.238	2.6505	1	7.0189	15.00	−7.693	0	128.22	−3.077	82.199	0.6409	−1.440
0.7	12.308	3.2916	1	5.5791	15.00	–2I.105	0	128.22	−3.077	55.506	0.4329	−2.695
0.8	9.231	3.7244	1	2.8840	15.00	−32.797	0	128.22	−3.077	13.773	0.1074	−3.789
0.9	6.154	3.8319	0	−0.9054	15.00	−42.990	12.35	140.57	−3.077	41.069	−0.2922	−3.899
1.0	3.077	3.5397	0	−4.8048	11.39	−34.998	12.35	140.57	−3.077	−86.162	−0.6130	−2.225
1.1	0	2.9268	0	−7.0295	3.825	−9.497	12.35	140.57	−10	−105.96	−0.7538	−1.051
1.2	−10	2.1729	0	−8.0806	−54.81	−11.525	12.35	140.57	5	−105.68	−0.7518	2.263
1.3	−5	1.4211	0	−5.8177	−14.76	56.784	12.35	140.57	5	−34.746	−0.2472	7.198
1.4	0	1.1739	−1	1.3860	−15.00	73.109	0	128.22	0	62.568	0.4880	6.842
1.5	0	1.6619	0	8.2227	−15.00	63.735	12.35	140 57	0	144.55	1.0283	2.995

6.8 Response for Elastoplastic Behavior Using MATLAB

The same as for the other programs presented in this book, MATLAB program can be used. After the user has selected one of these options, the program requests the name of the file and the necessary input data. The program continues by setting the initial values to the various constants and variables in the equations. Then by linear interpolation, values of the forcing function are computed at time increments equal to the selected time step Δt for the integration process. Results are presented in Table 6.1. Using values of force and time step, the displacement, velocity, and acceleration are computed at each time step. The nonlinear behavior of the restoring force is appropriately considered in the calculation by the variable KEY which is tested through a series of conditional statements in order to determine the correct expressions for the yield points and the magnitude of the restoring force in the system.

The output consists of a table giving the displacement, velocity, and acceleration at time increments Δt. The last column of the table shows the value of the index KEY which provides information about the state of the elastoplastic system. As indicated before, KEY $= 0$ for elastic behavior and KEY $= 1$ or KEY $= -1$ for plastic behavior, respectively, in tension or in compression.

Illustrative Example 6.5

Using the MATLAB, find the response of the structure in Example 6.4. Then repeat the calculation assuming elastic behavior. Plot and compare results for the elastoplastic behavior with the elastic response.

Solution:

Problem Data (from Illustrative Example 6.4)

Spring constant	$k = 12.35$ kip/in
Damping coefficient	$c = 0.274$ kip. sec/in
Mass	$m = 0.2$ (kip. sec^2/in)
Max. restoring force (tension)	$R_t = 15$ kip
Max. restoring force (compression)	$R_c = -15$ kip
Natural period	$T = 2\pi\sqrt{m/k} = 2\pi\sqrt{0.2/12.35} = 0.8$ sec
Select time step	$\Delta t = 0.1$ sec

```
close all
clear all
clc

%%%%-GIVEN VALUES-%%%%

load ForceData.txt
t = ForceData(:,1);      %Time
F = ForceData(:,2);      %Force

Dt = t(2)-t(1);          %Time interval
m=0.2;                   %Mass
k =12.35;                %Stiffness
c =0.274;                %Damping coefficient
xi =c/(2*sqrt(m*k));     %Damping ratio
omega = sqrt(k/m);       %Natural frequency (rad/sec)
Rt = 15;                 %Forces yielding in tension
Rc = -15;                %Forces yielding in compression
```

```
%%%%Linear acceleration method (Newmark beta method[Ch.6])

gamma =1/2;
beta = 1/6;
tt= length(t);

%%%Initial calculation

u(1)=0;                                    %Initial condition; Displ.
v(1)=0;                                    %Initial condition; Velocity
a(1)=(F(1)-c*v(1)-k*u(1))/m;

A = m/(beta*Dt)+gamma*c/beta;                %A in DFbar = DF + A*v0+B*a0 (Eq. 6.46)
B = m/(2*beta)+Dt*c*((0.5*gamma/beta)-1);    %B in DFbar = DF + A*v0+B*a0 (Eq. 6.46)
kbar = k +gamma*c/(beta*Dt)+m/(beta*Dt*Dt); %Eq.6.45
key = 0;                                     %Initial key=0 before Loop over
k_p = k;                                     %Initial stiffness before iteration

%%%Setting up for initial value of Loop over

u0=u;
v0=v;
a0=a;
t =t(1);

%%%% Calculate initial yield points %%%%
ut = Rt/k;
uc = Rc/k;
R(1)=0;

%%%%Iteration for each time step using Newmark beta method

ua =[]; va =[]; aa=[]; ta=[];
fori = 1:(tt-1)
DF=F(i+1)-F(i);
F1 = F(i+1);

[t,u,v,a, kbar, R, keyp, key, Du, k_p] =  NewmarkNon( t, DF, Dt, u0, v0, ut, uc, a0, F1, k,
c, m, Rt, Rc, R, gamma, beta, key, k_p);

keyi(:,i)=key(:,1)'; keypi(:,i)=keyp(:,1)'; F1i(:,i)=F1(:,1)'; Ri(:,i)= R(:,1)';k_pi(:,i)=
k_p(1,:)';

%%%Creating column of time,displ,velocity and acceleration
ta = [ta; t];
ua = [ua; u];
va = [va; v];
aa = [aa; a];

%%%New yielding point-Increasing displacement (Eq.6.49)
if v < 0 && v0 >0
ut = max(ua);
uc = ut-(Rt-Rc)/k;
else
ut = ut;
uc = uc;
end

%%%New yielding point-Decreasing displacement (Eq.6.50)
if v > 0 && v0 <0
uc = min(ua);
ut = uc+(Rt-Rc)/k;
else
ut = ut;
uc = uc;
end

%%%Setting up parameters for next iteration
u0 = u;
v0 = v;
a0 = a;
t = t;
R = R;
end
result = [ta,F1i',ua,keypi',keyi',va,Ri',aa,k_pi'];
```

```
%%%Plot response

figure (1)

plot (ta, ua);
grid on
xlabel ('t(sec)');
ylabel ('Displacement(in.)');
```

The function of Newmark-beta method of MATLAB is presented for considering elastoplastic behavior. The main program is the same as Newmark-beta method (Fig. 6.11).

```
function  [t,u,v,a, kbar, R, keyp, key, Du, k_p] = NewmarkNon( t, DF, Dt, u0, v0, ut, uc,

a0, F1, k, c, m, Rt, Rc, R, gamma, beta, key, k_p)

    %%%Algorithm-For each time step:(3),(4),(5),(6),(7)
    kbar = k_p +gamma*c/(beta*Dt)+m/(beta*Dt*Dt);        %Eq.6.57
    A = m/(beta*Dt)+gamma*c/beta;                        %A in DFbar (Eq. 6.46)
    B = m/(2*beta)+Dt*c*((0.5*gamma/beta)-1);            %B in DFbar (Eq. 6.46)

    DFbar = DF + A*v0+B*a0;                        %Incremental effective force (Eq.6.59)
    Du = DFbar/kbar;                               %Incremental displacement (Eq.6.60)
Dudot = gamma*Du/(beta*Dt)-gamma*v0/beta+ Dt*a0*(1-0.5*gamma/beta);  %Incremental velocity
(Eq.6.61)

    u=u0+Du ;               %Displacement at the end of time interval (Eq.6.62)
    v=v0+Dudot;             %Velocity at the end of time interval (Eq.6.63)

    %%%Algorithm-For each time step:(2)-(a)
            if u<ut && u>uc
                        keyp = 0;
                    elseif u > ut
                            keyp = 1;
                    else
                            keyp = -1;
            end

%%%Algorithm-For each time step:(2)-(b) and (2)-(c)
          if keyp == 1 && v > 0
                       key = 1;

          elseifkeyp == -1 && v < 0
                            key = -1;
                    else
                            key = 0;
             end

%%%Algorithm-For each time step:(8) (Eq. 6.65)

          if key ==0
                     if (R+(Du)*k)>=0
                            R = min(R+(Du)*k, Rt);
                       else
                            R = max(R+(Du)*k, Rc);
                       end

             elseif key ==1
                        R = Rt;
                        else
                        R = Rc;
             end
```

```
%%%Algorithm-For each time step:(3) (Eq.6.58)
                  if key == 0
                      k_p = k;
                  else
                      k_p = 0;

                  end

%%%Algorithm-For each time step:(8) (Eq.6.64)
    a=1/m*(F1-c*v-R);

    t=t+Dt;
```

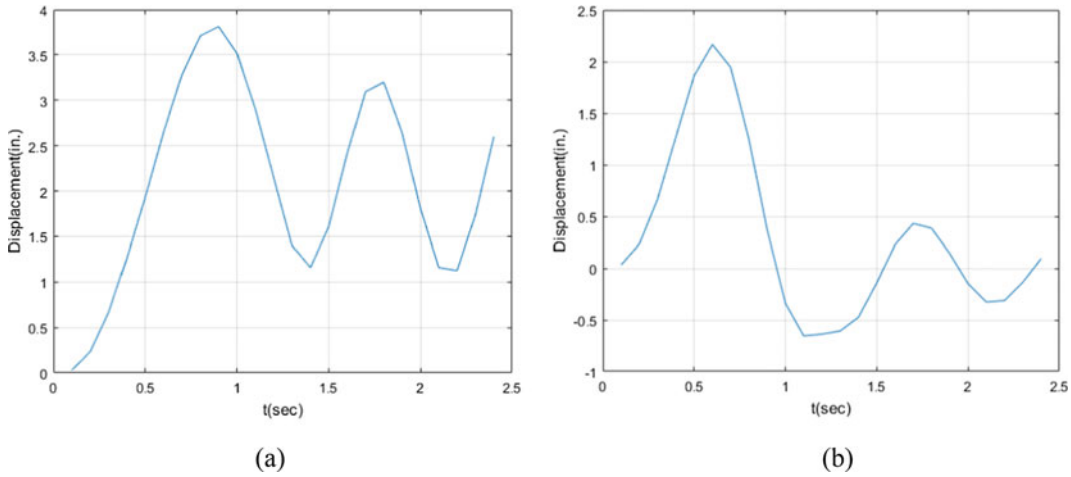

Fig. 6.11 Comparison of elastoplastic behavior with elastic response for Illustrative Example 6.2 (**a**) Nonlinear behavior; (**b**) Linear behavior

6.9 Summary

Structures are usually designed on the assumption that the structure is linearly elastic and that it remains linearly elastic when subjected to an expected dynamic excitation. However, there are situations in which the structure has to be designed for an eventual excitation of large magnitude such as the strong motion of an earthquake or the effects of nuclear explosion. In these cases, it is not realistic to assume that the structure will remain linearly elastic and it is then necessary to design the structure to withstand deformation beyond the elastic limit. The simplest and most accepted assumption for the design beyond the elastic limit is to assume an elastoplastic behavior. In this type of behavior, the structure is elastic until the restoring force reaches a maximum value (tension or compression) at which it remains constant until the motion reverses its direction and returns to an elastic behavior.

There are many methods to solve numerically the differential equation of this type of motion. The step-by-step linear acceleration presented in this chapter provides satisfactory results with relatively simple calculations. However, these calculations are tedious and time consuming when performed by hand. The use of a computer and the availability of a computer program, such as the one described in this chapter, reduce the effort to a simple routine of data preparation.

6.10 Problems

Problem 6.1
The single-degree-of-freedom of Fig. P6.1a is subjected to the foundation acceleration history in Fig. P6.1b Determine the maximum relative displacement of the columns. Assume elastoplastic behavior of Fig. P6.1c.

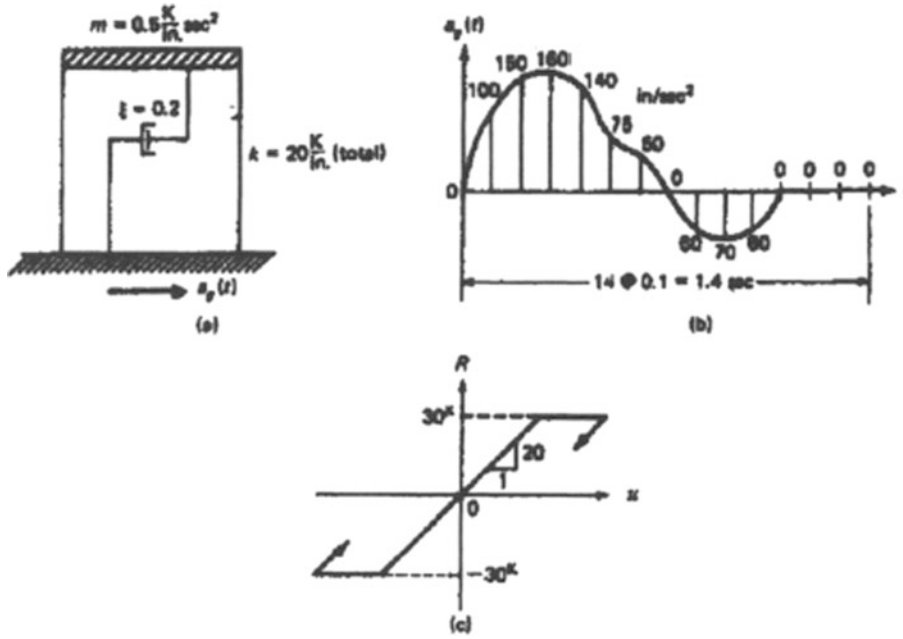

Fig. P6.1

Problem 6.2
Determine the displacement history for the structure in Fig. P6.1 when it is subjected to the impulse loading of Fig. P6.2 applied horizontally at the mass.

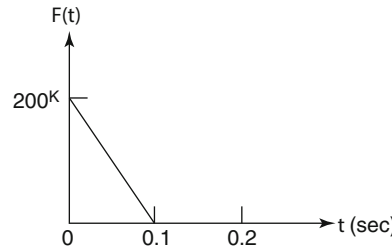

Fig. P6.2

Problem 6.3
Repeat Problem 6.2 for the impulse loading shown in Fig. P6.3 applied horizontally at the mass.

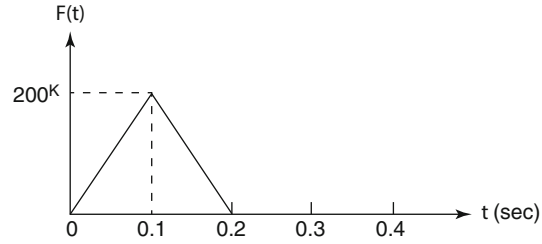

Fig. P6.3

Problem 6.4
Repeat Problem 6.2 for the acceleration history shown in Fig. P6.4 applied horizontally to the foundation.

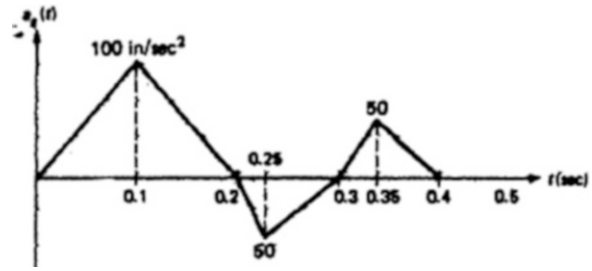

Fig. P6.4

Problem 6.5
Solve Problem 6.1 assuming elastic behavior of the structure. (Hint: Use computer Program 5 with $R_t = 200$ Kip and $R_c = -200$ Kip.)

Problem 6.6
Solve Problem 6.2 for elastic behavior of the structure. Plot the time-displacement response and compare with results from Problem 6.2.

Problem 6.7
Determine the ductility ratio from the results of Problem 6.2 (Ductility ratio is defined as the ratio of the maximum displacement to the displacement at the yield point).

Problem 6.8
A structure modeled as spring-mass shown in Fig. P6.8b is subjected to the loading force depicted in Fig. P6.8a. Assume elastoplastic behavior of Fig. P6.8c. Determine the response.

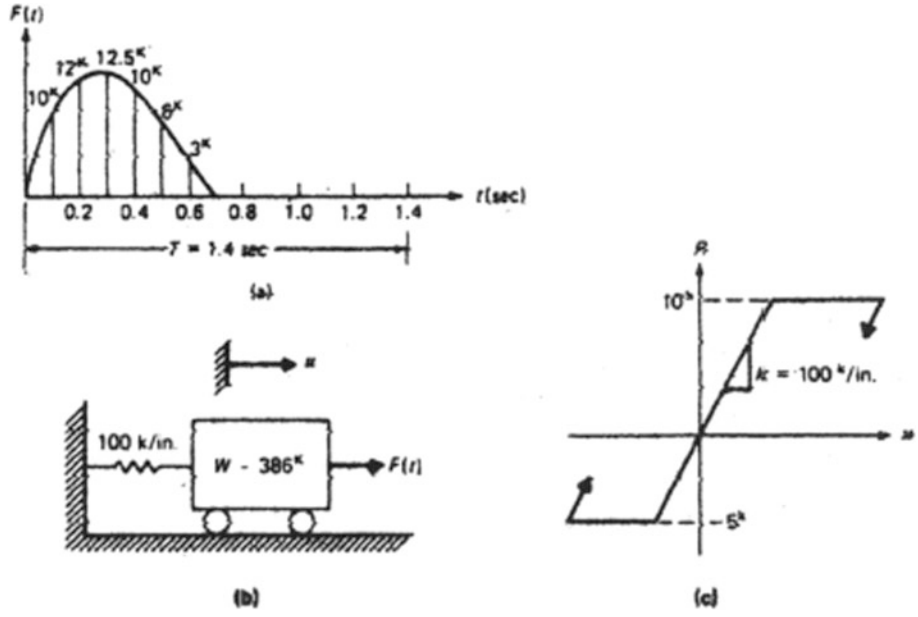

Fig. P6.8

Problem 6.9
Repeat Problem 6.8 assuming damping in the system equal to 20% of the critical damping.

Problem 6.10
Solve Problem 6.8 assuming elastic behavior of the system. (Hint: Use Program 5 with $R_T = 1000$ Kip and $R_C = -1000$ Kip.)

Problem 6.11
Solve Problem 6.9 assuming elastic behavior of the system.

Problem 6.12
A structure modeled as the damped spring-mass system shown in Fig. P6.12a is subjected to the time-acceleration excitation acting at its support. The excitation function is expressed as $a(t) = a_0 f(t)$, where $f(t)$ is depicted in Fig. P6.12b. Determine the maximum value that the factor a_0 may have for the structure to remain elastic. Assume that the structure has an elastoplastic behavior of Fig. P6.12c.

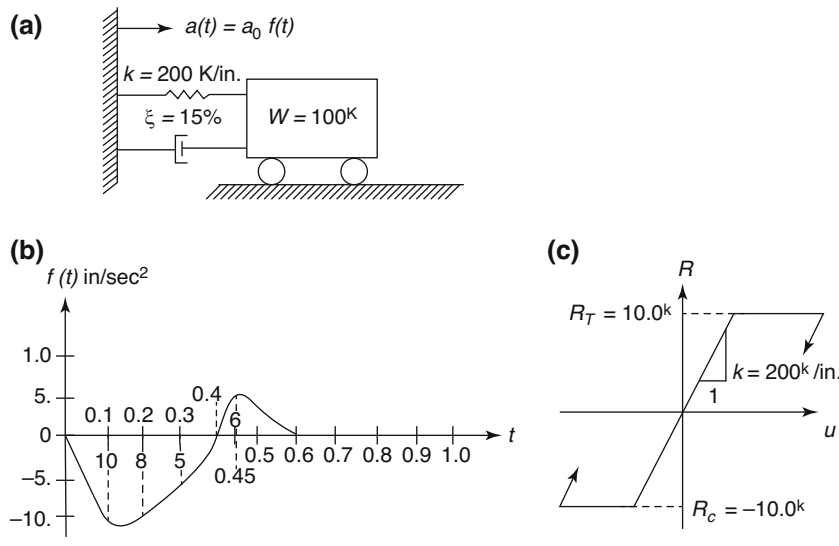
(a)
a(t) = a0 f(t)
k = 200 K/in.
ξ = 15%
W = 100K
(b)
f (t) in/sec2
1.0
5.
0
–5.
–10.
0.1 0.2 0.3
0.4
6
10 8 5
0.45
0.5 0.6 0.7 0.8 0.9 1.0
t
(c)
R
RT = 10.0k
k = 200k /in.
1
u
Rc = –10.0k

Fig. P6.12

Part II

Structures Modeled as Shear Buildings

Free Vibration of a Shear Building

7

In Part I we analyzed and obtained the dynamic response for structures modeled as a single-degree-of-freedom system. Only if the structure can assume a unique shape during its motion will the single-degree model provide the exact dynamic response. Otherwise, when the structure takes more than one possible shape during motion, the solution obtained from a single-degree model will be at best, only an approximation to the true dynamic behavior.

Structures cannot always be described by a single-degree-of-freedom model and, in general, have to be represented by multiple-degree models. In fact, structures are continuous systems and as such possess an infinite number of degrees of freedom. There are analytical methods to describe the dynamic behavior of continuous structures that have uniform material properties and regular geometry. These methods of analysis, though interesting in revealing information for the discrete modeling of structures, are rather complex and are applicable only to relatively simple actual structures. They require considerable mathematical analysis, including the solution of partial differential equations which will be presented in Part IV. For the present, we shall consider one of the most instructive and practical types of structure which involve many degrees of freedom, the multistory shear building.

7.1 Stiffness Equations for the Shear Building

A shear building may be defined as a structure in which there is no rotation of a horizontal section at the level of the floors. In this respect, the deflected building will have many of the features of a cantilever beam that is deflected by shear forces only, hence the name shear building. To accomplish such deflection in a building, we must assume that: (1) the total mass of the structure is concentrated at the levels of the floors; (2) the slabs or girders on the floors are infinitely rigid as compared to the columns; and (3) the deformation of the structure is independent of the axial forces present in the columns. These assumptions transform the problem from a structure with an infinite number of degrees of freedom (due to the distributed mass) to a structure that has only as many degrees as it has lumped masses at the floor levels. A three-story structure modeled as a shear building (Fig. 7.1a) will have three degrees of freedom, that is, the three horizontal displacements at the floor levels. The second assumption introduces the requirement that the joints between girders and columns are fixed against rotation. The third assumption leads to the condition that the rigid girders will remain horizontal during motion.

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M. Paz, Y. H. Kim, *Structural Dynamics*, https://doi.org/10.1007/978-3-319-94743-3_7

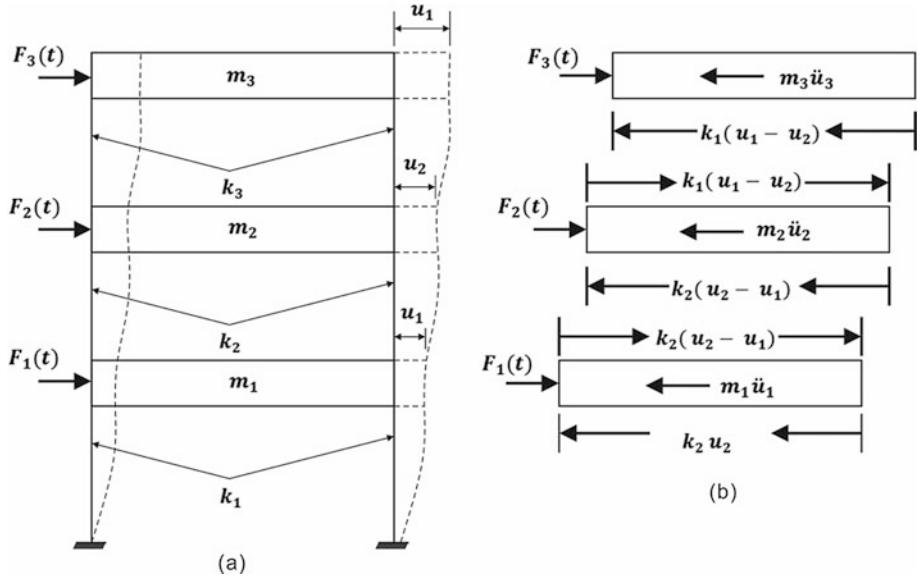

Fig. 7.1 (**a**) Single-bay model representation of a shear building. (**b**) Free body diagram

It should be noted that the building may have any number of bays and that it is only a matter of convenience that we represent the shear building solely in terms of a single bay. Actually, we can further idealize the shear building as a single column (Fig. 7.2a), having concentrated masses at the floor levels with the understanding that only horizontal displacements of these masses are possible. Another alternative is to adopt a multimass-spring system shown in Fig. 7.3a to represent the shear building. In any of the three representations depicted in these figures, the stiffness coefficient, or spring constant k_i, shown between any two consecutive masses is the force required to produce a relative unit displacement of the two adjacent floor levels

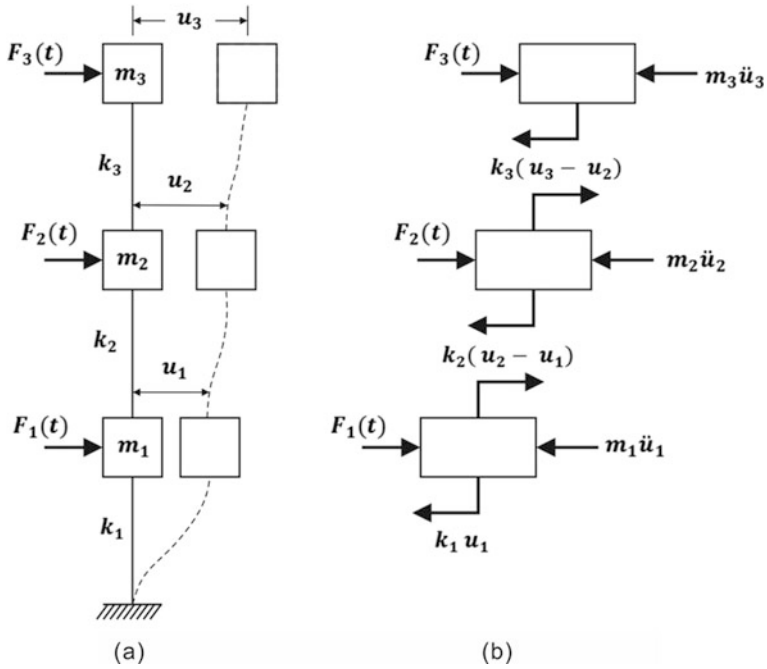

Fig. 7.2 (**a**) Single-column model representation of a shear building. (**b**) Free body diagram

For a uniform column with the two ends fixed against rotation, the stiffness or spring constant, k, is given by

$$k = \frac{12EI}{L^3} \tag{7.1a}$$

and for a column with one end fixed and the other pinned by

$$k = \frac{3EI}{L^3} \tag{7.1b}$$

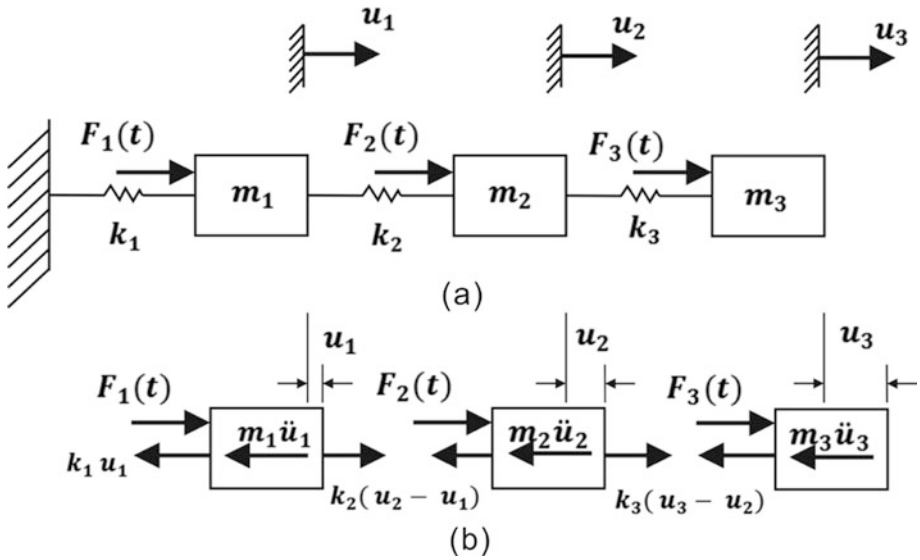

Fig. 7.3 (a) Multimass-spring model representation of a shear. (b) Free body diagram

where E is the material modulus of elasticity, I the cross-sectional moment of inertia, and L the length of the column.

It should be clear that all of the three representations shown in Figs. 7.1, 7.2 and 7.3 for the shear building are equivalent. Consequently, the following equations of motion for the thee-story shear building are obtained from any of the corresponding free body diagrams shown in these figures by equating to zero the sum of the forces acting on each mass. Hence

$$\begin{aligned} &m_1\ddot{u}_1 + k_1u_1 - k_2(u_2 - u_1) - F_1(t) = 0 \\ &m_2\ddot{u}_2 + k_2(u_2 - u_1) - k_3(u_3 - u_2) - F_2(t) = 0 \\ &m_3\ddot{u}_3 + k_3(u_3 - u_2) - F_3(t) = 0 \end{aligned} \tag{7.2}$$

This system of equations constitutes the stiffness formulation of the equations of motion for a three-story shear building. It may conveniently be written in matrix notation as

$$[M]\{\ddot{u}\} + [K]\{u\} = \{F\} \tag{7.3}$$

where $[M]$ and $[K]$ are the mass and stiffness matrices given, respectively, by

$$[M] = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \tag{7.4}$$

$$[K] = \begin{bmatrix} k_1 + k_2 & -k_2 & 0 \\ -k_2 & k_2 + k_3 & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix} \tag{7.5}$$

and $\{u\}$, $\{\ddot{u}\}$ and $\{F\}$ are, respectively, the displacement, acceleration and force vectors given by

$$\{u\} = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}, \qquad \{\ddot{u}\} = \begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \\ \ddot{u}_3 \end{Bmatrix}, \qquad \{F\} = \begin{Bmatrix} F_1(t) \\ F_2(t) \\ F_3(t) \end{Bmatrix} \tag{7.6}$$

It should be noted that the mass matrix, Eq. (7.4), corresponding to the shear building is a diagonal matrix (the non-zero elements are only in the main diagonal). The elements of the stiffness matrix, Eq. (7.5), are designated *stiffness coefficients*. In general, the stiffness coefficient, k_{ij}, is defined as the force at coordinate i when a unit displacement is given at j, all other coordinates being fixed. For example, the coefficient in the second row and second column of Eq. (7.5), $k_{22} = k_2 + k_3$, is the force required at the second floor when a unit displacement is given to this floor.

7.2 Natural Frequencies and Normal Modes

The problem of free vibration requires that the force vector $\{F\}$ be equal to zero in Eq. (7.3). Namely,

$$[M]\{\ddot{u}\} + [K]\{u\} = 0 \tag{7.7}$$

For free vibrations of the undamped structure, we seek solutions of Eq. (7.7) in the form

$$u_i = a_i \sin(\omega t - \alpha), \qquad i = 1, 2, \ldots, n$$

or in vector notation

$$\{u\} = \{a\} \sin(\omega t - \alpha) \tag{7.8}$$

where a_i is the amplitude of motion of the ith coordinate and n is the number of degrees of freedom. The substitution of Eq. (7.8) into Eq. (7.7) gives

$$-\omega^2[M]\{a\} \sin(\omega t - \alpha) + [K]\{a\} \sin(\omega t - \alpha) = 0$$

or factoring out $\sin(\omega t - \alpha)$ and rearranging terms

$$\left[[K] - \omega^2[M]\right]\{a\} = \{0\} \tag{7.9}$$

which for the general case, is set for n homogenous (right-hand side equal to zero) algebraic system of linear equations with n unknown displacements a_i and an unknown parameter ω^2. The formulation of Eq. (7.9) is an important mathematical problem known as an eigenproblem. Its nontrivial solution, that is, the solution for which not all $a_i = 0$, requires that the determinant of the matrix factor of $\{a\}$ be equal to zero; in this case

$$\left|[K] - \omega^2[M]\right| = 0 \tag{7.10}$$

In general, the expansion of the determinant in Eq. (7.10) results in a polynomial equation of degree n in ω^2 which should be satisfied for n values of ω^2. This polynomial is known as the characteristic equation of the system. For each of these values of ω^2 satisfying the characteristic Eq. (7.10) we can solve Eq. (7.9) for $a_1, a_2, \ldots, a_n$ in terms of an arbitrary constant. The necessary calculations are better explained through a numerical example.

Illustrative Example 7.1

The structure to be analyzed is the two-story steel rigid frame shown in Fig. 7.4. The weights of the floors and walls are indicated in the figure and are assumed to include the structural weight as well. The building consists of a series of frames spaced 15 ft. apart. It is further assumed that the structural properties are uniform along the length of the building and, therefore, the analysis to be made of an

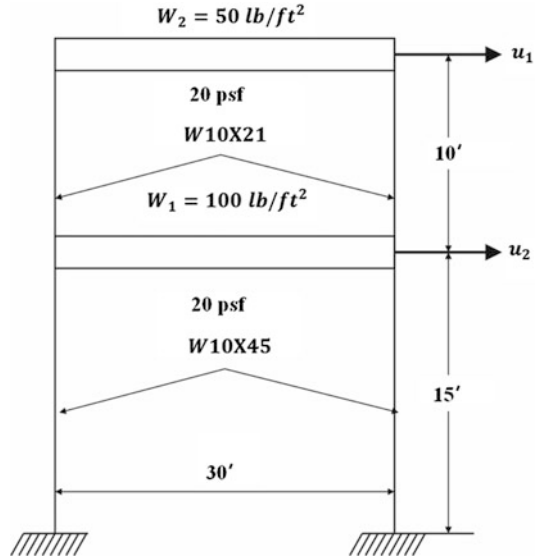

Fig. 7.4 Two-story shear building for Illustrative Example 7.1

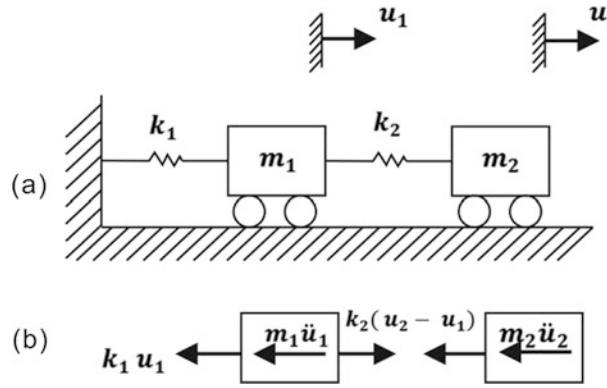

Fig. 7.5 Multimass-spring model for a two-story shear building of Illustrative Example 7.1. (**a**) Model, (**b**) Free body diagram

interior frame yields the response of the entire building. Determine (a) the natural frequencies and corresponding modal shapes, (b) the equations of motion with initial conditions for displacements u_{01}, u_{02}, and for velocities $\dot{u}_{01}$, and $\dot{u}_{02}$, respectively, for the first and second stories of the building.

Solution:

(a) Natural Frequencies and Modal Shapes
The building is modeled as a shear building and, under the assumptions stated, the entire building may be represented by the spring –mass system shown in Fig. 7.5. The concentrated weights, which are each taken as the total floor weight plus that of the tributary walls, are computed as follows:

$$W_1 = 100 \times 30 \times 15 + 20 \times 12.5 \times 15 \times 2 = 52,500 \text{ lb}$$
$$m_1 = 136 \text{ lb.sec}^2/\text{in}$$
$$W_2 = 50 \times 30 \times 15 + 20 \times 5 \times 15 \times 2 = 25,500 \text{ lb}$$
$$m_2 = 66 \text{ lb.sec}^2/\text{in}$$

Since the girders are assumed to be rigid and fixed at the two ends, the stiffness (spring constant) of each story is given by Eq. (7.1a) as

$$k = \frac{12E(2I)}{L^3}$$

and the individual values for the steel column sections indicated are thus

$$k_1 = \frac{12 \times 30 \times 10^6 \times 248 \times 2}{(15 \times 12)^3} = 30,700 \text{ lb/in}$$
$$k_2 = \frac{12 \times 30 \times 10^6 \times 118 \times 2}{(10 \times 12)^3} = 44,300 \text{ lb/in}$$

The equations of motion for the system, which are obtained by considering in Fig. 7.5b the dynamic equilibrium of each mass in free vibration, are

$$m_1\ddot{u}_1 + k_1u_1 - k_2(u_2 - u_1) = 0$$
$$m_2\ddot{u}_2 + k_2(u_2 - u_1) = 0$$

In the usual manner, these equations of motion are solved for free vibration by substituting

$$u_1 = a_1 \sin(\omega t - \alpha)$$
$$u_2 = a_2 \sin(\omega t - \alpha) \qquad \text{(a)}$$

for the displacements and

$$\ddot{u}_1 = -a_1\omega^2 \sin(\omega t - \alpha)$$
$$\ddot{u}_2 = -a_2\omega^2 \sin(\omega t - \alpha)$$

for the accelerations. In matrix notation, we obtain

$$\begin{bmatrix} k_1 + k_2 - m_1\omega^2 & -k_2 \\ -k_2 & k_2 - m_2\omega^2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \text{(b)}$$

For a nontrivial solution, we require that the determinant of the coefficients be equal to zero, that is,

$$\begin{vmatrix} k_1 + k_2 - m_1\omega^2 & -k_2 \\ -k_2 & k_2 - m_2\omega^2 \end{vmatrix} = 0 \tag{c}$$

The expansion of this determinant gives a quadratic equation in ω^2, namely

$$m_1 m_2 \omega^4 - [(k_1 - k_2)m_2 + m_1 k_2]\omega^2 + k_1 k_2 = 0 \tag{d}$$

or by introducing the numerical values for this example, we obtain

$$8976\omega^4 - 10,974,800\omega^2 + 1.36 \times 10^6 = 0 \tag{e}$$

The roots of this quadratic are

$$\omega_1^2 = 140$$
$$\omega_2^2 = 1082$$

Therefore, the natural frequencies of the structure are

$$\omega_1 = 11.83 \text{ rad/sec}$$
$$\omega_2 = 32.89 \text{ rad/sec}$$

or in cycles per second (cps)

$$f_1 = \omega_1/2\pi = 1.88 \text{ cps}$$
$$f_2 = \omega_2/2\pi = 5.24 \text{ cps}$$

and the corresponding natural periods:

$$T_1 = \frac{1}{f_1} = 0.532 \text{ sec}$$
$$T_2 = \frac{1}{f_2} = 0.191 \text{ sec}$$

To solve Eq. (b) for the amplitudes a_1 and a_2, we note that by equating the determinant to zero in Eq. (c), the number of independent equations is one less. Thus in the present case, the system of two equations is reduced to one independent equation. Considering the first equation in Eq. (b) and substituting the first natural frequency, $\omega_1 = 11.8$ rad/sec, we obtain

$$55,960a_{11} - 44,300a_{21} = 0 \tag{f}$$

We have introduced a second sub-index in a_1 and a_2 to indicate that the value ω_1 has been used in this equation. Since in the present case there are two unknowns and only one equation, we can solve Eq. (f) only for the relative value of a_{21} to a_{11}. This relative value is known as the normal mode or modal shape corresponding to the first frequency. For this example, Eq. (f) gives

$$\frac{a_{21}}{a_{11}} = 1.263$$

It is customary to describe the normal modes by assigning a unit value to one of the amplitudes; thus, for the first mode we set a_{11} equal to unity so that

$$\begin{aligned} a_{11} &= 1.000 \\ a_{21} &= 1.263 \end{aligned} \tag{g}$$

Similarly, substituting the second natural frequency, $\omega_2 = 32.9$ rad/sec into Eq. (b) we obtain the second normal mode as

$$\begin{aligned} a_{12} &= 1.000 \\ a_{22} &= -1.629 \end{aligned} \tag{h}$$

It should be noted that although we obtained only ratios, the amplitudes of motion could, of course, be found from initial conditions.

We have now arrived at two possible simple harmonic motions of the structure which can take place in such a way that all the masses move in phase in the same frequency, either ω_1 or ω_2. Such a motion of an undamped system is called a normal or natural mode of vibration. The shapes for these modes (a_{21}/a_{11} and a_{22}/a_{12}) for this example are called normal mode shapes of simply modal shapes for the corresponding natural frequencies ω_1 and ω_2. These two modes for this example are depicted in Fig. 7.6.

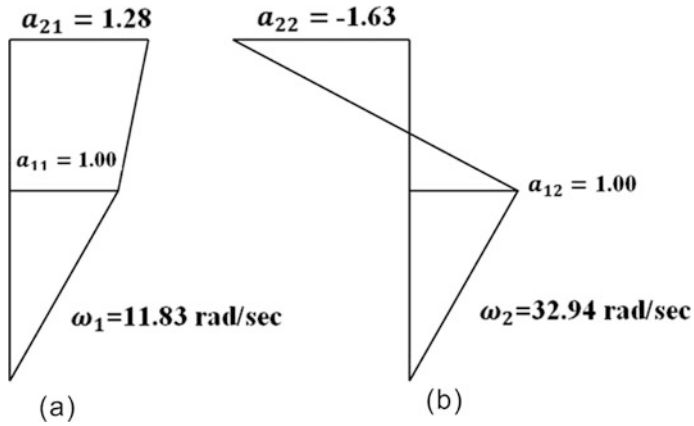

Fig. 7.6 Normal modes for Illustrative Example 7.1 (**a**) First mode (**b**) Second mode

We often use the designation first mode or fundamental mode to refer to the mode associated with the lowest frequency. The other modes are sometimes called harmonics or higher harmonics. It is evident that the modes of vibration, each having its own frequency, behave essentially as single-degree-of-freedom systems.

(b) Equations of Motion

The total motion of the system, that is, the total solution of the equations of motion, Eq. (7.7), is given by the superposition of the modal harmonic vibrations which in terms of arbitrary constants of integration may be written as

$$\begin{aligned} u_1(t) &= C_1' a_{11} \sin(\omega_1 t - \alpha_1) + C_2' a_{12} \sin(\omega_2 t - \alpha_2) \\ u_1(t) &= C_1' a_{21} \sin(\omega_1 t - \alpha_1) + C_2' a_{22} \sin(\omega_2 t - \alpha_2) \end{aligned} \tag{i}$$

Here C_1' and C_2' as well as α_1 and α_2 are four constants of integration to be determined from four initial conditions which are the initial displacement and velocity for each mass in the system. For a two-degree-of-freedom system, these initial conditions are

$$\begin{aligned} u_1(0) &= u_{01}, \qquad \dot{u}_1(0) = \dot{u}_{01} \\ u_2(0) &= u_{02}, \qquad \dot{u}_2(0) = \dot{u}_{02} \end{aligned} \tag{j}$$

For computational purposes, it is convenient to eliminate the phase angles [α_1 and α_2 in Eq. (i)] in favor of other constants. Expanding the trigonometric functions in Eq. (i) and renaming the constants, we obtain

$$\begin{aligned} u_1(t) &= C_1 a_{11} \sin \omega_1 t + C_2 a_{11} \cos \omega_1 t + C_3 a_{12} \sin \omega_2 t + C_4 a_{12} \cos \omega_2 t \\ u_2(t) &= C_1 a_{21} \sin \omega_1 t + C_2 a_{21} \cos \omega_1 t + C_3 a_{22} \sin \omega_2 t + C_4 a_{22} \cos \omega_2 t \end{aligned} \tag{k}$$

in which C_1, C_2, C_3 and C_4 are the new renamed constants of integration. From the first two initial conditions in Eq. (j), we obtain the following two equations:

$$\begin{aligned} u_{01} &= C_2 a_{11} + C_4 a_{12} \\ u_{02} &= C_2 a_{21} + C_4 a_{22} \end{aligned} \tag{l}$$

Since the modes are independent, these equations can always be solved for C_2 and C_4. Similarly, by expressing in Eq. (k) the velocities at time equal to zero, we find

$$\begin{aligned} \dot{u}_{01} &= \omega_1 C_1 a_{11} + \omega_2 C_3 a_{12} \\ \dot{u}_{02} &= \omega_1 C_1 a_{21} + \omega_2 C_3 a_{22} \end{aligned} \tag{m}$$

The solution of these two sets of Eqs. (l) and (m), allows us to express the motion of the system in terms of the two modal vibrations, each proceeding at its own frequency, completely independent of the other, the amplitudes and phases being determined by the initial conditions.

7.3 Orthogonality Property of the Normal Modes

We shall now introduce an important property of the normal modes, the orthogonality property. This property constitutes the basis of the most important method for solving dynamic problems, the Modal Superposition Method of multi-degree-of-freedom systems. We begin by rewriting the equations of motion in free vibration, Eq. (7.7) as

$$[K]\{a\} = \omega^2 [M]\{a\} \tag{7.11}$$

For the two-degree-of-freedom system, we obtain from Eq. (b) of Illustrative Example 7.1

$$\begin{aligned} (k_1 + k_2)a_1 - k_2 a_2 &= \omega^2 m_1 a_1 \\ -k_2 a_1 + k_2 a_2 &= \omega^2 m_2 a_2 \end{aligned} \tag{7.12}$$

These equations are exactly the same as Eq. (b) of Illustrative Example 7.1 but written in this form they may be given a static interpretation as the equilibrium equations for the system acted on by forces of magnitude $\omega^2 m_1 a_1$ and $\omega^2 m_2 a_2$ applied to masses m_1 and m_2, respectively. The modal shapes

may then be considered as the static deflections resulting from the forces on the right-hand side of Eq. (7.12) for any of the two modes. This interpretation, as a static problem, allows us to use the results of the general static theory of linear structures. In particular, we may use of Betti's theorem which states: For a structure acted upon by two systems of loads and corresponding displacements, the work done by the first system of loads moving through the displacements of the second system is equal to the work done by this second system of loads undergoing the displacements produced by the first load system. The two systems of loading and corresponding displacements which we shall consider are as follows:

System I:

Forces	$\omega_1^2 m_1 a_{11},$	$\omega_1^2 m_2 a_{21}$
Displacements	$a_{11},$	a_{21}

System II:

Forces	$\omega_2^2 m_1 a_{12}$	$\omega_2^2 m_2 a_{22}$
Displacements	$a_{12},$	a_{22}

The application of Betti's theorem for these two systems yields

$$\omega_1^2 m_1 a_{11} a_{12} + \omega_1^2 m_2 a_{21} a_{22} = \omega_2^2 m_1 a_{12} a_{11} + \omega_2^2 m_2 a_{22} a_{21}$$

or

$$\left(\omega_1^2 - \omega_2^2\right)\left(m_1 a_{11} a_{12} + m_2 a_{21} a_{22}\right) = 0$$

If the natural frequencies are different ($\omega_1 \neq \omega_2$), it follows that

$$m_1 a_{11} a_{12} + m_2 a_{21} a_{22} = 0$$

which is the so-called orthogonality relationship between modal shapes of a two degree-of-freedom system. For an n-degree-of-freedom system in which the mass matrix is diagonal, the orthogonality condition between any two modes i and j may be expressed as

$$\sum_{k=1}^{n} m_k a_{ki} a_{kj} = 0, \qquad \text{for } i \neq j \tag{7.13}$$

and in general for any n-degree-of-freedom system as

$$\{a\}_i^T [M] \{a\}_j = 0 \qquad \text{for } i \neq j \tag{7.14}$$

in which $\{a_i\}$ and $\{a_j\}$ are any two modal vectors and $[M]$ is the mass matrix of the system.

As mentioned before, the amplitudes of vibration in a normal mode are only relative values which may be scaled or normalized to some extent as a matter of choice. The following is an especially convenient normalization for a general system:

$$\phi_{ij} = \frac{a_{ij}}{\sqrt{\{a\}_i^T\,[M]\{a\}_j}} \tag{7.15}$$

which, for a system having a diagonal mass matrix, may be written as

$$\phi_{ij} = \frac{a_{ij}}{\sqrt{\sum_{k=1}^{n} m_k a_{kj}^2}} \tag{7.16}$$

in which ϕ_{ij} is the normalized i component of the j modal vector. For normalized eigenvectors, the orthogonality condition is given by

$$\begin{aligned} \{\phi\}_i^T[M]\{\phi\}_j &= 0 \quad \text{for } i \neq j \\ &= 1 \quad \text{for } i = j \end{aligned} \tag{7.17}$$

Another orthogonality condition is obtained by writing Eq. (7.9) for the normalized j mode as

$$[K]\{\phi\}_j = \omega_j^2[M]\{\phi\}_j \tag{7.18}$$

Then pre-multiplying Eq. (7.18) by $\{\phi\}_i^T$ we obtain, in view of Eq. (7.17), the following orthogonality condition between eigenvectors:

$$\begin{aligned} \{\phi\}_i^T[K]\{\phi\}_j &= 0 \quad \text{for } i \neq j \\ &= \omega_j^2 \quad \text{for } i = j \end{aligned} \tag{7.19}$$

Illustrative Example 7.2

For the two-story shear building of Illustrative Example 7.1 determine (a) the normalized modal shapes of vibration, and (b) verify the orthogonality condition between the modes.

Solution:

The substitution of Eqs. (g) and (h) from Illustrative Example 7.1 together with the values of the masses from Illustrative Example 7.1 into the normalization factor required in Eq. (7.16) gives

$$\sqrt{(136)(1.00)^2 + (66)(1.263)^2} = \sqrt{241.31}$$

$$\sqrt{(136)(1.00)^2 + (66)(-1.629)^2} = \sqrt{311.08}$$

Consequently, the normalized modes are

$$\phi_{11} = \frac{1.00}{\sqrt{241.31}} = 0.06437, \qquad \phi_{12} = \frac{1.00}{\sqrt{311.08}} = 0.0567$$

$$\phi_{21} = \frac{1.263}{\sqrt{241.31}} = 0.0813, \qquad \phi_{22} = \frac{-1.629}{\sqrt{311.08}} = -0.0924$$

The normal modes may be conveniently arranged in the columns of a matrix known as the modal matrix of the system. For the general case of n degrees of freedom, the modal matrix is written as

$$[\Phi] = \begin{bmatrix} \phi_{11} & \phi_{12} \cdots & \phi_{1n} \\ \phi_{21} & \phi_{22} \cdots & \phi_{2n} \\ \phi_{n1} & \phi_{n2} \cdots & \phi_{nn} \end{bmatrix} \tag{7.20}$$

The orthogonality condition may then be expressed in general as

$$[\Phi]^T [M][\Phi] = [I] \tag{7.21}$$

where $[\Phi]^T$ is the matrix transpose of $[\Phi]$, and $[M]$ the mass matrix of the system. For this example of two degrees of freedom, the modal matrix is

$$[\Phi] = \begin{bmatrix} 0.06437 & 0.0567 \\ 0.0813 & -0.0924 \end{bmatrix} \tag{a}$$

To check the orthogonality condition, we simply substitute the normal modes from Eq. (a) into Eq. (7.21) and obtain

$$\begin{bmatrix} 0.06437 & 0.0813 \\ 0.0567 & -0.0924 \end{bmatrix} \begin{bmatrix} 136 & 0 \\ 0 & 66 \end{bmatrix} \begin{bmatrix} 0.06437 & 0.0567 \\ 0.0813 & -0.0924 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

We have seen that to determine the natural frequencies and normal modes of vibration of a structural system, we have to solve an eigenvalue problem. The direct method of solution based on the expansion of the determinant and the solution of the resulting characteristic equation is limited in practice to systems having only a few degrees of freedom. For a system of many degrees of freedom, the algebraic and numerical work required for the solution of an eigenproblem becomes so immense as to make the direct method impossible. However, there are many numerical methods available for the calculation of eigenvalues and eigenvectors of an eigenproblem. The discussion of these methods belongs in a mathematical text on numerical methods rather than in a text such as this on structural dynamics. One of the most popular methods for the numerical solution of an eigenproblem is the Jacobi Method, which is an iterative method to calculate the eigenvalues and eigenvectors of the system. The basic Jacobi solution method has been developed for the solution of standard eigenproblems (i.e., $[M]$ being the identity matrix). The method was proposed over a century ago and has been used extensively. This method can be applied to all symmetric matrices $[K]$ with no restriction on the eigenvalues. It is possible to transform the generalized eigenproblem, $[[K] - \omega^2[M]] = \{\Phi\} = \{0\}$ into the standard form and still maintain the symmetry required for the Jacobi Method. However, this transformation can be dispensed with by using a generalized Jacobi solution method (Bathe, K. J. 1982) which operates directly on $[K]$ and $[M]$.

Examples 7.1 and 7.2 can be solved using MATLAB program. The function of MATLAB are used to solve eigenproblem using built-in function, eig(K, M). The natural frequencies and normal modes are estimated using the following MATLAB codes. The outcomes are the natural frequencies (natural periods) and normalized modal matrix. This MATLAB code adopts the framework proposed by Anderson and Naeim (2012).

```
clear all
close all

%%%%-GIVEN VALUES-%%%%

%%%Define Mass Matrix
M = [136 0; 0 66]

%%%Define Stiffness Matrix
K = [30700+44300 -44300;-44300 44300]

%%%Solve for eigenvalues (D) and eigenvectors (a)
[a, D] = eig(K, M)

[omegas,k] = sort(sqrt(diag(D)));

%%%Natural frequencies
omegas =sqrt(D)

%%%Natural periods
T =2*pi./omegas;

T1 = 2*pi./omegas(1,1);
T2 = 2*pi./omegas(2,2);

%%%{a}1before changing the unity in the first DOF.
a1 = a(:,1);

%%%{a}2before changing the unity in the first DOF.
a2 = a(:,2);

%%%Change the {a} w.r.t. the unity in the first DOF.
a11 = 1;
a21 = a1(2,1)./a1(1,1);
a12 = 1;
a22 = a2(2,1)./a2(1,1);

a =[];

%%%Calculate the {a}
a(:,1) = [a11, a21]                 %[a11,a21]
a(:,2) = [a12, a22]                 %[a12,a22]

%%%aMa = {a}'*[M]*{a}
aMa = a'*M*a;                       %Eq.7.14

%%%Normalization factor
norm_1 = sqrt(aMa(1,1));
norm_2 = sqrt(aMa(2,2));

%%%Normalized eigenvectors
nom_phi(:,1) = 1./norm_1.*a(:,1);   %Eq.7.16 for the first mode
nom_phi(:,2)= 1./norm_2.*a(:,2);    %Eq.7.16 for the 2nd mode
nom_phi
%Check the orthogonality condition for Mass Matrix
orth_M = nom_phi'*M*nom_phi;        %Eq.7.17

%Check the orthogonality condition for Stiffness Matrix
orth_K = nom_phi'*K*nom_phi;        %Eq.7.19
```

7.4 Rayleigh's Quotient

Several iterative methods for the solution of an eigenproblem make use of the Rayleigh's quotient. The Rayleigh's quotient may be obtained by pre-multiplying Eq. (7.18) by the transpose of the modal vector $\{\phi\}_j^T$. Hence,

$$\{\phi\}_j^T[K]\{\phi\}_j = \omega_j^2\{\phi\}_j^T[M]\{\phi\}_j$$

The property of the mass matrix $[M]$ being positive definite[1] renders the product $\{\phi\}_j^T[M]\{\phi\}_j \neq 0$, thus, it is permissible to solve for ω_j^2:

$$\omega^2 = \frac{\{\phi\}^T[K]\{\phi\}}{\{\phi\}^T[M]\{\phi\}} \tag{7.22}$$

in which for convenience the sub-index j has been omitted.

The ratio given by Eq. (7.22) is known as the Rayleigh's quotient. This quotient has the following properties: (1) It provides the eigenvalue ω_j^2 when the corresponding eigenvector $\{\phi\}_j$ is introduced in Eq. (7.22). (2) When a vector $\{\phi\}$ different from an eigenvector is used, then Eq. (7.22) provides a value ω^2 that lies between the smallest eigenvalue, ω_1^2 and the largest eigenvalue ω_N^2. (3) Finally, if a vector $\{\phi\}$ that is an approximation to eigenvector $\{\phi\}_j$, correct to d decimals is used, then the value of ω^2 obtained from Eq. (7.22) is accurate to $2d$ number of decimals as an approximation to ω_j^2.

Illustrative Example 7.3

Use Rayleigh's quotient to calculate an approximate value for the first eigenvalue of the structure in Illustrative Example 7.1 beginning with the approximate eigenvector for the first mode $\{\phi\}^T = \{1.00\ 1.50\}$, then iterate using Eqs. (7.12) and (7.22) to converge to the eigenvalue and eigenvector for the first mode.

Solution:

The substitution of the given vector $\{\phi\}^T = \{1.00\ 1.50\}$ and the matrices $[K]$ and $[M]$ into the numerator of Eq. (7.22) results in

$$\{1.00\ \ 1.50\}\begin{bmatrix} 75,000 & -44,300 \\ -44,300 & 44,300 \end{bmatrix}\begin{Bmatrix} 1.00 \\ 1.50 \end{Bmatrix} = 42,025$$

and also into the denominator

$$\{\,1.00\quad 1.50\,\}\begin{bmatrix} 136 & 0 \\ 0 & 66 \end{bmatrix}\begin{Bmatrix} 1.00 \\ 1.50 \end{Bmatrix} = 284.5$$

which substituted into Eq. (7.22) yields

$$\omega^2 = \frac{42,025}{284.5} = 147.9$$

[1] Matrix $[A]$ is defined as positive definite if it satisfies the condition that for any arbitrary nonzero vector $\{v\}$, the product $\{v\}^T[A]\{v\} > 0$.

The use of the calculated value $\omega^2 = 147.9$ together with $a_1 = 1.00$ into the first Eq. (7.12) yields

$$a_1 = 1.00 \quad \text{and} \quad a_2 = 1.24$$

A second iteration of Eq. (7.22) with $\{\Phi\}^T = \{1.00 \quad 1.24\}$ yields

$$\omega^2 = 140.02$$

This value of ω^2 is virtually equal to the solution $\omega_1 = 140.02$ obtained for the first mode in Illustrative Example 7.1

Another popular iterative method to solve an eigenproblem, that is, for structural dynamics, to calculate natural frequencies and modal shapes, is the Subspace Iteration Method.

7.5 Summary

The motion of an undamped dynamic system in free vibration is governed by a homogenous system of differential equations which in matrix notation is

$$[M]\{\ddot{u}\} + [K]\{u\} = 0$$

The process of solving this system of equations leads to a homogenous system of linear algebraic equations of the form

$$\left[[K] - \omega^2[M]\right]\{a\} = \{0\}$$

which mathematically is known as an eigenproblem.

For a nontrivial solution of this problem, it is required that the determinant of the coefficients of the unknown $\{a\}$ be equal to zero, that is,

$$\left|[K] - \omega^2[M]\right| = 0$$

The roots of this equation provide the natural frequencies ω_i, (i = 1, 2, ... n). It is then possible to solve for the unknowns $\{a\}_i$, in terms of relative values. The vectors $\{a\}_i$ corresponding to the roots ω_i^2 are the modal shapes (eigenvectors) of the dynamic system. The arrangement in matrix format of the modal shapes constitutes the modal matrix [Φ] of the system. It is particularly convenient to normalize the eigenvectors to satisfy the following condition:

$$\{\phi\}_i^T[M]\{\phi\}_i = [I] \quad i = 1, 2, \ldots, n$$

where the normalized modal vectors $\{\phi\}_i$ are obtained by dividing the components of the vector $\{a\}_{ij}$ by $\sqrt{\{a\}_i^T[M]\{a\}_j}$.

The normalized modal vectors satisfy the following important conditions of orthogonality:

$$\begin{aligned} \{\phi\}_i^T[M]\{\phi\}_j &= 0 \quad \text{for } i \neq j \\ \{\phi\}_i[M]\{\phi\}_j &= 1 \quad \text{for } i = j \end{aligned}$$

and

$$\begin{aligned} \{\phi\}_i^T[K]\{\phi\}_j &= 0 \quad \text{for } i \neq j \\ \{\phi\}_i[K]\{\phi\}_j &= \omega_i^2 \quad \text{for } i = j \end{aligned}$$

The above relations are equivalent to

$$[\Phi]^T[M][\Phi] = [I]$$

and

$$[\Phi]^T[K][\Phi] = [\Omega]$$

in which $[\Phi]$ is the modal matrix of the system and $[\Omega]$ is a diagonal matrix containing the eigenvalues ω_i^2 in the main diagonal.

For a dynamic system with only a few degrees of freedom, the natural frequencies and modal shapes may be determined by expanding the determinant and calculating the roots of the resulting characteristic equation. However, for a system with a large number of degrees of freedom, this direct method of solution becomes impractical. It is then necessary to resort to other numerical methods which usually require an iteration process.

7.6 Problems

Problem 7.1 Determine the natural frequencies and normal modes for the two-story shear building shown in Fig. P7.1.

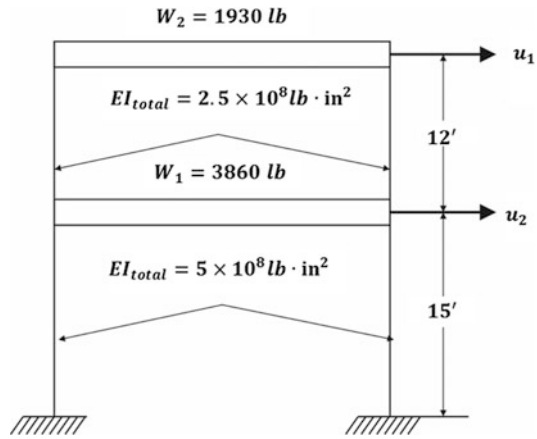

Fig. P7.1

Problem 7.2
A certain structure has been modeled as a three-degree-of-freedom system having the numerical values indicated in Fig. P7.2. Determine the natural frequencies of the structure and the corresponding normal modes. Check your answer using MATLAB.

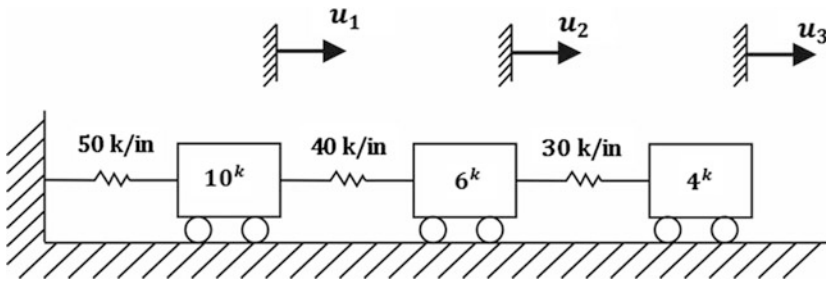

Fig. P7.2

Problem 7.3

Assume a shear building model for the frame shown in Fig. P7.3 and determine the natural frequencies and normal modes.

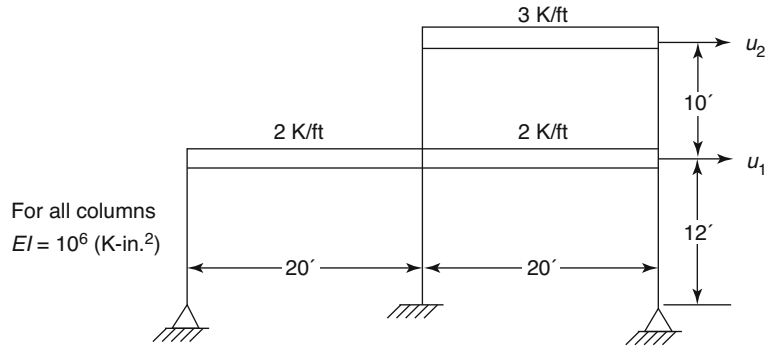

Fig. P7.3

Problem 7.4

Assume a shear building model for the frame shown in Fig. P7.4 and determine the natural frequencies and normal modes.

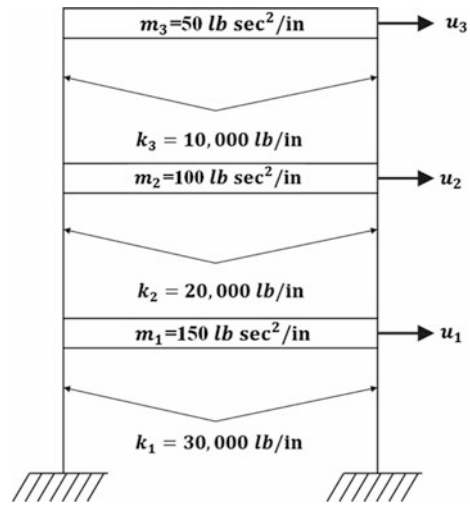

Fig. P7.4

Problem 7.5
Consider the uniform shear building in which the mass of each floor is m and the stiffness of each story is k. Determine the general form of the system of differential equations for a uniform shear building of N stories.

Problem 7.6
Find the natural frequencies and modal shapes for the three-degree-of-freedom shear building in Fig. P7.6.

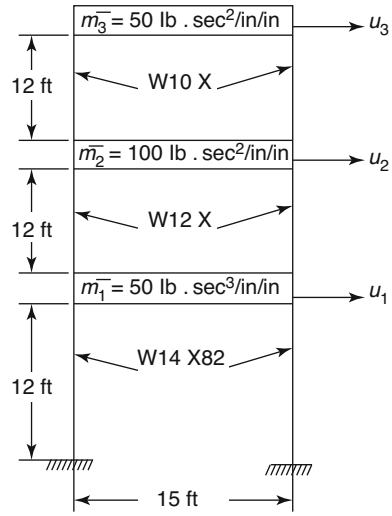

Fig. P7.6

Problem 7.7
Use the results of Problem 7.6 to write the expressions for the free vibration displacements u_1, u_2, and u_3 of the shear building in Fig. P7.6 in terms of constants of integration.

Problem 7.8
Use MATLAB to determine the natural frequencies for the six-story uniform shear building modeled as a column shown in Fig. P7.8.

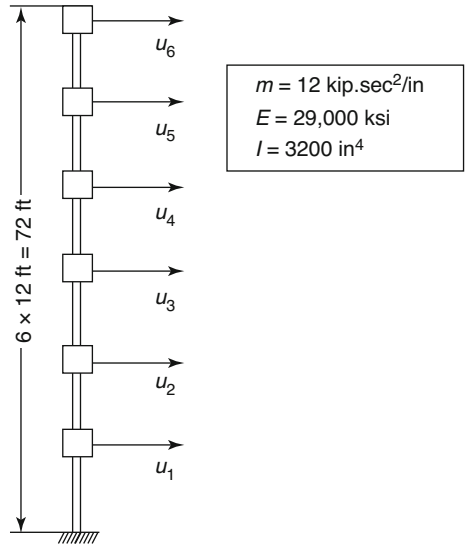
u_6
u_5
u_4
u_3
u_2
u_1
6 × 12 ft = 72 ft
m = 12 kip.sec^2/in
E = 29,000 ksi
I = 3200 in^4

Fig. P7.8

Check for updates

Forced Motion of Shear Buildings

8

In the preceding chapter, we have shown that the free motion of a dynamic system may be expressed in terms of the normal modes of vibration. Our present interest is to demonstrate that the forced motion of such a system may also be expressed in terms of the normal modes of vibration and that the total response may be obtained as the superposition of the solution of independent modal equations. In other words, our aim in this chapter is to show that the normal modes may be used to transform the system of coupled differential equations into a set of uncoupled differential equations in which each equation contains only one dependent variable. Thus the modal superposition method reduced the problem of finding the response of a multi-degree-of-freedom system to the determination of the response of single-degree-of-freedom systems.

8.1 Modal Superposition Method

In Chap. 6, we have shown that any free motion of a multi-degree-of-freedom system may be expressed in terms of normal modes of vibration. It will now be demonstrated that the forced motion of such a system may also be expressed in terms of the normal modes of vibration. We return to the equations of motion, Eq. (3.40), which for the particular case of a two-degree-of-freedom shear building are

$$\begin{aligned} m\ddot{u}_1 + (k_1 + k_2) - k_2 u_2 &= F_1(t) \\ m_2\ddot{u}_2 - k_2 u_1 + k_2 u_2 &= F_2(t) \end{aligned} \tag{8.1}$$

We seek to transform this coupled system of differential equations into a system of independent or uncoupled equations in which each equation contains only one unknown function of time. It is first necessary to express the solution in terms of the normal modes multiplied by some factors determining the contributions of each mode. In the case of free motion, these factors were sinusoidal functions of time; in the present case, for forced motion they are general functions of time which we designate as $q_i(t)$. Hence, the solution of Eq. (8.1) is assumed to be of the form

$$\begin{aligned} u_1(t) &= a_{11}q_1(t) + a_{12}q_2(t) \\ u_2(t) &= a_{21}q_1(t) + a_{22}q_2(t) \end{aligned} \tag{8.2}$$

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M. Paz, Y. H. Kim, *Structural Dynamics*, https://doi.org/10.1007/978-3-319-94743-3_8

Upon substitution into Eq. (8.1), we obtain

$$\begin{aligned} &m_1a_{11}\ddot{q}_1 + (k_1+k_2)a_{11}q_1 - k_2a_{21}q_1 + m_1a_{12}\ddot{q}_2 + (k_1+k_2)a_{12}q_2 - k_2a_{22}q_2 = F_1(t) \\ &m_2a_{21}\ddot{q}_1 - k_2a_{11}q_1 + k_2a_{21}q_1 + m_2a_{22}\ddot{q}_2 - k_2a_{12}q_2 + k_2a_{22}q_2 = F_2(t) \end{aligned} \tag{8.3}$$

To determine the appropriate functions $q_1(t)$ and $q_2(t)$ that will uncouple Eq. (8.3), it is necessary to make use of the orthogonality relations to separate the modes. The orthogonality relations are used by multiplying the first of Eq. (8.3) by a_{11} and the second by a_{21}. Addition of these equations after multiplication and simplification by using Eqs. (6.3) and (6.5) yields

$$\left(m_1a_{11}^2 + m_2a_{21}^2\right)\ddot{z}_1 + \omega_1^2\left(m_1a_{11}^2 + m_2a_{21}^2\right)z_1 = a_{11}F_1(t) + a_{21}F_2(t) \tag{8.4a}$$

Similarly, multiplying the first of Eq. (8.3) by a_{12} and the second by a_{22}, we obtain

$$\left(m_1a_{12}^2 + m_2a_{22}^2\right)\ddot{z}_2 + \omega_2^2\left(m_1a_{12}^2 + m_2a_{22}^2\right)z_2 = a_{12}F_1(t) + a_{22}F_2(t) \tag{8.4b}$$

The results obtained in Eq. (8.4) permit a simple physical interpretation. The force that is effective in exciting a mode is equal to the work done by the external force displaced by the modal shape in question. From the mathematical point of view, what we have accomplished is to separate or uncouple, by a change of variables, the original system of differential equations. Consequently, each of these equations, Eqs. (8.4a) and (8.4b), corresponds to a single-degree-of-freedom system which may be written as

$$\begin{aligned} M_1\ddot{q}_1 + K_1q_1 &= P_1(t) \\ M_2\ddot{q}_2 + K_2q_2 &= P_2(t) \end{aligned} \tag{8.5}$$

where $M_1 = m_1a_{11}^2 + m_2a_{21}^2$ and $M_2 = m_2a_{12}^2 + m_2a_{22}^2$ are the modal masses; $K_1 = \omega_1^2M_1$ and $K_2 = \omega_2^2M_2$, the modal spring constants; $P_1(t) = a_{11}F_1(t) + a_{21}F_2(t)$ and $P_2(t) = a_{12}F_1(t) + a_{22}F_2(t)$ the modal forces. Alternatively, if we make use of the previous normalization in Eq. (7.15) or (7.16), these equation may be written simply as

$$\begin{aligned} \ddot{q}_1 + \omega_1^2q_1 &= P_1(t) \\ \ddot{q}_2 + \omega_2^2q_2 &= P_2(t) \end{aligned} \tag{8.6}$$

where P_1 and P_2 are now given by

$$\begin{aligned} P_1 &= \phi_{11}F_1(t) + \phi_{21}F_2(t) \\ P_2 &= \phi_{12}F_1(t) + \phi_{22}F_2(t) \end{aligned} \tag{8.7}$$

The solution for the uncoupled differential equations, Eq. (8.5) or Eq. (8.6), may now be found by any of the methods presented in the previous chapters for the solution of a single degree of freedom system. In particular, Duhamel's integral provides a general solution for these equations regardless of the functions describing the forces acting on the structure. Also, maximum values of the response for each modal equation may readily be obtained using available response spectra. However, the superposition of modal maximum responses presents a problem. The fact is that these modal maximum values will in general not occur simultaneously as the transformation of coordinates, Eq. (8.2), requires. To obviate the difficulty, it is necessary to use an approximate method. An upper limit for the maximum response may be obtained by adding the absolute values of the maximum

modal contributions, that is, by substituting q_1 and q_2 in Eq. (8.2) for the maximum modal responses, $q_{1\max}$ and $q_{2\max}$, and adding the absolute values of the terms in these equations, so that

$$\begin{aligned} u_{1\max} &= |\phi_{11} q_{1\max}| + |\phi_{12} q_{2\max}| \\ u_{2\max} &= |\phi_{21} q_{1\max}| + |\phi_{22} q_{2\max}| \end{aligned} \tag{8.8}$$

The results obtained by this method will generally overestimate the maximum response. Another method, which is widely accepted and which generally gives a reasonable estimate of the maximum response from these spectral values, is the Square Root of the Sums of Squares of the modal contributions (SRSS method). Thus the maximum displacements may be approximated by

$$u_{1\max} = \sqrt{(\phi_{11} q_{1\max})^2 + (\phi_{12} q_{2\max})^2}$$

and

$$u_{2\max} = \sqrt{(\phi_{21} q_{1\max})^2 + (\phi_{22} q_{2\max})^2} \tag{8.9}$$

The results obtained by application of the SRSS method (Square Root of the Sum of the Squares of modal contributions) may substantially underestimate or overestimate the total response when two or more modes are closely spaced. In this case, another method known as the Complete Quadratic Combination (CQC) for combining modal responses to obtain the total response is recommended. The discussion of such a method is presented in this chapter in Sect. 8.6

The transformation from a system of two coupled differential equations, Eq. (8.1), to a set of two uncoupled differential equations, Eq. (8.6), may be extended to a general and system of N degrees of freedom. For such a system, it is particularly convenient to use matrix notation. With such notation, the equation of motion for a linear system of N degrees of freedom is given by Eq. (7.3) as

$$[M]\{\ddot{u}\} + [K]\{u\} = \{F(t)\} \tag{8.10}$$

where $[M]$ and $[K]$ are respectively the mass and the stiffness matrices of the system, $\{F(t)\}$ the vector of external forces, and $\{u\}$ the vector of unknown displacements at the nodal coordinates. Introducing into Eq. (8.10) the linear transformation of coordinates

$$\{u\} = [\Phi]\{q\} \tag{8.11}$$

in which $[\Phi]$ is the modal matrix of the system, yields

$$[M][\Phi]\{\ddot{q}\} + [K][\Phi]\{q\} = [F(t)\} \tag{8.12}$$

The pre-multiplication of Eq. (8.12) by the transpose of the ith modal vector, $\{\phi\}_i^T$, results in

$$\{\phi\}_i^T\ [M][\Phi]\{\ddot{q}\} + \{\phi\}_i^T [K][\Phi]\{q\} = \{\phi\}_i^T \{F(t)\} \tag{8.13}$$

The orthogonality conditions between normalized modes, Eqs. (7.17) and (7.19), imply that

$$\{\phi\}_i^T\ [M][\Phi] = 1 \tag{8.14}$$

and

$$\{\phi\}_i^T [K][\Phi] = \omega_i^2 \tag{8.15}$$

Consequently, Eq. (8.13) may be written as

$$\ddot{q}_i + \omega_i^2 q_i = P_i(t) \quad i = 1, 2, 3, \ldots, N \tag{8.16}$$

where the modal force $P_i(t)$ is given by

$$P_i(t) = \phi_{1i} F_1(t) + \phi_{2i} F_2(t) + \ldots + \phi_{Ni} F_N(t) \tag{8.17}$$

Equation (8.16) constitutes a set of N uncoupled or independent equations of motion in terms of the modal coordinates q_i. These uncoupled equations, as may be observed, may readily be written after the natural frequencies ω_i and the modal vectors, $\{\phi\}_i$ have been determined in the solution of the corresponding eigenproblem as presented in Chap. 7.

Illustrative Example 8.1

The two-story frame of Illustrative Example 7.1 is acted upon at the floor levels by horizontal triangular impulsive forces as shown in Fig. 8.1. For this frame, determine the maximum floor displacements and the maximum shear forces in the columns.

Solution:

The results obtained in Illustrative Examples 7.1 and 7.2 for the free vibration of this frame gave the following values for the natural frequencies and normalized modes:

$$\omega_1 = 11.83\,\text{rad/ sec}, \quad \omega_2 = 32.89\,\text{rad/ sec}$$
$$\phi_{11} = 0.06437, \qquad \phi_{12} = 0.0567$$
$$\phi_{21} = 0.08130, \qquad \phi_{22} = -0.0924$$

The forces acting on the frame which are shown in Fig. 8.1b may be expressed by

$$F_1(t) = 10,000(1 - t/t_d)\,\text{lb}$$
$$F_2(t) = 20,000(1 - t/t_d)\,\text{lb} \quad \text{for } t \leq 0.1\,\text{sec}$$

in which $t_d = 0.1$ sec and

$$F_1(t) = F_2(t) = 0, \qquad \text{for } t > 0.1\,\text{sec}$$

The substitution of these values into the uncoupled equations of motion, Eqs. (8.6) and (8.7) gives

$$\ddot{q}_1 + 140 q_1 = 2270 f(t)$$
$$\ddot{q}_2 + 1082.41 q_2 = -1281 f(t)$$

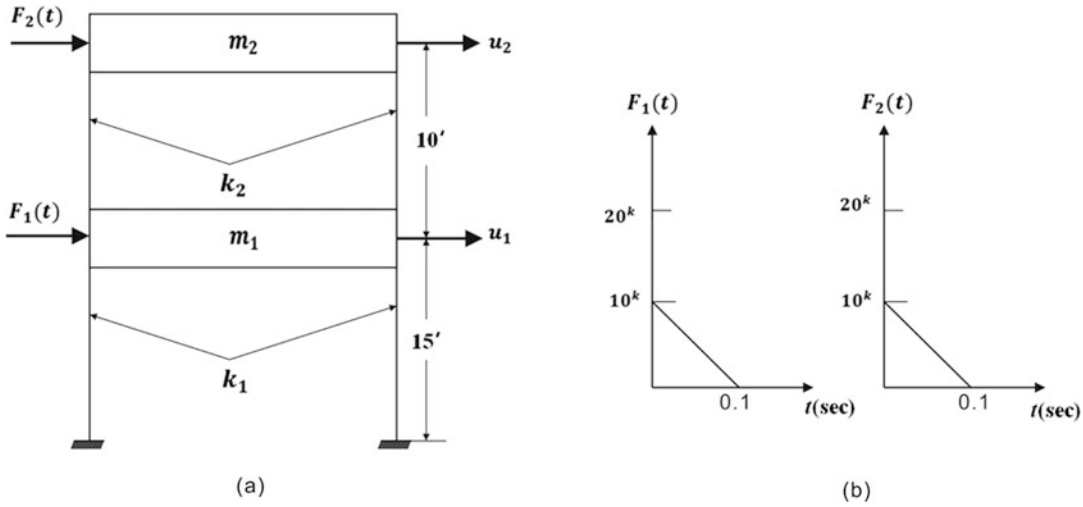

Fig. 8.1 Shear building with impulsive loadings. (**a**) Two-story shear building. (**b**) Impulsive loadings

in which $f(t) = 1 - t/t_d$ for $t \leq 0.1$ and $f(t) = 0$ for $t > 0.1$. The maximum values for q_1 and q_2 are then obtained from available spectral charts such as the one shown in Fig. 4.5. For this example,

$$\frac{t_d}{T_1} = \frac{0.1}{0.532} = 0.188$$

and

$$\frac{t_d}{T_2} = \frac{0.1}{0.191} = 0.524$$

in which the modal natural periods are calculated as

$$T_1 = \frac{2\pi}{\omega_1} = 0.532\,\text{sec} \quad \text{and} \quad T_2 = \frac{2\pi}{\omega_2} = 0.191\,\text{sec}$$

From Fig. 4.5, we obtain:

$$(DLF)_{1\max} = \frac{q_{1\max}}{q_{1st}} = 0.590$$

$$(DLF)_{2\max} = \frac{q_{2\max}}{q_{2st}} = 1.22$$

where the static deflections, q_{1st}, and q_{2st}, are calculated as

$$q_{1st} = \frac{F_{01}}{\omega_1^2} = \frac{2270}{140} = 16.3, \quad q_{2st} = \frac{F_{02}}{\omega_2^2} = \frac{-1281}{1082.41} = -1.18$$

Then the maximum values of the modal response are:

$$q_{1\max} = 0.590 \times 16.3 = 9.62, \quad q_{2\max} = 1.22 \times 1.18 = 1.44$$

As indicated above these maximum modal values do not occur simultaneously and therefore cannot simply be superimposed to obtain the maximum response of the system. However, an upper limit for the absolute maximum displacement may be calculated with Eq. (8.8) as

$$u_{1\max} = |0.06437 \times 9.62| + |0.0567 \times 1.44| = 0.70 \text{ in}$$

$$u_{2\max} = |0.08130 \times 9.62| + |-0.0924 \times 1.44| = 0.92 \text{ in}$$

A second acceptable estimate of the maximum response is obtained by taking the square root of the sum of the squared modal contributions as indicated by Eq. (8.9). For this example, we have

$$\begin{aligned} u_{1\max} &= \sqrt{(0.06437 \times 9.62)^2 + (0.0567 \times 1.44)^2} = 0.62 \text{ in} \\ u_{2\max} &= \sqrt{(0.08130 \times 9.62)^2 + (-0.0924 \times 1.44)^2} = 0.79 \text{ in} \end{aligned} \tag{a}$$

The maximum shear force $V_{\max}$ in the columns is given by

$$V_{\max} = k\Delta_u \tag{8.18}$$

in which k is the stiffness of the story and Δ_u the difference between the displacements at the two ends of the column. Since the maximum displacements calculated as in Eq. (a) may have positive or negative values, the relative displacement Δ_u cannot be determined as the difference of the absolute displacements of the two ends of the column. The maximum positive value for Δ_u could be estimated as the sum of the absolute maximum displacements at the ends of the columns. However, this procedure will in most cases greatly overestimate the actual forces in the columns. The recommended procedure is to calculate first the shear force in the columns for each mode, separately, and then combine these modal forces by a suitable method, such as the square root of the sum of the squares of modal contributions. This procedure is based on the fact that modal displacements are known with their correct relative sign and not as absolute values.

The maximum shear force V_{ij} at story i corresponding to mode j is given by

$$V_{ij} = q_{j\max}(\phi_{ij} - \phi_{i-1j})k_i \tag{8.19}$$

where $q_{j\max}$ is the maximum modal response, $(\phi_{ij} - \phi_{i-1j})$ the relative modal displacement of story i (with $\phi_{0j} = 0$), and k_i the stiffness of the story. For this example we have for the first story

$$k_1 = \frac{12EI_1}{L_1^3} = \frac{12 \times 30 \times 10^6 \times 248.6}{(15 \times 12)^3} = 15,345 \text{ lb/in}$$

$$V_{11} = 9.62 \times 0.06437 \times 15,345 = 9502 \text{ lb}$$

$$V_{12} = 1.44 \times 0.0567 \times 15,345 = 1253 \text{ lb}$$

$$V_{1\max} = \sqrt{9502^2 + 1253^2} = 9584 \text{ lb}$$

and for a column in the second story

$$k_2 = \frac{12EI_2}{L_2^3} = \frac{12 \times 30 \times 10^6 \times 106.3}{(10 \times 12)^3} = 22,146 \text{ lb}$$

$$V_{21} = 9.62 \times (0.08130 - 0.06437) \times 22,146 = 3607 \text{ lb}$$

$$V_{22} = 1.44 \times (-0.0924 - 0.0567) \times 22,146 = -4755 \; lb$$

$$V_{2\max} = \sqrt{3607^2 + 4755^2} = 5968 \text{ lb}$$

8.2 Response of a Shear Building to Base Motion

The response of a shear building to the base or foundation motion is conveniently obtained in terms of floor displacements relative to the base motion. For the two-story shear building of Fig. 8.2a, which is modeled as shown in Fig.8.2b, the equations of motion obtained by equating to zero the sum of forces in the free body diagrams of Fig. 8.2c are the following:

$$\begin{aligned} m_1\ddot{u}_1 + k_1(u_1 - u_s) - k_2(u_2 - u_1) &= 0 \\ m_2\ddot{u}_2 + k_2(u_2 - u_1) &= 0 \end{aligned} \tag{8.20}$$

where $u_s = u_s(t)$ is the displacement imposed at the foundation of the structure. Expressing the floor displacements relative to the base motion, we have

$$\begin{aligned} u_{r1} &= u_1 - u_s \\ u_{r2} &= u_2 - u_s \end{aligned} \tag{8.21}$$

The differentiation yields

$$\begin{aligned} \ddot{u}_1 &= \ddot{u}_{r1} + \ddot{u}_s \\ \ddot{u}_2 &= \ddot{u}_{r2} + \ddot{u}_s \end{aligned} \tag{8.22}$$

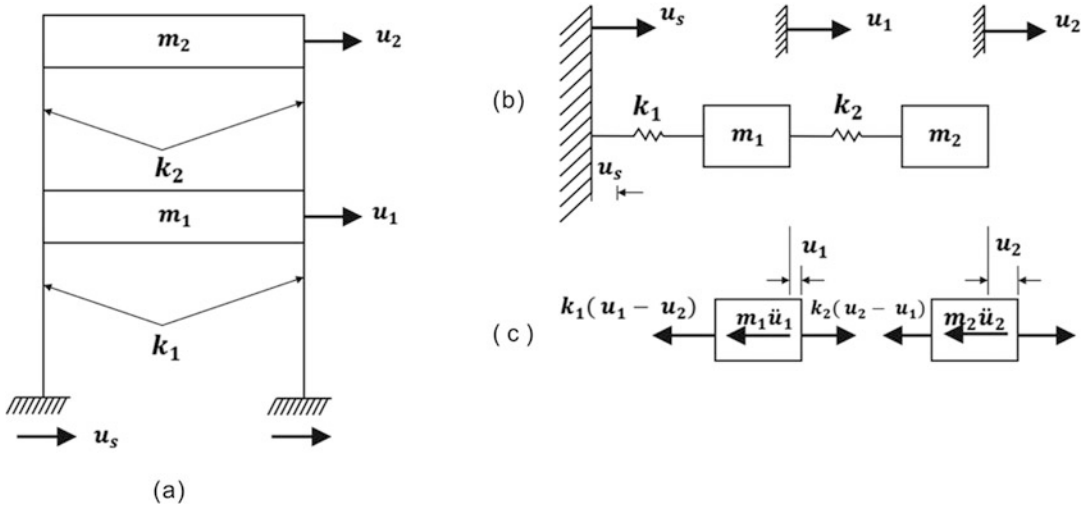

Fig. 8.2 Shear building with base motion. (**a**) Two-story shear building. (**b**) Mathematical model. (**c**) Free-body diagrams

Substitution of Eqs. (8.21) and (8.22) into Eq. (8.20) results in

$$\begin{aligned} m_1\ddot{u}_{r1} + (k_1 + k_2)u_{r1} - k_2u_{r2} &= -m_1\ddot{u}_s \\ m_2\ddot{u}_{r2} - k_2u_{r1} + k_2u_{r2} &= m_2\ddot{u}_s \end{aligned} \tag{8.23}$$

We note that the right-hand sides of Eq. (8.23) are proportional to the same function of time, $\ddot{u}_s(t)$. This fact leads to a somewhat simpler solution compared to the solution of Eq. (8.6), which may contain different functions of time in each equation. For the base motion of the shear building, Eq. (8.4) may be written as

$$\begin{aligned} \ddot{q}_1 + \omega_1^2 q_1 &= -\frac{m_1a_{11} + m_2a_{211}}{m_1a_{11}^2 + m_2a_{21}^2}\ddot{u}_s(t) \\ \ddot{q}_2 + \omega_2^2 q_2 &= -\frac{m_1a_{12} + m_2a_{22}}{m_1a_{12}^2 + m_2a_{22}^2}\ddot{u}_s(t) \end{aligned} \tag{8.24}$$

or

$$\begin{aligned} \ddot{q}_1 + \omega_1^2 q_2 &= \Gamma_1\ddot{u}_s(t) \\ \ddot{q}_2 + \omega_2^2 q_2 &= \Gamma_2\ddot{u}_s(t) \end{aligned} \tag{8.25}$$

where Γ_1 and Γ_2 are called participation factors and are given by

$$\begin{aligned} \Gamma_1 &= -\frac{m_1a_{11} + m_2a_{21}}{m_1a_{11}^2 + m_2a_{21}^2} \\ \Gamma_2 &= -\frac{m_1a_{12} + m_2a_{22}}{m_1a_{12}^2 + m_2a_{22}^2} \end{aligned} \tag{8.26}$$

The relation between the modal displacements q_1, q_2 and the relative displacement u_{r1}, u_{r2} is given from Eq. (8.2) as

$$\begin{aligned} u_{r1} &= a_{11}q_1 + a_{12}q_2 \\ u_{r2} &= a_{21}q_1 + a_{22}q_2 \end{aligned} \tag{8.27}$$

In practice, it is convenient to introduce a change of variables in Eq. (8.25) such that the second members of these equations equal $\ddot{u}_s(t)$. The required change of variables to accomplish this simplification is

$$\begin{aligned} q_1 &= \Gamma_1 g_1 \\ q_2 &= \Gamma_2 g_2 \end{aligned} \tag{8.28}$$

which when introduced into Eq. (8.25) gives

$$\begin{aligned} \ddot{g}_1 + \omega_1^2 g_1 &= \ddot{u}_s(t) \\ \ddot{g}_2 + \omega_2^2 g_2 &= \ddot{u}_s(t) \end{aligned} \tag{8.29}$$

Finally, solving for $g_1(t)$ and $g_2(t)$ in the uncoupled Eq. (8.29) and substituting the solution into Eqs. (8.27) and (8.28) give the response as

$$u_{r1}(t) = \Gamma_1 a_{11} g_1(t) + \Gamma_2 a_{12} g_2(t)$$
$$u_{r2}(t) = \Gamma_1 a_{21} g_1(t) + \Gamma_2 a_{22} g_2(t) \tag{8.30}$$

When the maximum modal response $g_{1\max}$ and $g_{2\max}$ are obtained from spectral charts, we may estimate the maximum values $u_{r1\max}$ and $u_{r2\max}$ by the SRSS combination method as

$$u_{r1\max} = \sqrt{(\Gamma_1 a_{11} g_{1\max})^2 + (\Gamma_2 a_{12} g_{2\max})^2}$$
$$u_{r2\max} = \sqrt{(\Gamma_1 a_{21} g_{1\max})^2 + (\Gamma_2 a_{22} g_{2\max})^2} \tag{8.31}$$

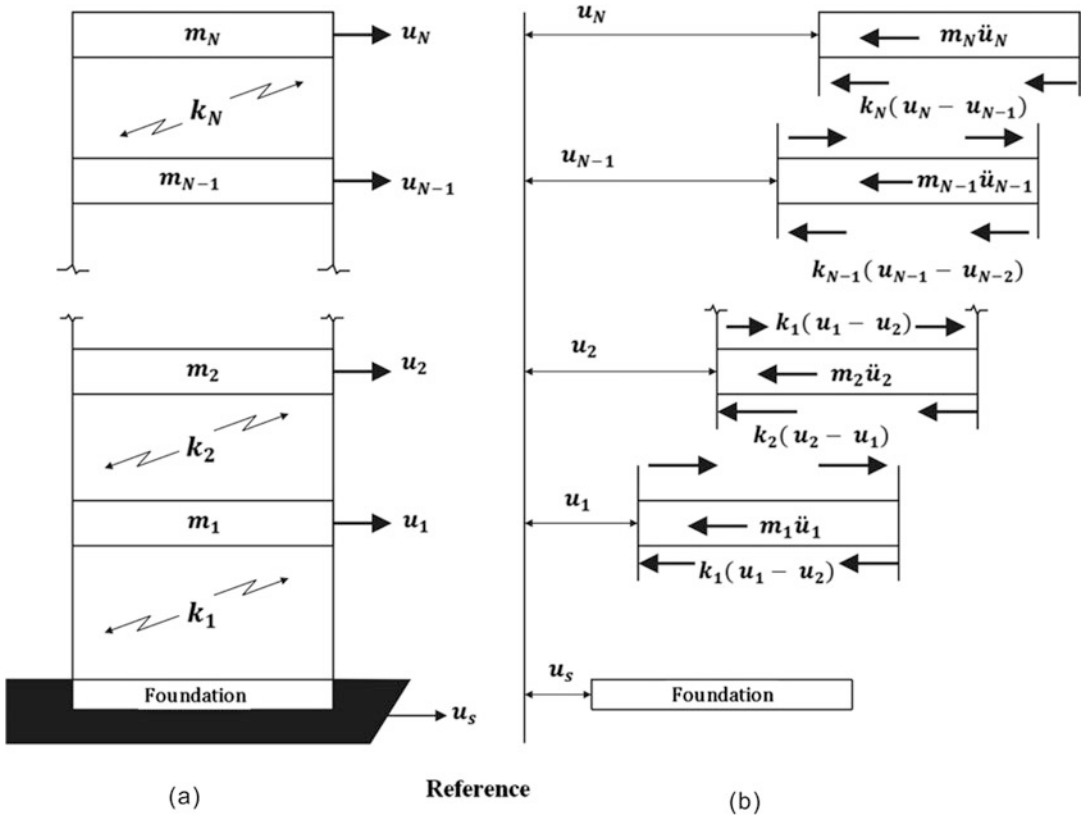

Fig. 8.3 Multistory shear building excited at the foundation (**a**) Structural Model. (**b**) Free body diagrams

The equations of motion for an N-story shear building (Fig. 8.3a) subjected to excitation motion at its base are obtained by equating to zero the sum of forces shown in the free body diagrams of Fig. 8.3b, namely

$$
\begin{aligned}
&m_1\ddot{u}_1 + k_1(u_1 - u_s) - k_2(u_2 - u_1) = 0\\
&m_2\ddot{u}_2 + k_2(u_2 - u_1) - k_3(u_3 - u_2) = 0\\
&\qquad \cdots\\
&m_{N-1}\ddot{u}_{N-1} + k_{N-1}(u_{N-1} - u_{N-2}) - k_N(u_N - u_{N-1}) = 0\\
&m_N\ddot{u}_N + k_N(u_N - u_{N-1}) = 0
\end{aligned}
\tag{8.32}
$$

Introducing into Eq. (8.32)

$$
u_{ri} = u_i - u_s \quad (i = 1, 2, \ldots, N) \tag{8.33}
$$

results in

$$
\begin{aligned}
&m_1\ddot{u}_{r1} + k_1u_{r1} - k_2(u_{r2} - u_{r1}) = -m_1\ddot{u}_s\\
&m_2\ddot{u}_{r2} + k_2u_{r2} - k_3(u_{r3} - u_{r2}) = -m_2\ddot{u}_s\\
&\qquad \cdots\\
&m_{N-1}\ddot{u}_{rN-1} + k_{N-1}(u_{rN-1} - u_{rN-2}) - k_N(u_{rN} - u_{rN-1}) = -m_{N-1}\ddot{u}_s\\
&m_N\ddot{u}_{rN} + k_N(u_{rN} - u_{rN-1}) = -m_N\ddot{u}_s
\end{aligned}
\tag{8.34}
$$

where $\ddot{u}_s = \ddot{u}_s(t)$ is the acceleration function exciting the base of the structure.

Equations (8.34) may be conveniently be written in matrix notation as

$$
[M]\{\ddot{u}_r\} + [K]\{u_r\} = -[M]\{1\}\ddot{u}_s(t) \tag{8.35}
$$

in which $[M]$, the mass matrix, is a symmetric matrix, $\{1\}$ is a vector with all its elements equal to 1, $\ddot{u}_s = \ddot{u}_s(t)$ is the applied acceleration at the foundation of the building, and $\{u_r\}$ and $\{\ddot{u}_r\}$ are, respectively, the displacement and acceleration vectors relative to the motion of the foundation.

As has been demonstrated, the system of differential equations (8.35) can be uncoupled through the transformation given by Eq. (8.11) as

$$
\{u_r\} = [\Phi]\{q\} \tag{8.36}
$$

where $[\Phi]$ is the modal matrix obtained in the solution of corresponding eigenproblem $[[K] - \omega^2[M]\{\phi\} = \{0\}$.

The substitution of Eq. (8.36) into Eq. (8.35) followed by premultiplication by the transpose of the ith eigenvector, $\{\phi\}_i^T$ (the ith modal shape), results in

$$
\{\phi\}_i^T[M][\Phi]\{\ddot{q}\} + \{\phi\}_i^T[K][\Phi]\{q\} = -\{\phi\}_i^T[M]\{1\}\ddot{u}_s(t) \tag{8.37}
$$

which upon introduction of orthogonality property of the normalized eigenvectors [Eqs. (8.14) and (8.15)] results in the modal equations

$$
\ddot{z}_i + \omega_1^2 z_i = \Gamma\ddot{u}_s(t) \qquad (i = 1, 2, \ldots, N) \tag{8.38}
$$

Where the modal participation factor Γ_i is given in general by

$$\Gamma_i = \frac{\sum_{j=1}^{N} m_j \phi_{ji}}{\sum_{j=1}^{N} m_j \phi_{ji}^2} \tag{8.39}$$

and for normalized eigenvectors by

$$\Gamma_i = \sum_{j=1}^{N} m_j \phi_{ji} \quad (i = 1, 2, \ldots, N) \tag{8.40}$$

The maximum response in terms of maximum values for displacements ($u_{\mathrm{r\ i\ max}}$) or for acceleration ($\ddot{u}_{i\ \mathrm{max}}$) at the modal coordinates calculated by the SRSS method, is then given, respectively, by

$$u_{r i \max} = \sqrt{\sum_{j=1}^{N} \left(\Gamma_j \phi_{ij} S_{Dj}\right)^2} \tag{8.41}$$

and

$$\ddot{u}_{i \max} = \sqrt{\sum_{j=1}^{N} \left(\Gamma_j \phi_{ij} S_{Aj}\right)^2} \tag{8.42}$$

where S_{Dj} and S_{Aj} are, respectively, the spectral displacement and spectral acceleration for the jth mode.

The participation factors Γ_j indicated in Eqs. (8.39) and (8.40) are the coefficients of the excitation function $\ddot{u}_s(t)$ in Eq. (8.38). As presented in Chap. 5, response spectral charts are prepared as the solution of Eq. (8.38) (with $\Gamma_i = 1$). Therefore, the spectral values obtained from these charts S_{Dj} or S_{Aj} should be multiplied as indicated in Eqs. (8.41) and (8.42) by the participation factor T_j, which was omitted in the calculation of spectral values.

Illustrative Example 8.2

Determine the response of the frame of Illustrative Example 8.1 shown in Fig. 8.2 when it is subjected to a suddenly applied constant acceleration $\ddot{u}_s = 0.28$ g at its base.

Solution:

The natural frequencies and corresponding normal modes from calculations in Examples 7.1 and 7.2 are

$$\omega_1 = 11.83 \text{ rad/ sec}, \qquad \omega_2 = 32.89 \text{ rad/ sec}$$
$$\phi_{11} = 0.06437, \qquad \phi_{12} = 0.0567$$
$$\phi_{21} = 0.08130, \qquad \phi_{22} = -0.0924$$

The acceleration acting at the base of this structure is

$$\ddot{u}_s = 0.28 \times 386 = 108.47 \text{ in/ sec}^2$$

The participation factors are calculated from Eq. (8.39) with the denominators set equal to unity since the modes are normalized. These factors are then

$$\begin{aligned} \Gamma_1 &= -(136 \times 0.06437 + 66 \times 0.08130) = -14.120 \\ \Gamma_2 &= -(136 \times 0.0567 - 66 \times 0.0924) = -1.613 \end{aligned} \tag{a}$$

The modal equations (8.29) are

$$\begin{aligned} \ddot{g}_1 + 140 g_1 &= 108.47 \\ \ddot{g}_2 + 1082 g_2 &= 108.47 \end{aligned} \tag{b}$$

and their solution, assuming zero initial conditions for velocity and displacement, is given by Eq. (4.5) as

$$\begin{aligned} g_1(t) &= \frac{108.47}{140}(1 - \cos 11.83t) \\ g_2(t) &= \frac{108.47}{1082}(1 - \cos 32.89t) \end{aligned} \tag{c}$$

The response in terms of the relative motion of the stories at the floor levels with respect to the displacement of the base is given as a function of time by Eqs. (8.27) and (8.28), as

$$u_{r1}(t) = -14.120 \times 0.06437 \times 0.775(1 - \cos\ 11.83t) - 1.613 \times 0.0567 \times 0.100(1 - \cos\ 32.89t)$$
$$u_{r2}(t) = -14.120 \times 0.08130 \times 0.775(1 - \cos\ 11.83t) + 1.613 \times 0.0924 \times 0.100(1 - \cos\ 32.89t)$$

or, upon simplification, as

$$\begin{aligned} u_{r1} &= -0.7135 + 0.704 \cos\ 11.83t + 0.009 \cos\ 32.89t \\ u_{r2} &= -0.874 + 0.900 \cos\ 11.83t - 0.015 \cos\ 32.89t \end{aligned} \tag{d}$$

In this example, due to the simple excitation function (a constant acceleration), it was possible to obtain a closed solution of the problem as a function of time. For a complex excitation function such as the one produced by an actual earthquake, it would be necessary to resort to numerical integration to obtain the response or to use response spectra if available. The maximum modal response is obtained for the present example when the cosine functions in Eq. (c) are set equal to -1. In this case the maximum modal response is then

$$\begin{aligned} g_{1\max} &= 1.55 \\ g_{2\max} &= 0.20 \end{aligned} \tag{e}$$

and the maximum response, calculated from the approximate formulas (8.31), is

$$\begin{aligned} u_{r1\max} &= 1.409 \text{ in} \\ u_{r2\max} &= 1.800 \text{ in} \end{aligned} \tag{f}$$

The possible maximum values for the response calculated from Eq. (d) by setting the cosines function to their maximum value results in

$$u_{1\max} = 1.426\,in$$
$$u_{2\max} = 1.789\,in \tag{g}$$

which for this particular example certainly compares very well with the approximate results obtained in Eq. (f) above.

8.3 Response by Modal Superposition Using MATLAB

MATLAB calculates the response of a linear system by superposition of the solutions of the modal equations. Before one can use this program, it is necessary to solve an eigenproblem to determine the natural frequencies and modal shapes of the structure. The program determines the response of the structure excited either by time-dependent forces applied at nodal coordinates or a time-dependent acceleration at the support of the structure.

Illustrative Example 8.3

$$\{u\} = [\Phi]\{q\}$$

$$\ddot{q}_i + \omega_i{}^2 q_i = \{\phi\}_i{}^T\{F\} = P_i$$

$$\Gamma_i{}' = \frac{\{\phi\}_i{}^T\{F\}}{\{\phi\}_i{}^T[M]\{\phi\}_i} = \{\phi\}_i{}^T\{F\}\ (i = 1, 2, \ldots N)$$

$m_1 = 136$ #-sec^2/in., $m_2 = 66$ #-sec^2/in., $k_1 = 30{,}700$ #/in., $k_2 = 44{,}300$ #/in.
$F_2 = 5000$ #, $F_1 = 0$
Use computer MATLAB to determine natural frequencies and normalized mode shapes, and the participation factors.

Solution:
The natural frequencies and the modal matrix for this structure as calculated in Example 10.1 are

$$\omega_1 = 11.83\ \text{rad/sec}$$
$$\omega_2 = 32.89\ \text{rad/sec}$$

and

$$[\Phi] = \begin{bmatrix} 0.0644 & 0.0567 \\ 0.0813 & -0.0924 \end{bmatrix}$$

The mass matrix is

$$[M] = \begin{bmatrix} 136 & 0 \\ 0 & 66 \end{bmatrix}$$

```
clc
clear all
close all

%%%%-GIVEN VALUES-%%%%

%%%Define Mass Matrix
M = [136 0; 0 66]

%%%Define Stiffness Matrix
K = [30700+44300 -44300;-44300 44300]

%%%Define Force Matrix
F =[0; 5000];
%_____________________________________________________________________
% Solve the eignevalue problem and normalized eigenvectors
%_____________________________________________________________________

%%%Solve for eigenvalues (D) and eigenvectors (a)
[a, D] = eig(K, M)

[omegas,k] = sort(sqrt(diag(D)));

%%%Natural frequencies
omegas =sqrt(D)

%%%{a}1 before changing the unity in the first DOF.
a1 = a(:,1);

%%%{a}2 before changing the unity in the first DOF.
a2 = a(:,2);

%%%Change the {a} wrt the unity in the first DOF.
a11 = 1;
a21 = a1(2,1)./a1(1,1);
a12 = 1;
a22 = a2(2,1)./a2(1,1);

a =[];

%%%Calculate the {a}
a(:,1) = [a11, a21];                  %[a11,a21]
a(:,2) = [a12, a22];                  %[a12,a22]

%%%aMa = {a}'*[M]*{a}
aMa = a'*M*a;                         %Eq.7.14

%%%Normalization factor
norm_1 = sqrt(aMa(1,1));
norm_2 = sqrt(aMa(2,2));

%%%Normalized eigenvectors
nom_phi(:,1) = 1./norm_1.*a(:,1);    %Eq.7.16 for the first mode
nom_phi(:,2)= 1./norm_2.*a(:,2);     %Eq.7.16 for the 2nd mode
nom_phi


%fMf = {f}'*[M]*{f}
fMf=nom_phi'*M*nom_phi

%Omega: [Omega]
Omega = D'*fMf;
%P
P = nom_phi'*F

%q_st
q_st = inv(Omega)*P
```

8.4 Harmonic Force Excitation

When the excitation, that is, the external forces or base motion, is harmonic (sine or cosine function), the analysis is quite simple and the response can readily be found without the use of modal analysis. Let us consider the two-story shear building as shown in Fig. 8.4 subjected to a single harmonic force $F = F_0 \sin \overline{\omega} t$ which is applied at the level of the second floor. In this case, Eq. (8.1) with $F_1(t) = 0$ and $F_2 = F_0 \sin \overline{\omega} t$ become

$$\begin{aligned} m_1 \ddot{u}_1 + (k_1 + k_2) u_1 - k_2 u_2 &= 0 \\ m_2 \ddot{u}_2 - k_2 u_1 + k_2 u_2 &= F_0 \sin \overline{\omega} t \end{aligned} \tag{8.43}$$

For the steady-state response we seek a solution of the form

$$\begin{aligned} u_1 &= U_1 \sin \overline{\omega} t \\ u_2 &= U_2 \sin \overline{\omega} t \end{aligned} \tag{8.44}$$

After substitution of Eq. (8.44) into Eq. (8.43) and cancellation of the common factor $\sin \overline{\omega} t$, we obtain

$$\begin{aligned} \left(k_1 + k_2 - m_1 \overline{\omega}^2\right) U_1 - k_2 U_2 &= 0 \\ -k_2 U_1 + \left(k_2 - m_2 \overline{\omega}^2\right) U_2 &= F_0 \end{aligned} \tag{8.45}$$

which is a system of two equation in two unknowns, U_1, and U_2. This system always has a unique solution except in the case when the determinant formed by the coefficients of the unknowns is equal to zero. The reader should remember that in this case the forced frequency $\overline{\omega}$ would equal one of the natural frequencies, since this determinant when equated to zero is precisely the condition used for determining the natural frequencies. In other words, unless the structure is forced to vibrate at one of the resonant frequencies, the algebraic system of Eq. (8.43) has a unique solution for U_1, and U_2.

Illustrative Example 8.4

Determine the steady-state response of the two-story shear building of Illustrative Example 7.1 when a force $F_2(t) = 10{,}000 \sin 20t$ is applied to the second story as shown in Fig. 8.4.

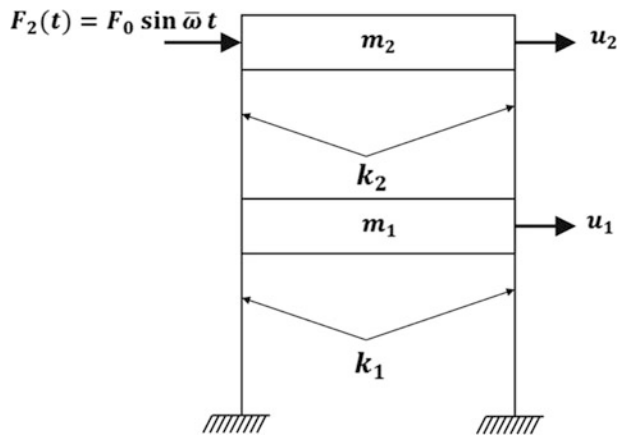

Fig. 8.4 Shear building with harmonic load

Solution:
The natural frequencies for this frame were determined in Illustrative Example 7.1 to be

$$\omega_1 = 11.83 \text{ rad/ sec}$$
$$\omega_2 = 32.89 \text{ rad/ sec}$$

Since the forcing frequency is 20 rad/sec, the system is not at resonance. The steady-state response is then given by solving Eq. (8.45) for U_1 and U_2. substituting numerical values in this system or equations, we have

$$\left(75,000 - 136 \times 20^2\right)U_1 - 44,300U_2 = 0$$
$$-44,300U_1 + \left(44,300 - 66 \times 20^2\right)U_2 = 10,000$$

Solving these equations simultaneously results in

$$U_1 = -0.28 \text{ in}, \qquad U_2 = -0.13 \text{ in}$$

Therefore, according to Eq. (8.44), the steady-state response is

$$u_1 = -0.28 \sin 20t \text{ in}$$
$$u_2 = -0.13 \sin 20t \text{ in} \qquad \text{(Ans)}$$

Damping may be considered in the analysis by simply including damping elements in the model as it is shown in Fig. 8.5 for a two-story shear building. The equations of motion which are obtained by equating to zero the sum of the forces in the free body diagram shown in Fig. 8.5c are

$$\begin{aligned} m_1\ddot{u}_1 + (c_1 + c_2)\dot{u}_1 + (k_1 + k_2)u_1 - c_2\dot{u}_2 - k_2u_2 &= F_1(t) \\ m_2\ddot{u}_2 - c_2\dot{u}_1 - k_2u_1 + c_2\dot{u}_2 + k_2u_2 &= F_2(t) \end{aligned} \tag{8.46}$$

Now, considering the general case of applied forces of the form given by

$$F(t) = F_c \cos \overline{\omega} t + F_s \sin \overline{\omega} t \tag{8.47}$$

we, conveniently, express such force in complex form as

$$F(t) = (F_c - iF_s)e^{i\omega t} \tag{8.48}$$

with the tacit understanding that only the real part of the Eq. (8.48) is the applied force. We show that the real part of the complex force in Eq. (8.48) is precisely the force in Eq. (8.47). Using Euler's formula $e^{i\overline{\omega}t} = \cos \overline{\omega} t + i \sin \overline{\omega} t$, we obtain

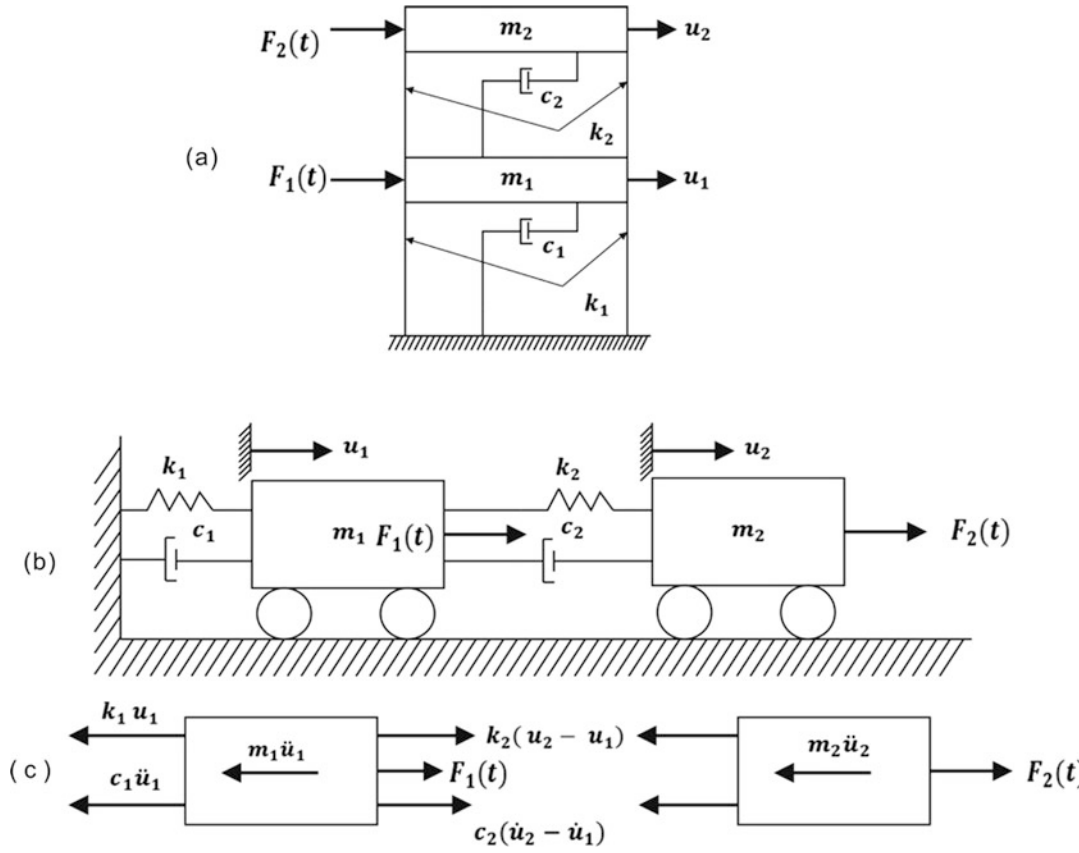

Fig. 8.5 (**a**) Damped shear building with harmonic load. (**b**) Multi-degree mass-spring model. (**c**) Free body diagrams

$$\begin{aligned}\text{Real}\left\{(F_c - iF_s)e^{i\overline{\omega}t}\right\} &= \text{Real}\left\{(F_c - i\text{F}_\text{s})\left(\cos\overline{\omega}t + i\sin\ \overline{\omega}t\right)\right\} \\ &= F_c\ \cos\overline{\omega}t + F_s\sin\overline{\omega}t\end{aligned} \tag{8.49}$$

which is equal to the expression in Eq. (8.47).

Assuming that the forces $F_1(t)$ and $F_2(t)$ in Eq. (8.46) are in the form given by Eq. (8.47), we substitute Eq. (8.48) into Eq. (8.46) to obtain

$$\begin{aligned}m_1\ddot{u}_1 + (c_1 + c_2)\dot{u}_1 + (k_1 + k_2)u_1 - c_2\dot{u}_2 - k_2u_2 &= (F_{c1} - iF_{s1})e^{i\overline{\omega}t} \\ m_2\ddot{u}_2 - c_2\dot{u}_1 - k_2u_1 + c_2\dot{u}_2 + k_2u_2 &= (F_{c2} - iF_{s2})e^{i\overline{\omega}t}\end{aligned} \tag{8.50}$$

The solution of the complex system of Eq. (8.50) will, in general, be of the form of

$$\begin{aligned}\text{u}_1(\text{t}) &= (\text{U}_{\text{c1}} + \text{iU}_{\text{s1}}).e^{i\overline{\omega}t} \\ \text{u}_2(\text{t}) &= (\text{U}_{\text{c2}} + \text{iU}_{\text{s2}})\,e^{i\overline{\omega}t}\end{aligned} \tag{8.51}$$

The substitution of Eq. (8.51) together with the first and second derivatives of u_1 and u_2 into Eq. (8.50) results in the following system of complex algebraic equations:

$$\begin{aligned}&\left\{\left(k_1+k_2-m\overline{\omega}^2\right)+\mathrm{i}\overline{\omega}\left(c_1+c_{2)}\right\}(U_{ci}+iU_{si})-\left\{\left(k_2+\mathrm{i}\overline{\omega}c_2\right)(U_{c2}+iU_{s2})=F_{c1}-F_{s1}\right.\\&-\left(k_2+i\overline{\omega}c_2\right)(U_{ci}+\mathrm{i}U_{si})+\left\{\left(k_2-m_2\overline{\omega}^2\right)+\mathrm{i}\overline{\omega}c_2\right)\}(U_{c2}+\mathrm{i}U_{s2})=F_{c2}-F_{s2}\end{aligned}\tag{8.52}$$

As already stated, the response is then found by solving the complex system of equations (11.54) and retaining only the real part of the solution. Hence, analogously to Eq. (8.49),

$$\begin{aligned}u_1(t)&=U_{c1}\cos\ \overline{\omega}t-U_{s1}\sin\ \overline{\omega}t\\u_2(\mathrm{t})&=U_{c2}\cos\ \overline{\omega}t-U_{s2}\sin\ \overline{\omega}t\end{aligned}\tag{8.53}$$

in which U_{c1}, U_{s2}, U_{c2}, U_{s2}, is the solution of the complex equations (8.52). The necessary calculations are better explained through the use of a numerical example.

Illustrative Example 8.5

Determine the steady state response for the two-story shear building of Illustrative Example 8.4 in which damping is considered in the analysis (Fig. 8.5). Assume for this example that the damping constants c_1 and c_2 are, respectively, proportional to the magnitude of spring constants k_1 and k_2 in which the factor of proportionality, $a_1 = 0.01$.

Solution:

The damping constants are calculated as

$$\begin{aligned}c_1&=a_1k_1=307\ \text{lb. sec}/\text{in}\\c_2&=a_1k_2=443\ \text{lb. sec}/\text{in}\end{aligned}\tag{a}$$

The substitution of numerical values for this example into Eq. (8.54) results in the following system of equations:

$$\begin{aligned}&(20,600+15,000i)U_1-(44,300+8860i)U_2=0\\&-(44,300+8860i)U_1+(17,900+8860i)U_2=-10,000i\end{aligned}\tag{b}$$

The solution of this system of equations is

$$\begin{aligned}\mathrm{U}_1&=0.000681+\mathrm{i}\ 0.26865\\\mathrm{U}_2&=-0.06390+\mathrm{i}\ 0.13777\end{aligned}$$

The, it follows from Eq. (8.55),

$$\begin{aligned}u_1(t)&=0.0006814\cos 20t-0.26865\sin 20t\\u_2(t)&=-0.0639\cos 20t-0.137775\sin 20t\end{aligned}\tag{8.54}$$

which may also be written as

$$\begin{aligned}u_1&=0.2686\sin\left(20t+3.144\right)\ \text{in}\\u_2&=0.1516\sin\left(20t+3.571\right)\ \text{in}\end{aligned}\tag{Ans}$$

When the results are compared with those obtained for the undamped structure in Illustrative Example 8.4, we note only a small change in the amplitude of motion. This is always the case for systems lightly damped and subjected to harmonic excitation of a frequency that is not close to one of the natural frequencies of the system. For this example, the forced frequency $\overline{\omega} = 20$ rad/sec is relatively far from the natural frequencies, $\omega_1 = 11.83$ rad/sec or $\omega_2 = 32.89$ rad/sec which were calculated in Illustrative Example 5.1.

8.5 Harmonic Response: MATLAB Program

MATLAB calculates the response to harmonic excitations of a structural system having the stiffness and mass matrices.

Damping in the system is assumed to be proportional to the stiffness and/or mass coefficients, that is, the damping matrix is calculated as

$$[C] = a_0[M] + a_1[K] \tag{8.55}$$

in which a_0 and a_1 are constants specified in the input data. The program calculates the steady-state response for structures subjected to harmonic forces applied at the nodal coordinates or a harmonic acceleration applied at the base of the structure.

Illustrative Example 8.6
Obtain the response of the damped two-degree-of-freedom shear building of Illustrative Example 8.5 using MATLAB.

Solution:

```
clc
clear all
close all

%_____________________________________________________________________
% Inputs:
%   M, K
%   F = forcing function
%   t = Time period
%   u0 = initial displacement
%   v0 = initial velocity
%_____________________________________________________________________

%%%%-GIVEN VALUES-%%%%

%%%Time for Response
t = 0:0.01:10;

%%%Define Mass Matrix
M = [136 0; 0 66]

%%%Define Stiffness Matrix
k1=30700;
k2=44300;

K = [k1+k2 -k2;
    -k2 k2];

%%%Determine #s of DOFs
[n,n]= size(M);

%%%Define Force Matrix
F = zeros(n,1); F(2)=10000;

nstep = size(t');

%_____________________________________________________________________
% Initial conditions
%_____________________________________________________________________

u0 = zeros(n,1); u0(1) =0;
v0 = zeros(n,1); v0(1) =0;
[n,n]= size(M);
```

```
%_____________________________________________________________________
% Solve the eignevalue problem and normalized eigenvectors
%_____________________________________________________________________

%%%Solve for eigenvalues (D) and eigenvectors (a)
[a, D] = eig(K, M)

%%%Natural Frequencies
[omegas,k] = sort(sqrt(diag(D)));

%%%Eigenvectors
a = a(:,k)

%%%Natural Periods
T = 2*pi./omegas;

%%%aMa = {a}'*[M]*(a}
aMa = diag(a'*M*a)

%%%Normalized modal matrix
nom_phi = (a)*inv(sqrt(diag(aMa)))

%%%Normalized force, P = nom_F
P = nom_phi'*F;
q0 = nom_phi'*M*u0
dq0 = nom_phi'*M*v0

%_____________________________________________________________________
% Damping matrix using the proportional damping matrix
% [C] = a0[M]+a1[K] (Eq. 8.55)
% zetas = damping ratios
%_____________________________________________________________________
a0 = 0;
a1 = 0.01;
nom_C = nom_phi'*(a0*M+a1*K)*nom_phi;
zetas = diag((1/2)*nom_C*inv(diag(omegas)));

save ('temp1.mat', 'omegas', 'P' ,'zetas');
q = [];
r = [];

%_____________________________________________________________________
% Solve uncoupled equations of motions
%_____________________________________________________________________

%%%Iteration for uncoupled equations of motion (Eq. 8.6)
fori=1:n
q0_i = q0(i,:);
dq0_i = dq0(i,:);

load temp1.mat
omega = omegas(i,:);
P = P(i,:);
m = M(i,i);
zeta = zetas(i,:);

save ('temp2.mat', 'omega', 'P', 'm', 'zeta');

[t,q] = ode45(@MDOFP, t, [q0_i dq0_i]',[]);

r(:,i) = q(:,1);
save ('temp3.mat', 'r')

end

load ('temp3.mat', 'r');

%%%Response using Modal Superposition Method
yim = nom_phi*[r'];                 %Eq.8.11

%%%Response
figure
```

```
subplot(2,1,1);
plot(t, yim(1,:))                    %Response @ 1DOF
title('(a) u_1')
xlabel ('Time (sec)');
ylabel ('u_1(in.)');
grid on

subplot(2,1,2);
plot(t, yim(2,:))                    %Response @ 2DOF
title('(b) u_2')
xlabel ('Time (sec)');
ylabel ('u_2(in.)');
grid on

%%%Maximum response
umax_1=max(abs(yim(1,:)))            %u_max @ 1DOF
umax_2=max(abs(yim(2,:)))            %u_max @ 2DOF
```

This function of MATLAB is similar to the function defined in Chap. 3 (MATLAB function file: SDPF.m). This function is used to solve the partial differential equations for solving uncoupled equations of motions. The function of force is defined here except for participation factors which is found in the main MATLAB program.

```
function  q = MDOFP(t, q)
load ('temp2.mat', 'omega', 'P', 'm' ,'zeta')

%_____________________________________
%illustrative Example 8.4
%_____________________________________
P = P*sin(20*t);

%zeta =0 ;
q = [q(2); -omega*omega*q(1)-2*zeta*omega*q(2)+P];
```

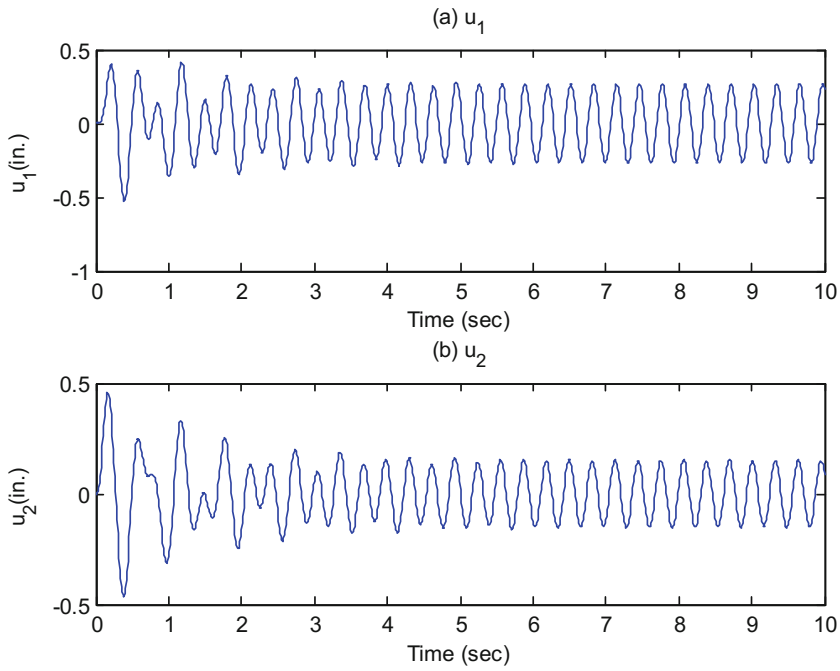

Fig. 8.6 (**a**) Response of 1st degree-of-freedom. (**b**) Response of 2nd degree-of-freedom

The results given by the MATLAB, as expected, are the same as the values calculated in Example 8.6. The steady-state maximum displacement is the same (Fig. 8.6).

8.6 Combining Maximum Values of Modal Response

The square root of the sum of squared contributions (SRSS), to estimate the total response from calculated maximum modal values, may be expressed, in general, from Eq. (11.41) or Eq. (11.42), as

$$R = \sqrt{\sum_{i=1}^{N} R_i^2} \tag{8.56}$$

where R is the estimated response (force, displacement, etc.) at a specified coordinate and R_I is the corresponding maximum response of the ith mode at that coordinate.

Application of the SRSS method for combining modal response generally provides an acceptable estimation of the total maximum response. However, when some of the modes are closely spaced, the use of the SRSS method may result in grossly underestimating or overestimating the maximum response. In particular, large errors have been found in the analysis of three-dimensional structures in which torsional effects are significant. The term "closely spaced" referring to modes, may be arbitrarily define the case when the difference between two natural frequencies is within 10% of the smallest of the two frequencies.

A formulation known as the Complete Quadratic Combinations (CQC), which is based on the theory of random vibrations, has been proposed by Kiureghian (1980) and by Wilson, et al. (1981). The CQC method, which may be considered as an extension of the SRSS method, is given by the following equation:

$$R = \sqrt{\sum_{i=1}^{N} \sum_{j=1}^{N} R_j \rho_{ij} R_j} \tag{8.57}$$

in which the cross-modal coefficient ρ_{ij}, may be approximated by

$$\rho_{ij} = \frac{8\left(\xi_i \xi_j\right)^{1/2}\left(\xi_i + r\xi_j\right) r^{3/2}}{\left(1 - r^2\right)^2 + 4\xi_i \xi_j r\left(1 - r^2\right) + 4\left(\xi_i^2 + \xi_j^2\right) r^2} \tag{8.58}$$

where $r = \omega_j/\omega_i$ is the ratio of the natural frequencies or order i and j and ξ_i and ξ_j the corresponding damping ratios for modes i and j. For constant modal damping ξ, Eq. (11.58) reduces to

$$\rho_{ij} = \frac{8\xi^2(1 - r) r^{3/2}}{\left(1 - r^2\right)^2 + 4\xi^2 r(1 + r)^2}) \tag{8.59}$$

It is important to note that, for i = j, Eq. (11.58) or Eq. (11.59) yields $\rho_{ij} = 0$ for any value of the damping ratio, including.$\xi = 0$ Thus, for an undamped structure, the CQC method (Eq. 11.57) is identical to the SRSS method (Eq. 11.56).

8.7 Summary

For the solution of linear equations of motion, we may employ either the modal superposition method of dynamic analysis or a step-by-step numerical integration procedure. The modal superposition method is restricted to the analysis of structures governed by linear systems of equations whereas the

step-by-step methods of numerical integration are equally applicable to systems with linear or nonlinear behavior. We have deferred the presentation of the multi-degree-of-freedom systems.

In the present chapter, we have introduced the modal superposition method in obtaining the responses of the shear building subjected to either force excitation or to base motion and have demonstrated that the use of normal modes of free vibration for transforming the coordinates leads to a set of uncoupled differential equations. The solution of these equations may then be obtained by any of the methods presented in Part 1 for the single-degree-of-freedom system.

When use is made of response spectra to determine maximum values for modal responses, these values are usually combined by the square root of the sum of squares (SRSS) method. However, the SRSS method could seriously overestimate or underestimate the total response when some of the natural frequencies are closely spaced. A more precise method of combining maximum values of the modal response is the Complete Quadratic Combination (CQC). This method has been strongly recommended in lieu of the SRSS method.

In the particular case of harmonic excitation, the response may be obtained in closed form by simply solving a system of algebraic equations in which the unknowns are the amplitudes of the response at the various coordinates.

8.8 Problems

Problem 8.1

Determine the response as a function of time for the two-story shear building of Problem 7.1 when a constant force of 500 lb is suddenly applied at the level of the second floor as shown in Fig. P8.1. Bays are 15 ft. apart.

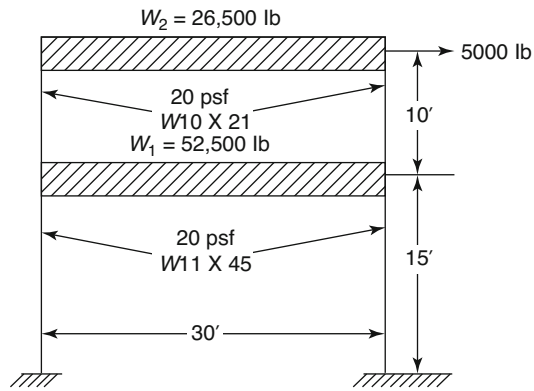

Fig. P8.1

Problem 8.2

Repeat Problem 8.1 if the excitation is applied to the base of the structure in the form of a suddenly applies acceleration of magnitude 0.5 g.

Problem 8.3

Determine the maximum displacement a the floor levels of the three-story shear building (Fig P8.3a) subjected to impulsive triangular loads as shown in Fig. P8.3b. The total stiffness of the columns of each story is $k = 1500$ lb/in and the mass at each floor is $m = 0.386$ lb. sec^2/in.

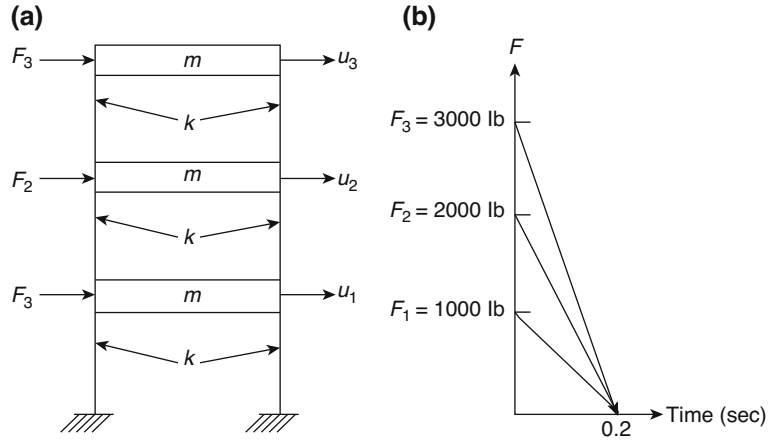

Fig. P8.3

Problem 8.4

Determine the maximum shear force in the columns of the second story of Problem 8.3. (Hint: Calculate modal shear forces and combine contributions using method of square root sum of squares.)

Problem 8.5

Use SAP 2000 to obtain the time history response of the three-story building in Fig. P8.5a subjected to the support acceleration plotted in Fig. P8.5b. Determine the response for a total time of 1.0 sec using time step) t = 0.05 sec, and modal damping coefficient of 10% for all the modes ($E = 30 \times 10^6$ psi).

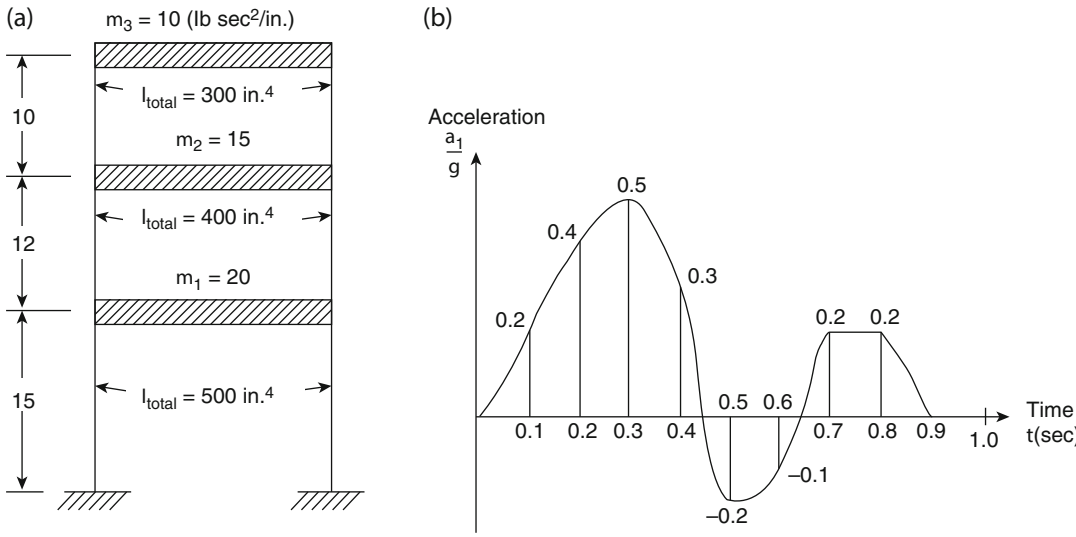

Fig. P8.5

Problem 8.6

Find the steady-state response of the shear building shown in Fig. P8.6 subjected to the harmonic forces indicated in the figure. Neglect damping.

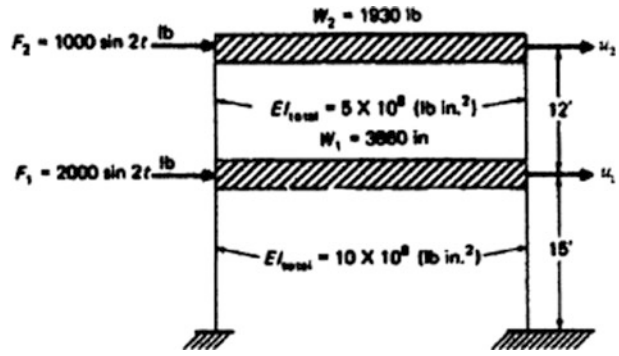

Fig. P8.6

Problem 8.7

Solve Problem 8.6 assuming damping coefficients proportional to the story stiffness, $c_i = 0.05\ K_i$.

Problem 8.8

For the structure (shear building) shown in Fig. P8.8 determine the steady-state motion for the following load systems (loads in pounds):

$$(a)\ F_1(\mathrm{t}) = 1000\sin t, \quad F_2(\mathrm{t}) = 2000\sin t, F_3(\mathrm{t}) = 1500\sin t$$

$$(b)\ F_1(\mathrm{t}) = 2000\cos t, \quad F_2(\mathrm{t}) = 3000\cos t, F_3(\mathrm{t}) = 4000\cos t$$

Also load the structure simultaneously with load systems (a) and (b) and verify the superposition of results.

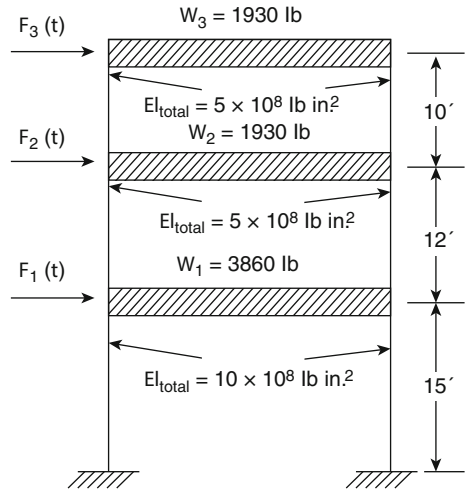

Fig. P8.8

Problem 8.9

For the structure modeled as a four-story shear building shown in Fig. P8.9 determine the steady-state response when it is subjected to a force F = 10,000 sin 20 t (lb) applied at the top floor of the building. The modulus of elasticity is $E = 2.0 \times 10^6$ psi. Assume damping in the system is proportional to the stiffness coefficient ($c_o = 0.01$)

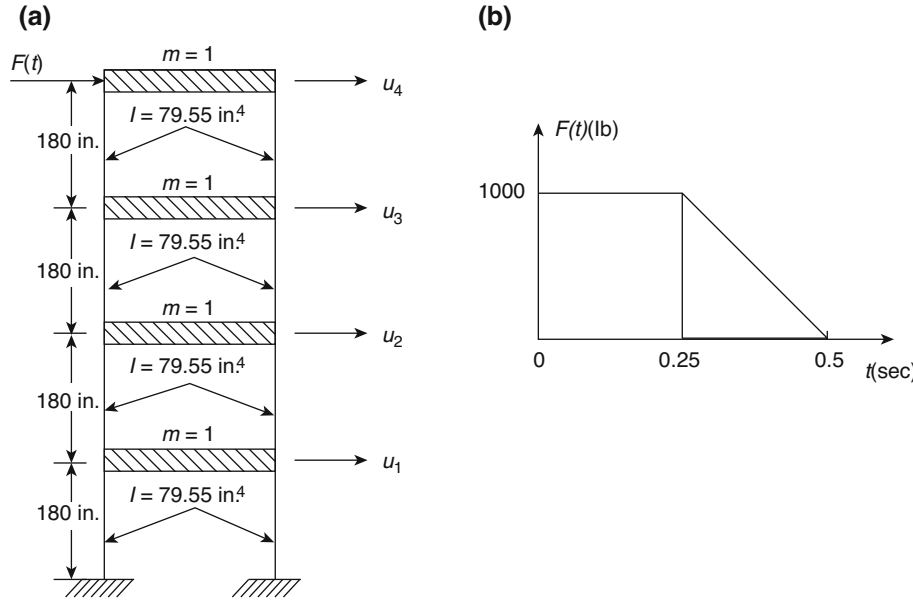

Fig. P8.9

Reduction of Dynamic Matrices

9

In the discretization process it is sometimes necessary to divide a structure into a large number of elements because of changes in geometry, loading, or material properties. When the elements are assembled for the entire structure, the number of unknown displacements, that is, the number of degrees of freedom, may be quite large. As a consequence, the stiffness, mass, and damping matrices will be of large dimensions. The solution of the corresponding eigenproblem to determine natural frequencies and modal shapes will be difficult and, in addition, expensive. In such cases it is desirable to reduce the size of these matrices in order to make the solution of the eigenproblem more manageable and economical. Such reduction is referred to as condensation.

A popular method of reduction is the Static Condensation Method This method, though simple to apply, is only approximate and may produce relatively large errors in the results when applied to dynamic problems. An improved method for condensation of dynamic problems, which gives virtually exact results, has recently been proposed. This method, called the Dynamic Condensation Method, will be presented in this chapter after the introduction of the Static Condensation Method.

9.1 Static Condensation

A practical method of accomplishing the reduction of the stiffness matrix is to identify those degrees of freedom to be condensed as dependent or secondary degrees of freedom, and express them in the term of the remaining independent or primary degrees of freedom. The relationship between the secondary or primary degrees of freedom is found by establishing the static relation between them, hence the name Static Condensation Method (Guyan 1965). This relationship provides the means to reduce the stiffness matrix. This method is also used in the static problems to eliminate unwanted degrees of freedom such as the internal degrees of freedom of an element used with the Finite Element Method. In order to describe the Static Condensation Method, let us assume that those (secondary) degrees of freedom to be reduced or condensed are arranged as the first s coordinates, and the remaining (primary) degrees of freedom are the last ρ coordinates. With this arrangement, the stiffness equation for the structure may be written using partition matrices as

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M. Paz, Y. H. Kim, *Structural Dynamics*, https://doi.org/10.1007/978-3-319-94743-3_9

$$\begin{bmatrix} [K_{ss}] & \vdots & [K_{sp}] \\ \cdots & \vdots & \cdots \\ [K_{ps}] & \vdots & [K_{pp}] \end{bmatrix} \begin{Bmatrix} \{u_s\} \\ \cdots \\ u_p \end{Bmatrix} = \begin{Bmatrix} \{0\} \\ \cdots \\ \{F_p\} \end{Bmatrix} \tag{9.1}$$

where $\{u_s\}$ is the displacement vector corresponding to the s degrees of freedom to be reduced and $\{u_p\}$ is the vector corresponding to the reining p independent degrees of freedom. In Eq. (8.1), it is assumed that the external forces were zero at the dependent (i.e., secondary) degrees of freedom; this assumption is not mandatory (Gallagher 1975), but serves to simplify explanations without affecting the final results. A simple multiplication of the matrices on the left side of Eq. (9.1) expands this equation into two matrix equations, namely,

$$[K_{ss}]\{u_s\} + K_{sp}\{u_p\} = \{0\} \tag{9.2}$$

$$K_{ps}\{u_s\} + K_{pp}\{u_p\} = \{F_p\} \tag{9.3}$$

Equation (9.2) is equivalent to

$$\{u_s\} = [\bar{T}]\{u_p\} \tag{9.4}$$

where $[\bar{T}]$ is the transformation matrix given by

$$[\bar{T}] = -[K_{ss}]^{-1}[K_{sp}] \tag{9.5}$$

Substituting Eq. (9.4) and using Eq. (9.5) in Eq. (9.3) results in the reduced stiffness equation relating forces and displacements at the primary coordinates, that is,

$$[\bar{K}]\{u_p\} = \{F_p\} \tag{9.6}$$

where $[\bar{K}]$ is the reduced stiffness matrix given by

$$[\bar{K}] = [K_{pp}] - [K_{ps}][K_{ss}]^{-1}[K_{sp}] \tag{9.7}$$

Equation (9.4), which expresses that static relationship between the secondary coordinates $\{u_s\}$ and primary coordinates $\{u_p\}$, may also be written using the identity $\{u_p\} = [I]\{u_p\}$ as

$$\begin{Bmatrix} \{u_s\} \\ \cdots \\ \{u_p\} \end{Bmatrix} = \begin{bmatrix} [\bar{T}] \\ \cdots \\ [I] \end{bmatrix} \{u_p\}$$

or

$$\{u\} = [T]\{u_p\} \tag{9.8}$$

where

$$\{u\} = \begin{Bmatrix} \{u_s\} \\ \{u_p\} \end{Bmatrix} \quad \text{and} \quad [T] = \begin{bmatrix} [T] \\ [I] \end{bmatrix} \tag{9.9}$$

Substituting Eqs. (9.8) and (9.9) into Eq. (9.1) and pre-multiplying by the transpose of $[T]$ results in

$$[T]^T[K][T]\{u_p\} = \left[[T]^T[I]\right]\begin{Bmatrix}\{0\}\\ \{F_p\}\end{Bmatrix}$$

or

$$[T]^T[K][T]\{u_p\} = \{F_p\}$$

and using Eq. (9.6)

$$\left[\bar{K}\right] = [T]^T[K][T] \tag{9.10}$$

thus showing that the reduced stiffness matrix $\left[\bar{K}\right]$ can be expressed as a transformation of the system stiffness matrix $[K]$.

It may appear that the calculation of the reduced stiffness matrix $\left[\bar{K}\right]$ given by Eq. (9.7) requires the inconvenient calculation of the inverse matrix $[K_{ss}]^{-1}$. However, the practical application of the Static Condensation Method does not require a matrix inversion. Instead, the standard Gauss-Jordan elimination process, as it used in the solution of a system of linear equations, is applied systematically on the system's stiffness matrix $[K]$ up to the elimination of the secondary coordinates $\{u_s\}$. At this stage of the elimination process the stillness Eq. (9.1) has been reduced to

$$\begin{bmatrix}[I] & -\left[\bar{T}\right]\\ [0] & \left[\bar{K}\right]\end{bmatrix}\begin{Bmatrix}\{u_s\}\\ \{u_p\}\end{Bmatrix} = \begin{Bmatrix}\{0\}\\ \{F_p\}\end{Bmatrix} \tag{9.11}$$

It may be seen by expanding Eq. (9.11) that the partition matrices $\left[\bar{T}\right]$ and $\left[\bar{K}\right]$ are precisely the transformation matrix and the reduced stiffness matrix defined by Eqs. (9.4) and (9.6), respectively. In this way, the Gauss-Jordan elimination process yields both the transformation matrix $\left[\bar{T}\right]$ and the reduced stiffness matrix $\left[\bar{K}\right]$. There is thus no need to calculate $[K_{ss}]^{-1}$ in order to reduce the secondary coordinates of the system.

Illustrative Example 9.1

Consider the two-degree-of-freedom system represented by the model shown in Fig. 9.1 and use static condensation to reduce the first coordinate.

Solution:

For this system the equations of equilibrium are readily obtained as

$$\begin{bmatrix}2k & -k\\ -k & k\end{bmatrix}\begin{Bmatrix}u_1\\ u_2\end{Bmatrix} = \begin{Bmatrix}0\\ F_2\end{Bmatrix} \tag{9.12}$$

The reduction of u_1 using Gauss elimination leads to

$$\begin{bmatrix}1 & -1/2\\ 0 & k/2\end{bmatrix}\begin{Bmatrix}u_1\\ u_2\end{Bmatrix} = \begin{Bmatrix}0\\ F_2\end{Bmatrix} \tag{9.13}$$

Comparing Eq. (9.13) with Eq. (9.11), we identify in this example

$$\begin{aligned}\left[\bar{T}\right] &= \frac{1}{2}\\ \left[\bar{K}\right] &= k/2\end{aligned} \tag{9.14}$$

Consequently, from Eq. (9.9) the transformation matrix is

$$[T] = \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} \tag{9.15}$$

We can now check Eq. (9.10) by simple performing the indicated multiplications, namely

$$[\bar{K}] = [1/2 \;\; 1] \begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix} \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} = \frac{k}{2} \tag{9.16}$$

which agrees with the result given in Eq. (9.14).

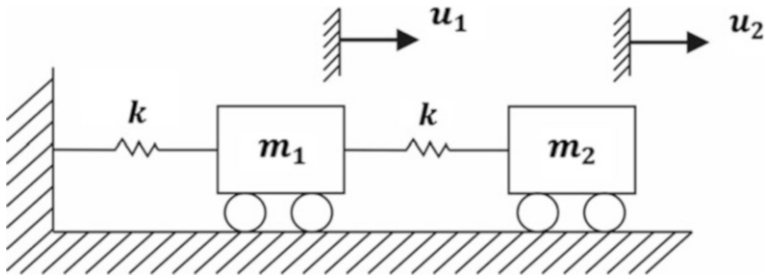

Fig. 9.1 Mathematical model for a two-degree-of-freedom system

9.2 Static Condensation Applied to Dynamic Problems

In order to reduce the mass and the damping matrices, it is assumed that the same static relationship between the secondary and primary degrees of freedom remains valid in the dynamic problem. Hence the same transformation based on static condensation for the reduction of the stiffness matrix is also used in reducing the mass and damping matrices. In general this method of reducing the dynamic problem is not exact and introduces errors in the results. The magnitude of these errors depends on the relative number of degrees of freedom reduced as well as on the specific selection of these degrees of freedom for a given structure.

We consider first the case in which the discretization of the mass results in a number of massless degrees of freedom selected to be condensed. In this case it is only necessary to carry out the static condensation of the stiffness matrix and to delete from the mass matrix the rows and columns corresponding to the massless degrees of freedom. The Static Condensation Method in this case does not alter the original problem, and thus results in an equivalent eigenproblem without introducing any error.

In the general case, that is, the case involving the condensation of degrees of freedom to which the discretization process has allocated mass, the reduced mass and damping matrices are obtained using transformations analogous to Eq. (9.10). Specifically, if $[M]$ is the mass matrix of the system, then the reduced mass matrix is given by

$$[\bar{M}] = [T]^T [M][T] \tag{9.17}$$

where $[T]$ is the transformation matrix defined by Eq. (9.9).

Analogously, for a damped system, the reduced damping matrix is given by

$$[\bar{C}] = [T]^T [C][T] \tag{9.18}$$

where $[C]$ is the damping matrix of the system.

This method of reducing the mass and damping matrices may be justified as follows: The potential elastic energy V and the kinetic energy KE of the structure may be written, respectively, as

$$V = \frac{1}{2}\{u\}^T[K]\{u\} \tag{9.19}$$

$$KE = \frac{1}{2}\{\dot{u}\}^T[M]\{\dot{u}\} \tag{9.20}$$

Analogously, the virtual work of δW_d done by the damping forces $F_d = [C]\{\dot{u}\}$ corresponding to the virtual displacement $\{\delta u\}$ may be expressed as

$$\delta W_d = \{du\}^T[C]\{\dot{u}\} \tag{9.21}$$

Introduction of the transformation Eq. (9.8) in the above equation results in

$$V = \frac{1}{2}\{u_p\}^T[T]^T[K][T]\{u_p\} \tag{9.22}$$

$$KE = \frac{1}{2}\{\dot{u}_p\}^T[T]^T[M][T]\{\dot{u}_p\} \tag{9.23}$$

$$\delta W_d = \{\delta u_p\}^T[T]^T[C][T]\{\dot{u}_p\} \tag{9.24}$$

The respective substitution of $[\bar{K}]$ $[\bar{M}]$ and $[\bar{C}]$ from Eqs. (9.10), (9.17), and (9.18) for the product of the three central matrices in the Eqs. (9.22), (9.23), and (9.24) yields

$$V = \frac{1}{2}\{u_p\}^T[\bar{K}][T]\{u_p\} \tag{9.25}$$

$$KE = \frac{1}{2}\{\dot{u}_p\}^T[\bar{M}]\{\dot{u}_p\} \tag{9.26}$$

$$\delta W_d = \{\delta u_p\}^T[\bar{C}]\{\dot{u}_p\} \tag{9.27}$$

These last three equations express the potential energy, the kinetic energy, and the virtual work of the damping forces in terms of the primary coordinates $\{u_p\}$. Hence the matrices $[\bar{K}]$, $[\bar{M}]$ and $[\bar{C}]$ may be interpreted, respectively, as the stiffness, mass, and damping matrices of the structure corresponding to the primary degrees of freedom $\{u_p\}$ resulting in the same potential energy, kinetic energy and virtual work calculated with all the original nodal coordinate.

Illustrative Example 9.2

Determine the natural frequencies and modal shapes for the three-degree-of-freedom shear building shown in Fig. 9.2; then condense the first degree of freedom and compare the resulting values obtained for natural frequencies and modal shapes. The stiffness of each story and the mass at each floor level are indicated in the figure. MATALB program is used to demonstrate the natural frequencies using static condensation method.

Solution:

1. Calculation of Natural Frequencies and Modal Shapes:

The equation of motion in free vibration for this structure is given by Eq. (8.3) with the force vector $\{F\} = \{0\}$, namely,

$$[M]\{\ddot{u}\} + [K]\{u\} = 0 \tag{a}$$

where the matrices $[M]$ and $[K]$ are given, respectively, by Eqs. (7.4) and (7.5). Substituting the corresponding numerical value in Eq. (a) yields

$$\begin{bmatrix} 25 & 0 & 0 \\ 0 & 50 & 0 \\ 0 & 0 & 100 \end{bmatrix} \begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \\ \ddot{u}_3 \end{Bmatrix} + \begin{bmatrix} 40,000 & -10,000 & 0 \\ -10,000 & 20,000 & -10,000 \\ 0 & -10,000 & 10,000 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

Upon substitution $u_i = U_i \sin \omega t$ and cancellation of the factor $\sin \omega t$, we obtain the following system of equations:

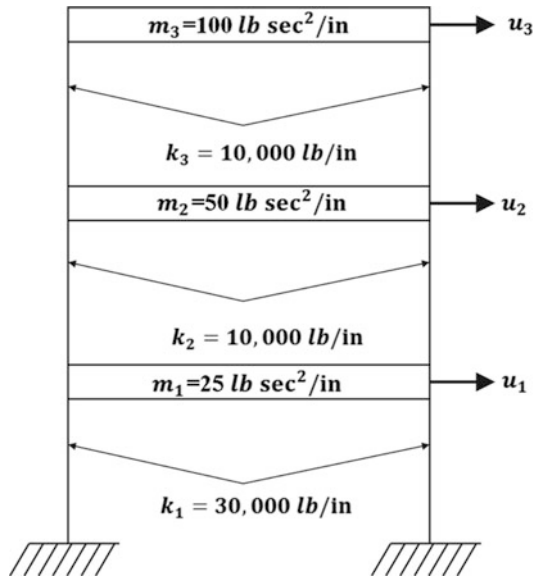

Fig. 9.2 Shear building for Illustrative Example 9.2

$$\begin{bmatrix} 40,000 - 25\omega^2 & -10,000 & 0 \\ -10,000 & 20,000 - 50\omega^2 & -10,000 \\ 0 & -10,000 & 10,000 - 100\omega^2 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \end{Bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \tag{b}$$

which for a nontrivial solution requires that the determinant of the coefficients of the unknowns be equal to zero, that is,

$$\begin{vmatrix} 40,000 - 25\omega^2 & -10,000 & 0 \\ -10,000 & 20,000 - 50\omega^2 & -10,000 \\ 0 & -10,000 & 10,000 - 100\omega^2 \end{vmatrix} = 0$$

The expansion of this determinant results in a third degree equation in terms of ω^2 having the following roots:

$$\omega_1^2 = 36.1 \qquad \omega_2^2 = 400.0 \qquad \omega_3^2 = 1664.0 \tag{c}$$

The natural frequencies are calculated as $f = \omega/2\pi$, so that

$$f_1 = 0.96\,\text{cps} \quad f_2 = 3.18\,\text{cps} \quad f_3 = 264.8\,\text{cps}$$

The modal shapes are then determined by substituting in turn each of the values for the natural frequencies into Eq. (b), deleting a redundant equation, and solving the remaining two equations for two of the unknowns in terms of the third. As we mentioned previously, in solving for these unknowns it is expedient to set the first nonzero unknown equal to one. Performing these operations, we obtain from Eqs. (b) and (c) the following values for the modal shapes:

$$\begin{aligned} U_{11} &= 1.00, \quad U_{12} = 1.00, \quad U_{13} = 1.00 \\ U_{21} &= 3.91, \quad U_{22} = 3.00, \quad U_{23} = 3.338 \\ U_{31} &= 6.11, \quad U_{32} = -1.00, \quad U_{33} = -2.025 \end{aligned}$$

in which the second sub-index in U refers to the modal order.

2. Condensation of Coordinate u_1:

The stiffness matrix for this structure is

$$\begin{bmatrix} 40,000 & -10,000 & 0 \\ -10,000 & 20,000 & -10,000 \\ 0 & -10,000 & 10,000 \end{bmatrix}$$

Gauss elimination applied to the first row gives

$$\left[\begin{array}{c|cc} 1 & -0.25 & 0 \\ \hdashline 0 & 17,500 & -10,000 \\ 0 & -10,000 & 10,000 \end{array}\right] \tag{d}$$

A comparison of Eq. (d) with Eq. (9.11) indicates that

$$\begin{aligned} [\bar{T}] &= [0.25 \quad 0] \\ [\bar{K}] &= \begin{bmatrix} 17,500 & -10,000 \\ -10,000 & 10,000 \end{bmatrix} \end{aligned} \tag{e}$$

Then from Eq. (9.9)

$$[T] = \begin{bmatrix} 0.25 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{f}$$

To check results, we use Eq. (9.10) to compute $[\bar{K}]$. Hence

$$[\bar{K}] = \begin{bmatrix} 0.25 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 40,000 & -10,000 & 0 \\ -10,000 & 20,000 & -10,000 \\ 0 & -10,000 & 10,000 \end{bmatrix} \begin{bmatrix} 0.25 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$[\bar{K}] = \begin{bmatrix} 17,500 & -10,000 \\ -10,000 & 10,000 \end{bmatrix}$$

which checks with Eqs. (e).

The reduced mass matrix is calculated by substituting matrix [T] and its transpose from Eq. (f) into Eq. (9.17), so that

$$[\bar{M}] = \begin{bmatrix} 0.25 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 25 & 0 & 0 \\ 0 & 50 & 0 \\ 0 & 0 & 100 \end{bmatrix} \begin{bmatrix} 0.25 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

which results in

$$[\bar{M}] = \begin{bmatrix} 51.6 & 0 \\ 0 & 100 \end{bmatrix}$$

The condensed dynamic problem is then

$$\begin{bmatrix} 51.6 & 0 \\ 0 & 100 \end{bmatrix} \begin{Bmatrix} \ddot{u}_2 \\ \ddot{u}_3 \end{Bmatrix} + \begin{bmatrix} 17,500 & -10,000 \\ -10,000 & 10,000 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

The natural frequencies and modal shapes are determined from the solution of the following reduced eigenproblem:

$$\begin{bmatrix} 17,500 - 51.6\omega^2 & -10,000 \\ -10,000 & 10,000 - 100^2 \end{bmatrix} \begin{Bmatrix} U_2 \\ U_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{g}$$

Equating to zero the determinant of the coefficient matrix in Eq. (g) and solving the resulting quadratic equation in ω^2 gives

$$\omega_1^2 = 36.1 \qquad \omega_2^2 = 403.3 \tag{h}$$

from which

$$f_1 = \sqrt{36.1/2\pi} = 0.95\,\text{cps}$$
$$f_2 = \sqrt{403.3/2\pi} = 3.20\,\text{cps}$$

Corresponding modal shapes are obtained from Eq. (g) after substituting in turn the numerical values for ω_1^2 or ω_2^2 and solving the first equation for U_3 with $U_2 = 1$,. We then obtain

$$U_{21} = 1.00, \quad U_{22} = 1.00$$
$$U_{31} = 1.56, \quad U_{32} = -0.33$$

in which the second subindexes in U serve to indicate the mode 1 or 2.

Application of Eq. (9.8) for the first mode gives

$$\begin{Bmatrix} U_{11} \\ U_{21} \\ U_{31} \end{Bmatrix} = \begin{bmatrix} 0.25 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} 1.00 \\ 1.56 \end{Bmatrix} = \begin{Bmatrix} 0.25 \\ 1.00 \\ 1.56 \end{Bmatrix}$$

or, after normalizing so that the first component is 1,

$$U_{11} = 1.00, \quad U_{21} = 4.00, \quad U_{31} = 6.24$$

and analogously for the second mode

$$U_{12} = 1.00, \quad U_{22} = 4.00, \quad U_{32} = -1.32$$

For this system of only three degrees of freedom, the reduction of one coordinate gives natural frequencies that compare rather well for the first two modes [Eqs. (h) and (c)]. However, experience shows that static condensation may produce large errors in the calculation of eigenvalues and eigenvectors obtained from the reduced system. A general recommendation given by users of this method is to assume that the static condensation process results in an eigenproblem that provides acceptable approximations of only about a third of the calculated eigenvalues (natural frequencies) and eigenvectors (modal shapes).

The MATLAB program is used to obtain the natural frequencies in Eq. (h).

```
clc
clear all
close all

%_____________________________________________________________________
% Inputs:
%   M, K
%   m =  Number of row to apply the elimination process
%_____________________________________________________________________


%%%%-GIVEN VALUES-%%%%

%%%Mass Matrix
M = [25 0 0; 0 50 0; 0 0 100]

%%%Stiffness Matrix
K = [40000, -10000,0; -10000, 20000, -10000; 0, -10000, 10000]

%%%Elimination of the first row: m=1
m = 1;

N= length(K);   %Total number of row

K_full = K;     %[K] before the elimination

%%%Partition Matrix for Static Condensation (Eq.9.1)
K_pp_full = K_full(m+1:N,m+1:N);
K_ps_full = K_full(m+1:N,1:m);
K_ss_full = K_full(1:m,1:m);
K_sp_full = K_full(1:m,m+1:N);

K_bar_full = K_pp_full-K_ps_full*inv(K_ss_full)*K_sp_full;  %Eq.9.7

T_bar_full = -inv(K_ss_full)*K_sp_full;                     %Eq.9.5
T = [T_bar_full; eye(N-m,N-m)];

K_bar = T'*K_full*T                                         %Eq.9.10
M_bar_full = T'*M*T                                         %Eq.9.17

%_____________________________________________________________________
% Solve the eignevalue problem and normalized eigenvectors
%_____________________________________________________________________

%%%Solve for eigenvalues (D) and eigenvectors (a)
[a, D] = eig(K_bar, M_bar_full)
```

Illustrative Example 9.3

Figure 9.3 shows a uniform four-story shear building. For this structure, determine the following: (a) the natural frequencies and corresponding modal shapes as a four-degree-of-freedom system, (b) the natural frequencies and modal shapes after static condensation of coordinates u_1 and u_3.

Solution:

(a) Natural Frequencies and Modal Shapes as a Four-Degree-of Freedom System:

The stiffness and the mass matrices for this structure are respectively

$$[K] = 327.35 \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \tag{a}$$

and

$$[M] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{b}$$

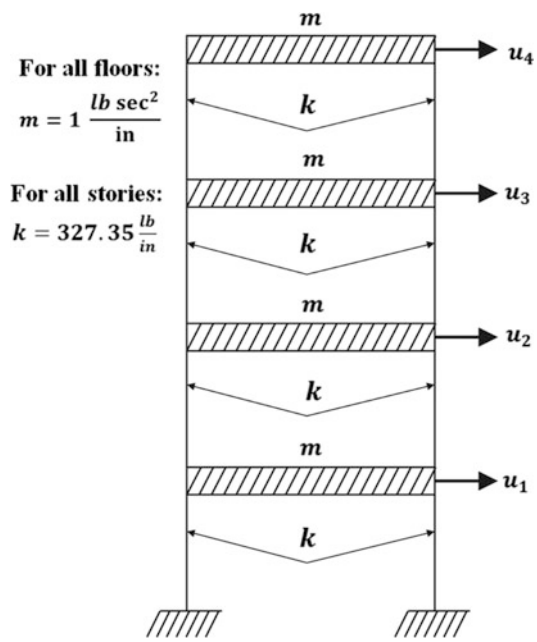

Fig. 9.3 Uniform four-story shear building for Illustrative Example 9.3

Substituting Eqs. (a) and (b) into Eq. (9.3) and solving the corresponding eigenvalue problem (using Program 8) yields

$$\omega_1^2 = 39.48, \quad \omega_2^2 = 327.35, \quad \omega_3^2 = 768.3, \quad \text{and} \quad \omega_4^2 = 1156.00$$

corresponding to the natural frequencies

$$\begin{aligned} \mathrm{f}_1 &= \frac{\omega_1}{2\pi} = 1.00\,\mathrm{cps} \quad \mathrm{f}_2 = \frac{\omega_2}{2\pi} = 2.88\,\mathrm{cps} \\ \mathrm{f}_3 &= \frac{\omega_3}{2\pi} = 4.41\,\mathrm{cps} \quad \mathrm{f}_4 = \frac{\omega_4}{2\pi} = 5.41\,\mathrm{cps} \end{aligned} \tag{c}$$

and the normalized modal matrix (see Sect. 7.2)

$$[\Phi] = \begin{bmatrix} 0.2280 & 0.5774 & -0.6565 & 0.4285 \\ 0.4285 & 0.5774 & 0.2280 & -0.6565 \\ 0.5774 & 0 & 0.5774 & 0.5774 \\ 0.6565 & -0.5774 & 0.4285 & -0.2280 \end{bmatrix} \tag{d}$$

(b) Natural Frequencies and Modal Shapes after Reduction to Two-Degree-of Freedom System:

To reduce coordinates u_1 and u_3, for convenience, we first rearrange the stiffness matrix in Eq. (a) to have the coordinates in order u_1, u_3, u_2, u_4

$$[K] = 327.35 \begin{bmatrix} 2 & 0 & -1 & 0 \\ 0 & 2 & -1 & -1 \\ -1 & -1 & 2 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \tag{e}$$

Applying Gauss-Jordan elimination to the first two rows of the matrix in Eq. (e) result in

$$[D] = \left[\begin{array}{cc:cc} 1 & 0 & -0.5 & 0 \\ 0 & 1 & -0.5 & -0.5 \\ \hdashline 0 & 0 & 327.35 & -163.70 \\ 0 & 0 & -163.70 & 163.70 \end{array}\right] \tag{f}$$

A comparison of Eq. (f) with Eq. (9.11) reveals that

$$[\bar{T}] = \begin{bmatrix} 0.5 & 0 \\ 0.5 & 0.5 \end{bmatrix} \tag{g}$$

and

$$[\bar{K}] = \begin{bmatrix} 327.35 & -163.70 \\ -163.70 & 163.70 \end{bmatrix} \tag{h}$$

Use of Eq. (9.9) gives

$$[T] = \begin{bmatrix} 0.5 & 0 \\ 0.5 & 0.5 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The reduced mass matrix can now be calculated by Eq. (9.17) as

$$\left[\bar{M}\right] = [T]^T[M][T] = \begin{bmatrix} 1.5 & 0.25 \\ 0.25 & 1.25 \end{bmatrix} \tag{i}$$

The condensed eigenproblem is then

$$\begin{bmatrix} 327.35 - 1.5\omega^2 & -163.70 - 0.25\omega^2 \\ -163.70 - 0.25\omega^2 & 163.70 - 1.25\omega^2 \end{bmatrix} \begin{Bmatrix} U_2 \\ U_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{j}$$

and its solution is

$$\omega_1^2 = 40.39, \quad \omega_2^2 = 365.98 \tag{k}$$

$$[U]_p = \begin{bmatrix} 0.4380 & 0.7056 \\ 0.6723 & -0.6128 \end{bmatrix} \tag{l}$$

where $[U]_p$ is the modal matrix corresponding to the primary degrees of freedom.

The eigenvectors for the four-degree-of-freedom system are calculated for the first mode using Eq. (9.8) as

$$\begin{Bmatrix} U_1 \\ U_3 \\ U_2 \\ U_4 \end{Bmatrix}_1 = \begin{bmatrix} 0.5 & 0 \\ 0.5 & 0.5 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} 0.4380 \\ 0.6723 \end{Bmatrix} = \begin{Bmatrix} 0.2190 \\ 0.5552 \\ 0.4380 \\ 0.6723 \end{Bmatrix}_1$$

or rearranging the component of the first modal shape:

$$\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{Bmatrix}_1 = \begin{Bmatrix} 0.2190 \\ 0.4380 \\ 0.5552 \\ 0.6723 \end{Bmatrix}_1 \tag{m}$$

and for the second mode

$$\begin{Bmatrix} U_1 \\ U_3 \\ U_2 \\ U_4 \end{Bmatrix}_2 = \begin{bmatrix} 0.5 & 0 \\ 0.5 & 0.5 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} 0.7056 \\ -0.6128 \end{Bmatrix} = \begin{Bmatrix} 0.3528 \\ 0.0464 \\ 0.7056 \\ -0.6128 \end{Bmatrix}_2$$

or

$$\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{Bmatrix}_2 = \begin{Bmatrix} 0.3528 \\ 0.7056 \\ 0.0464 \\ -0.6128 \end{Bmatrix}_2$$

The MATLAB program is used to obtain the natural frequencies in Eq. (k).

```
clc
clear all
close all

%_____________________________________________________________________
% Inputs:
%   M, K
%   m =  Number of row to apply the static elimination process
%_____________________________________________________________________


%%%%-GIVEN VALUES-%%%%

%%%Mass Matrix
M = [1, 0 0 0; 0, 1, 0, 0; 0, 0, 1,0; 0,0,0, 1];

%%%Stiffness Matrix
K = 327.35*[2,0,-1,0; 0,2,-1,-1; -1,-1,2,0; 0,-1,0,1];

%%%Elimination of the first and second rows: m=2
m = 2;


N= length(K);       %Total number of row


K_full = K;         %[K] before the elimination

%%%Partition Matrix for Static Condensation (Eq.9.1)
K_pp_full = K_full(m+1:N,m+1:N);
K_ps_full = K_full(m+1:N,1:m);
K_ss_full = K_full(1:m,1:m);
K_sp_full = K_full(1:m,m+1:N);

K_bar_full = K_pp_full-K_ps_full*inv(K_ss_full)*K_sp_full;      %Eq.9.7

T_bar_full = -inv(K_ss_full)*K_sp_full;                         %Eq.9.5
T = [T_bar_full; eye(N-m,N-m)];

K_bar = T'*K_full*T;                                            %Eq.9.10
M_bar_full = T'*M*T;             %Eq.9.17

%_____________________________________________________________________
% Solve the eignevalue problem and normalized eigenvectors
%_____________________________________________________________________

%%%Solve for eigenvalues (D) and eigenvectors (a)
[a, D] = eig(K_bar, M_bar_full)
```

Illustrative Example 9.4

The shear building of Illustrative Example 9.3 is subjected to an earthquake motion at its foundation. For design purposes, use the response spectrum of Fig. 5.10 Sect. (5.4) and determine the maximum horizontal displacements of the structure at the level of the floors.

Solution:
The participation factor for the ith mode of a shear building with N stories is given by Eq. (8.40) as

$$\Gamma_i = -\sum_{j=1}^{N} \left(m_j \phi_{ji}\right) \tag{a}$$

where m_j is the mass at the jth floor and the ϕ_{ji} the jth element of the normalized ith eigenvector of the mode.

(a) Response Considering Four Degrees of Freedom:

The substitution into Eq. (a) of the corresponding numerical results from Illustrative Example 9.3 gives

$$\Gamma_1 = -1.890, \quad \Gamma_2 = -0.5775, \quad \Gamma_3 = -0.2797, \quad \text{and} \quad \Gamma_4 = -0.1213 \tag{b}$$

The spectral displacements corresponding to the values of the natural frequencies of this building [Eq. (c) of Illustrative Example 9.3] are obtained from the response spectrum, Fig. 5.10, as

$$S_{D1} = 14.32, \quad S_{D2} = 3.240, \quad S_{D3} = 1.433, \quad \text{and} \quad S_{D4} = 0.969 \tag{c}$$

The maximum displacements at the floor levels relative to the displacement at the base of the building are calculated using Eq. (8.41), namely,

$$u_{i\max} = \sqrt{\sum_{j=1}^{n} \left(T_j S_{Dj} \phi_{ij}\right)^2} \tag{d}$$

to obtain

$$u_{1\max} = 6.274\,\text{in}, \quad u_{2\max} = 11.65\,\text{in}, \quad u_{3\,\max} = 15.64\,\text{in}, \quad \text{and} \quad u_{4\max} = 17.81\,\text{in}$$

(b) Response Considering the System Reduce to Two Degrees of Freedom:

The natural frequencies, calculated from Eq. (k) in Illustrative Example 9.3, are

$$\begin{aligned} f_1 &= \sqrt{40.39}/2\pi = 1.011\,\text{cps} \\ f_2 &= \sqrt{365.98}/2\pi = 3.044\,\text{cps} \end{aligned} \tag{e}$$

Upon introducing, into Eq. (a), the corresponding eigenvectors give in Eqs. (m) and (n) of Illustrative Example 9.3, we obtain the participation factors

$$\Gamma_1 = -1.884 \qquad \Gamma_2 = -0.492$$

The values of spectral displacements corresponding to the frequencies in Eq. (e) can be read from Fig. 5.10:

$$S_{D1} = 14.16 \qquad S_{D4} = 2.913$$

Use of Eq. (d) gives the relative maximum displacements at the level of the floors as

$$u_{1\max} = \sqrt{(1.884 \times 14.16 \times 0.2190)^2 + (0.4920 \times 2.913 \times 0.3528)^2} = 5.864\,\text{in}$$

$$u_{2\max} = \sqrt{(1.884 \times 14.16 \times 0.4380)^2 + (0.4920 \times 2.913 \times 0.7056)^2} = 11.73\,\text{in}$$

$$u_{3\max} = \sqrt{(1.884 \times 14.16 \times 0.5552)^2 + (0.4920 \times 2.913 \times 0.0464)^2} = 14.81\,\text{in}$$

$$u_{4\max} = \sqrt{(1.884 \times 14.16 \times 0.6723)^2 + (0.4920 \times 2.913 \times 0.6128)^2} = 17.97\,\text{in}$$

9.3 Dynamic Condensation

A method of reduction that may be considered an extension of the Static Condensation Method has been proposed (Paz 1984). The algorithm for this method starts by assigning an approximate value (e.g., zero) to the first eigenvalue ω_1^2, applying dynamic condensation to the dynamic matrix of the system $[D_1] = [K] - \omega_1^2[M]$ and then solving the reduced eigenproblem to determine the first and second eigenvalues, ω_1^2 and ω_2^2. The process continues in this manner, with one virtually exact eigenvalue and an approximate value for the next order eigenvalue calculated at each step.

The Dynamic Condensation Method requires neither matrix inversion nor series expansion. To demonstrate this fact, consider the eigenvalues problem of a discrete structural system for which it is desired to reduce the secondary degrees of freedom $\{u_o\}$ and retain the primary degrees of freedom $\{u_p\}$. In this case, the equations of free motion may be written in partitioned matrix form as

$$\left[\begin{array}{c|c} [M_{ss}] & [M_{sp}] \\ \hline [M_{ps}] & [M_{pp}] \end{array}\right] \left\{\begin{array}{c} \{\ddot{u}_s\} \\ \hline \{\ddot{u}_p\} \end{array}\right\} + \left[\begin{array}{c|c} [K_{ss}] & [K_{sp}] \\ \hline [K_{ps}] & [K_{pp}] \end{array}\right] \left\{\begin{array}{c} \{u_s\} \\ \hline u_p \end{array}\right\} = \left\{\begin{array}{c} \{0\} \\ \hline \{0\} \end{array}\right\} \tag{9.28}$$

The substitution of $\{u\} = \{U\}\sin\omega_i t$ in Eq. (9.28) results in the generalized eigenproblem

$$\left[\begin{array}{c|c} [K_{ss}] - \omega_i^2 [M_{ss}] & [K_{sp}] - \omega_i^2 [M_{sp}] \\ \hline [K_{ps}] - \omega_i^2 [M_{ps}] & [K_{pp}] - \omega_i^2 [M_{pp}] \end{array}\right] \left\{\begin{array}{c} \{u_s\} \\ \hline \{u_p\} \end{array}\right\} = \left\{\begin{array}{c} \{0\} \\ \hline \{0\} \end{array}\right\} \tag{9.29}$$

where ω_i^2 is the approximation of the ith eigenvalues which was calculated in the preceding step of the process. To start the process one takes an approximate or zero value for the first eigenvalue ω_1^2.

The following three steps are executed to calculate the ith eigenvalue ω_i^2 and the corresponding eigenvector $\{U\}_i$ as well as an approximation of the eigenvalue of the next order $\omega_{i=1}^2$:Step 1: The approximation of ω_i^2 is introduced in Eq. (9.29). Gauss-Jordan elimination of the secondary coordinates $\{U_s\}$ is then used to reduce Eq. (9.29) to

$$\left[\begin{array}{c:c} [I] & -[\bar{T}_i] \\ \hdashline [0] & [D_i] \end{array}\right]\left\{\begin{array}{c} \{U_s\} \\ \hdashline \{U_p\} \end{array}\right\} = \left\{\begin{array}{c} \{0\} \\ \hdashline \{0\} \end{array}\right\} \tag{9.30}$$

The first equation in Eq. (9.30) can be written as

$$\{U_s\} = [\bar{T}_i]\{U_p\} \tag{9.31}$$

Consequently, the *i*th modal shape $\{\mathrm{U}\}_i$ can be expressed as

$$\{U\}_{\mathrm{i}} = [T]\{U_p\} \tag{9.32}$$

where

$$[T_i] = \left[\begin{array}{c} [\bar{T}] \\ \hdashline [\bar{I}] \end{array}\right] \quad \text{and} \quad \{U_i\} = \left\{\begin{array}{c} \{U_s\} \\ \hdashline \{U_p\} \end{array}\right\}_i \tag{9.33}$$

Step 2: The reduces mass matrix $[\bar{M}_i]$ and the reduced stiffness matrix $[\bar{K}_i]$ are calculated as

$$[\bar{M}_i] = [T_i]^T[M][T_i] \tag{9.34}$$

and

$$[\bar{K}_i] = [\bar{D}_i] + \omega_i^2[\bar{M}_i] \tag{9.35}$$

where the transformation matrix $[\mathrm{T_i}]$ is given by Eq. (9.33) and the reduced dynamic matrix $[D_i]$ is defined Eq. (9.30).Step 3: The reduced eigenproblem

$$[[\bar{K}_i] - \omega^2[\bar{M}_i]]\{U_p\} = \{0\} \tag{9.36}$$

is then solved to obtain improved eigenvalues ω_i^2, its corresponding eigenvector $\{Up\}_i$, and also an approximation for the next order eigenvalue ω_{i+1}^2.

These three-step process may be applied iteratively; that is, the value of ω_i^2 obtained in step 3 may be used as an improved approximate value in step 1 to obtain a further improved value of ω_i^2, in step 3. Experience has shown that one or two such iterations will produce virtually exact eigensolutions.

Illustrative Example 9.5

Repeat Example 9.3 of Sect. 9.2 using the Dynamic Condensation Method.

Solution:

The stiffness matrix and the mass matrix with the coordinates in the order u_1 u_3, u_2, u_4 are given, respectively, by Eqs. (e) and (b) of Illustrative Example 9.3. Substitution of these matrices into Eq. (9.29) results in the dynamic matrix for the system:

$$[D]=\left[\begin{array}{cc:cc} 654.70-\omega_i^2 & 0 & -327.35 & 0 \\ 0 & 654.70-\omega_i^2 & -327.35 & -327.35 \\ \hdashline -327.35 & -327.35 & 654.70-\omega_i^2 & 0 \\ 0 & -327.35 & 0 & 327.35-\omega_i^2 \end{array}\right] \tag{a}$$

Step 1: Assuming we have no initial approximation of ω_1^2, we start step 1 by setting $\omega_1^2 = 0$ and substituting this value into Eq. (a):

$$[D]=\left[\begin{array}{cc:cc} 654.70 & 0 & -327.35 & 0 \\ 0 & 654.70 & -327.35 & -327.35 \\ \hdashline -327.35 & -327.35 & 654.70 & 0 \\ 0 & -327.35 & 0 & 327.35 \end{array}\right] \tag{b}$$

Application of the Gauss-Jordan elimination process to the first two rows gives

$$\left[\begin{array}{cc:cc} 1 & 0 & -0.5 & 0.0 \\ 0 & 1 & -0.5 & -0.5 \\ \hdashline 0 & 0 & 327.35 & -163.67 \\ 0 & 0 & -163.67 & 163.67 \end{array}\right]$$

from which, by Eqs. (9.30) and (9.33)

$$[T_1] = \begin{bmatrix} 0.5 & 0.0 \\ 0.5 & 0.5 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$[\bar{D}_1] = \begin{bmatrix} 327.35 & -163.67 \\ -163.67 & 163.67 \end{bmatrix}$$

Step 2: The reduced mass and stiffness matrices by Eqs. (9.34) and (9.35), are

$$[\bar{M}_1] = [T_1]^T[M][T_1] = \begin{bmatrix} 1.5 & 0.25 \\ 0.25 & 1.25 \end{bmatrix}$$

and

$$[\bar{K}_1] = [\bar{D}_1] + \omega_1^2[\bar{M}_1] = \begin{bmatrix} 327.35 & -163.67 \\ -163.67 & 163.67 \end{bmatrix}$$

Step 3: The solution of the reduced eigenproblem $\left[[\bar{K}_1] - \omega^2[\bar{M}_1]\right]\{U_p\}_1 = \{0\}$ yields

$$\omega_1^2 = 40.39 \quad \text{and} \quad \omega_2^2 = 365.98$$

These values for ω_1^2 and ω_2^2 may be improved by iterating the calculations, that is, by introducing $\omega_1^2 = 40.39$ into Eq. (9.29). This substitution results in

$$[D_1] = \left[\begin{array}{cc:cc} 614.31 & 0 & -327.35 & 0 \\ 0 & 614.31 & -327.35 & -327.35 \\ \hdashline -327.35 & -327.35 & 614.31 & 0 \\ 0 & -327.35 & 0 & 286.96 \end{array}\right]$$

Applications of Gauss-Jordan elimination to the first two rows gives

$$\left[\begin{array}{cc:cc} 1 & 0 & -0.533 & 0.0 \\ 0 & 1 & -0.533 & -0.533 \\ \hdashline 0 & 0 & 265.44 & -174.44 \\ 0 & 0 & -174.44 & 112.53 \end{array}\right]$$

from which

$$[T_1] = \begin{bmatrix} 0.533 & 0.0 \\ 0.533 & 0.533 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$[\bar{D}_1] = \begin{bmatrix} 265.44 & -174.44 \\ -174.44 & 112.53 \end{bmatrix}$$

The reduced mass and stiffness matrices are then

$$[\bar{M}_1] = [T_1]^T [M][T_1] = \begin{bmatrix} 1.568 & 0.284 \\ 0.284 & 1.284 \end{bmatrix}$$

and

$$[\bar{K}_1] = \bar{D}_1 + \omega_1^2 [\bar{M}_1] = \begin{bmatrix} 328.76 & -162.97 \\ -162.67 & 164.39 \end{bmatrix}$$

The solution of the reduced eigenproblem,

$$[[\bar{K}_1] - \omega^2 [\bar{M}_1]]\{U_p\} = \{0\}$$

yields the eigenvalues

$$\omega_1^2 = 39.48, \quad \omega_2^2 = 360.21 \tag{c}$$

and corresponding eigenvectors

$$\{U_p\}_1 = \begin{bmatrix} 0.4283 \\ 0.6562 \end{bmatrix} \qquad \{U_p\}_2 = \begin{bmatrix} 0.6935 \\ -0.6171 \end{bmatrix} \tag{d}$$

The same process is now applied to the second mode, starting by substituting into Eq. (9.29) the approximate eigenvalues $\omega_2^2 = 360.21$ calculated for the second mode in Eq. (c). In this case we obtain

$$[D_2] = \left[\begin{array}{cc:cc} 294.49 & 0 & -327.35 & 0 \\ 0 & 294.49 & -327.35 & -327.35 \\ \hdashline -327.35 & -327.35 & 294.49 & 0 \\ 0 & -327.35 & 0 & -32.86 \end{array}\right]$$

Gauss-Jordan elimination of the first two rows yields

$$\left[\begin{array}{cc:cc} 1 & 0 & -1.112 & 0.0 \\ 0 & 1 & -1.112 & -1.112 \\ \hdashline 0 & 0 & -433.27 & -363.88 \\ 0 & 0 & -363.88 & -396.74 \end{array}\right]$$

from which

$$[T_2] = \begin{bmatrix} 1.112 & 0.0 \\ 1.112 & 1.112 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad [\bar{D}_2] = \begin{bmatrix} -433.27 & -363.88 \\ -363.88 & -396.74 \end{bmatrix}$$

The reduced mass and stiffness matrices are

$$[\bar{M}_2] = [T_2]^T [M][T_2] = \begin{bmatrix} 3.471 & 1.236 \\ 1.236 & 2.236 \end{bmatrix}$$

$$[\bar{K}_2] = [\bar{D}_2] + \omega_2^2 [\bar{M}_2] = \begin{bmatrix} 817.12 & 81.21 \\ 81.21 & 408.56 \end{bmatrix}$$

The solution of the reduced eigenproblem $\left[[\bar{K}_2] - \omega^2 [\bar{M}_2^2]\right]\{U_p\}_2 = \{0\}$ yields for the second mode

$$\omega_2^2 = 328.61$$

An iteration is performed by introducing $\omega_2^2 = 328.61$ into Eq. (9.29) to obtain the following:

$$[D_2] = \left[\begin{array}{cc:cc} 326.09 & 0 & -327.35 & 0 \\ 0 & 326.09 & -327.35 & -327.35 \\ \hdashline -327.35 & -327.35 & 326.09 & 0 \\ 0 & -327.35 & 0 & -1.26 \end{array}\right]$$

Applying Gauss-Jordan elimination to the first two rows yields

$$\left[\begin{array}{cc:cc} 1 & 0 & -1.004 & 0.0 \\ 0 & 1 & -1.004 & -1.004 \\ \hdashline 0 & 0 & -331.14 & -328.62 \\ 0 & 0 & -328.62 & -329.88 \end{array}\right]$$

from which

$$[T_2] = \begin{bmatrix} 1.004 & 0.0 \\ 1.004 & 1.004 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$[\bar{D}_2] = \begin{bmatrix} -331.14 & -328.62 \\ -328.62 & -329.88 \end{bmatrix}$$

The reduced mass and stiffness matrices are:

$$[\bar{M}_2] = [T_2]^T [M][T_2] = \begin{bmatrix} 3.015 & 1.008 \\ 1.008 & 2.008 \end{bmatrix}$$

and

$$[\bar{K}_2] = [\bar{D}_2] + \omega_2^2 [\bar{M}_2] = \begin{bmatrix} 659.78 & 2.54 \\ 2.54 & 329.89 \end{bmatrix}$$

The solution of the reduced eigenproblem $\left[[\bar{K}_2] - \omega^2 [\bar{M}_2]\right]\{U_p\}_2 = \{0\}$ now gives for the second mode

$$\{U_p\}_2 = \begin{Bmatrix} 0.5766 \\ -0.5766 \end{Bmatrix} \quad \omega_2^2 = 327.35 \tag{e}$$

Therefore, from Eqs. (c), (d), and (e) we have obtained for the first two eigenvalues

$$\omega_1^2 = 39.48 \quad \text{and} \quad \omega_2^2 = 327.35 \tag{f}$$

and corresponding eigenvectors

$$\{U_p\}_1 = \begin{Bmatrix} 0.4283 \\ 0.6562 \end{Bmatrix}, \quad \{U_p\}_2 = \begin{Bmatrix} 0.5766 \\ -0.5766 \end{Bmatrix}$$

The eigenvectors of the system are then computed using Eq. (9.32) as follows:

$$\begin{bmatrix} U_1 \\ U_3 \\ U_2 \\ U_4 \end{bmatrix} = \begin{bmatrix} 0.533 & 0.0 \\ 0.533 & 0.533 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} 0.4283 \\ 0.6562 \end{Bmatrix} = \begin{Bmatrix} 0.2283 \\ 0.5780 \\ 0.4283 \\ 0.6562 \end{Bmatrix}_1 \tag{g}$$

Hence ordering the elements of the modal vector:

$$\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{Bmatrix}_1 = \begin{Bmatrix} 0.2283 \\ 0.4283 \\ 0.5780 \\ 0.6562 \end{Bmatrix}_1 \tag{h}$$

and

$$\begin{Bmatrix} U_1 \\ U_3 \\ U_2 \\ U_4 \end{Bmatrix}_2 = \begin{bmatrix} 1.004 & 0.0 \\ 1.004 & 1.004 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} 0.5766 \\ -0.5766 \end{Bmatrix} = \begin{Bmatrix} 0.5789 \\ 0.0 \\ 0.5766 \\ -0.5766 \end{Bmatrix}_2$$

Hence ordering the modal vector:

$$\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{Bmatrix}_2 = \begin{Bmatrix} 0.5789 \\ 0.5766 \\ 0.0 \\ -0.5766 \end{Bmatrix}_2 \tag{i}$$

The eigenvalues and eigenvectors [Eqs. (f), (h), and (i)] calculated for the first two modes in this example using dynamic condensation are virtually identical to the exact solution determined in Eqs. (c) and (d) of Illustrative Example 9.3.

It should be noted that normalization of the eigenvectors is not needed in Eqs. (h) and (i) if the reduced vectors are normalized with respect to the reduced mass of the system, that is, if a reduced eigenvector $\{U_p\}$ satisfies the normalizing equation

$$\{U_p\}^T [\bar{M}] \{U_p\} = 1$$

then by Eq. (9.34)

$$\{U_p\}^T [T]^T [M][T] \{U_p\} = 1 \tag{9.37}$$

and since by Eq. (9.32)

$$\{U\}_i = [T]\{U_p\}$$

and

$$\left\{U_i^T = \{U\}_p^T [T]^T\right.$$

Upon substitution of $\{U\}_\mathrm{I}$ and $\{U\}_i^T$ into Eq. (9.37) results in

$$\{U\}_i^T [M]\{U\}_i = 1$$

thus demonstrating that the eigenvector $\{U\}_i$ is normalized with respect to the mass matrix [M] of the system, if $\{U_p\}$ has been normalized with respect to the reduced mass matrix $\left[\bar{M}\right]$.

The MATLAB program is used to obtain the natural frequencies in Eq. (e).

```
clear all
close all

%_____________________________________________________________________
% Inputs:
%   M, K
%   m =  Number of row to apply the dynamic elimination process
%_____________________________________________________________________

%%%%-GIVEN VALUES-%%%%

%%%Mass Matrix
M = [1, 0 0 0; 0, 1, 0, 0; 0, 0, 1,0; 0,0,0, 1];

%%%Stiffness Matrix
K = 327.35*[2,0,-1,0; 0,2,-1,-1; -1,-1,2,0; 0,-1,0,1];
% m = 0;        %Exact Solution;

%%%Elimination of the first and second rows: m=2
m = 2;          %Dynamic Condensation

N= length(K);   %Total number of row

s=0;
for i=1:2
W =[];
w(i)= sqrt(s(1,1));
W = diag(w(i)^2);

    D = K - W.*M;

    %%%Partition Matrix for Dynamic Condensation (Eq.9.28)
    D_pp = D(m+1:N,m+1:N);
    D_ps = D(m+1:N,1:m);
    D_ss = D(1:m,1:m);
    D_sp = D(1:m,m+1:N);


    D_bar = D_pp-D_ps*inv(D_ss)*D_sp;
    T_bar = -inv(D_ss)*D_sp;
    T = [T_bar; eye(N-m,N-m)];
    M_bar = T'*M*T;
    K_bar = D_bar+W.*M_bar;

    %%%% Solve the eigneproblem (Eq.9.36)
    [a, s] = eig(K_bar, M_bar);

    s;

    a;
end
```

```
s1 = s(1,1)           % Eigenvalue, w1 in Eq.(c)
s2 = s(2,2)           % Eigenvalue, w2 in Eq.(c)
a1 = a(:,1)           % Eigenvector, {U_p}_1 in Eq.(d)

% Use s(2,2) in Eq.(c) for the following iterations

for i=1:2
w(i)= sqrt(s(2,2));
W = diag(w(i)^2);

    D = K - W.*M;

    %%%Partition Matrix for Dynamic Condensation (Eq.9.28)
    D_pp = D(m+1:N,m+1:N);
    D_ps = D(m+1:N,1:m);
    D_ss = D(1:m,1:m);
    D_sp = D(1:m,m+1:N);


    D_bar = D_pp-D_ps*inv(D_ss)*D_sp;
    T_bar = -inv(D_ss)*D_sp;
    T = [T_bar; eye(N-m,N-m)];
    M_bar = T'*M*T;
    K_bar = D_bar+W.*M_bar;

    %%%% Solve the eigenproblem (Eq.9.36)
    [a, s] = eig(K_bar, M_bar);

    s;
    a;
end

s2 = s(2,2);          % Eigenvalue, w2 in Eq.(f)
a2 = a(:,2);          % Eigenvalue, w2 in Eq.(c)

aa = [a1,a2]          % Eigenvalues in Eq.(f)
ss = [s1,s2]          % Eigenvectors in Eq.(f)
```

9.4 Modified Dynamic Condensation

The dynamic condensation method requires the application of elementary operations, as it is routinely done to solve a linear system of algebraic equations, using the Gauss-Jordan elimination process. The elementary operations are required to transform Eq. (9.29) to the form given by Eq. (9.30). However, the method also requires the calculation of the reduced mass matrix by Eq. (9.34). This last calculation involves the multiplication of three matrices of dimensions equal to the total number of coordinates in the system. Thus, for a system defined with many degrees of freedom, the calculation of the reduced mass matrix $[M]$ requires a large number of numerical operations. A modification (Paz 1989), recently proposed, obviates such large number of numerical operations. This modification consists of calculating the reduced stiffness matrix $\left[\bar{K}\right]$ only once by simple elimination of s displacements in Eq. (9.29) after setting $\omega^2 = 0$, thus making unnecessary the repeated calculation of $\left[\bar{K}\right]$ for each mode using Eq. (9.35). Furthermore, it also eliminates the time consumed in calculating the reduced mass matrix $\left[\bar{M}\right]$ using Eq. (9.34). In the modified method, the reduced mass matrix for any mode i is calculated from Eq. (9.35) as

$$\left[\bar{M}_i\right] = \frac{1}{\omega_i^2}\left[\left[\bar{K}\right] - \left[\bar{D}_i\right]\right] \tag{9.38}$$

Where $\left[\bar{K}\right]$ is the reduced stiffness matrix, already calculated, and $\left[\bar{D}_i\right]$ is the dynamic matrix given in the partitioned matrix of Eq. (9.30).

As can be seen, the modified algorithm essentially requires, for each eigenvalues calculated, only the application of the Gauss-Jordan process to eliminate s unknowns in a linear system of equations such as the system in Eq. (9.29).

Illustrative Example 9.6
Repeat Illustrative Example 9.5 using the modified dynamic condensation method.

Solution:
The initial calculations of the modified method are the same as those in Illustrative Example 9.5. Thus, from Illustrative Example 9.5 we have

$$\left[\bar{K}_1\right] = \begin{bmatrix} 327.35 & -163.67 \\ -163.67 & 163.67 \end{bmatrix} \tag{a}$$

$$\left[\bar{M}_1\right] = \begin{bmatrix} 1.5 & 0.25 \\ 0.25 & 1.25 \end{bmatrix} \tag{b}$$

$$\omega_1^2 = 40.39 \qquad \omega_2^2 = 365.98, \tag{c}$$

$$\left[\bar{D}_i\right] = \begin{bmatrix} 265.44 & -174.44 \\ -174.44 & 112.53 \end{bmatrix} \tag{d}$$

and

$$\left[\bar{T}_1\right] = \begin{bmatrix} 0.533 & 0 \\ 0.533 & 0.533 \end{bmatrix} \tag{e}$$

Now, the reduced mass matrix $\left[\bar{M}_1\right]$ is calculated from Eq. (9.38), after substitution in this equation of $\left[\bar{K}_1\right]$ from Eq. (a) and $\left[\bar{D}_1\right]$ from Eq. (d), as

$$\left[\bar{M}_1\right] = \frac{1}{\omega_1^2}\left[\left[\bar{K}_1\right] - \left[\bar{D}_1\right]\right] = \begin{bmatrix} 1.530 & 0.267 \\ 0.267 & 1.266 \end{bmatrix} \tag{f}$$

Then the reduced stiffness and mass matrix given by Eqs. (a) and (f) are used to solve the reduced eigenproblem:

$$\left[\left[\bar{K}_1\right] - \omega^2\left[\bar{M}_1\right]\left\{U_p\right\}_1 = \{0\}\right.$$

to obtain the eigenvalues

$$\omega_1^2 = 39.46 \qquad \omega_2^2 = 363.67 \tag{g}$$

and the eigenvector for the first mode

$$\begin{Bmatrix} U_2 \\ U_4 \end{Bmatrix}_1 = \begin{Bmatrix} 0.43359 \\ 0.66424 \end{Bmatrix}_1 \tag{h}$$

The eigenvector for the first mode, in terms of the original four coordinate, is then obtained from Eq. (9.31) as

$$\begin{Bmatrix} U_1 \\ U_3 \end{Bmatrix} = \begin{bmatrix} 0.533 & 0 \\ 0.533 & 0.533 \end{bmatrix} \begin{Bmatrix} 0.43359 \\ 0.66424 \end{Bmatrix} = \begin{Bmatrix} 0.23110 \\ 0.58514 \end{Bmatrix} \tag{i}$$

The combination of Eqs. (h) and (i) give the eigenvector for the first mode as

$$\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{Bmatrix}_1 = \begin{Bmatrix} 0.23110 \\ 0.43359 \\ 0.58514 \\ 0.66424 \end{Bmatrix}_1 \tag{j}$$

Analogously, for the second mode, we substitute $\omega_2^2 = 363.67$ from Eqs. (g) into (a) of Illustrative Example 9.5, to obtain the following matrices after reducing the first two coordinates:

$$[\bar{T}_2] = \begin{bmatrix} 1.1248 & 0 \\ 1.1248 & 1.1248 \end{bmatrix} \tag{k}$$

$$[\bar{D}_2] = \begin{bmatrix} -445.43 & -368.20 \\ -368.20 & -404.54 \end{bmatrix} \tag{l}$$

The reduced mass matrix $[\bar{M}_2]$ is the calculated from Eq. (9.37) as

$$\begin{aligned} [\bar{M}_2] &= \frac{1}{363.67}\left\{ \begin{bmatrix} 327.35 & -163.67 \\ -163.67 & 163.67 \end{bmatrix} - \begin{bmatrix} -445.43 & -368.20 \\ -368.20 & -404.54 \end{bmatrix} \right\} \\ [\bar{M}_2] &= \begin{bmatrix} 2.1250 & 0.5624 \\ 0.5624 & 1.5624 \end{bmatrix} \end{aligned} \tag{m}$$

Then, for the second mode, the solution of the corresponding reduced eigenproblem gives

$$\omega_2^2 = 319.41, \qquad \begin{Bmatrix} U_2 \\ U_4 \end{Bmatrix}_2 = \begin{Bmatrix} 0.61894 \\ -0.63352 \end{Bmatrix} \tag{n}$$

$$\begin{Bmatrix} U_1 \\ U_3 \end{Bmatrix}_2 = \begin{bmatrix} 1.1248 & 0 \\ 1.1248 & 1.1248 \end{bmatrix} \begin{Bmatrix} 0.61894 \\ -0.63352 \end{Bmatrix} = \begin{Bmatrix} 0.69618 \\ -0.01640 \end{Bmatrix} \tag{o}$$

and

$$\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \end{Bmatrix}_2 = \begin{Bmatrix} 0.69618 \\ 0.61894 \\ -0.01640 \\ -0.63352 \end{Bmatrix}_2 \tag{p}$$

The results obtained for this example using the modified method, although sufficiently approximate, are not as close to the exact solution as those obtained in Example 9.5 by the direct application of the dynamic condensation method. Table 9.1 shows and compares eigenvalues calculated in Illustrative Examples 9.5 and 9.6 with the exact solution obtained previously in Illustrative Example 9.3.

Table 9.1 Comparison of results in Illustrative Examples 9.5 and 9.6 using dynamic condensation and modified dynamic condensation

Eigenvalue					
Mode	Exact solution	Dynamic condensation	Error %	Modified method	Error %
1	39.48	39.48	0.00	39.46	0.05
2	327.35	327.35	0.00	319.41	2.42

The following MATLAB program is to reproduce the solution of Example 6.5 using the modified dynamic condensation method.

```
clc
clear all
close all

%_____________________________________________________________________
% Inputs:
%   M, K
%   m =  Number of row to apply the dynamic elimination process
%_____________________________________________________________________

%%%%-GIVEN VALUES-%%%%

%%%Mass Matrix
M = [1, 0 0 0; 0, 1, 0, 0; 0, 0, 1,0; 0,0,0, 1];

%%%Stiffness Matrix
K = 327.35*[2,0,-1,0; 0,2,-1,-1; -1,-1,2,0; 0,-1,0,1]; %u1,u3,u2,u4
% m = 0;         % Exact Solution;

%%%Elimination of the first and second rows: m=2
m = 2;           % Dynamic Condensation

N= length(K);    %Total number of row

s=0;
for i=1:1
W =[];
w(i)= sqrt(s(1,1));
W = diag(w(i)^2);

    D = K - W.*M;

    %%%Partition Matrix for Dynamic Condensation (Eq.9.28)
    D_pp = D(m+1:N,m+1:N);
    D_ps = D(m+1:N,1:m);
    D_ss = D(1:m,1:m);
    D_sp = D(1:m,m+1:N);


    D_bar = D_pp-D_ps*inv(D_ss)*D_sp;
    T_bar = -inv(D_ss)*D_sp;
    T = [T_bar; eye(N-m,N-m)];
    M_bar = T'*M*T;
    K_bar = D_bar+W.*M_bar;

    %%%% Solve the eigneproblem (Eq.9.36)
    [a, s] = eig(K_bar, M_bar);

    a;

    s;

end


w = sqrt(s(1,1));
M_bar = M_bar;
K_bar = K_bar;


for i=1:1
W = diag(w^2);

    D = K - W.*M;

    %%%Partition Matrix for Dynamic Condensation (Eq.9.28)
    D_pp = D(m+1:N,m+1:N);
    D_ps = D(m+1:N,1:m);
    D_ss = D(1:m,1:m);
    D_sp = D(1:m,m+1:N);

    D_bar = D_pp-D_ps*inv(D_ss)*D_sp;
    T_bar = -inv(D_ss)*D_sp;


end
```

```
D_bar = D_bar;                          % Eq.(d)
T_bar = T_bar;                          % Eq.(e)

%%%Modified Dynamic Condensation (Eq.9.38)
M_bar = 1/W.*(K_bar-D_bar);             % Eq.(f)

[a_1, s_1] = eig(K_bar, M_bar);

s_11 = s(1,1)                           % Eq.(g)
a_11 = (a_1(:,1));                      % Eq.(h) for u2, u4
a_12 = T_bar*(a_11);                    % Eq.(i) for u1, u3

a_1 = [a_11; a_12]                      % Eq.(j) u2,u4,u1,u3


%%%Analogously, for the second mode

w = sqrt(s_1(2,2));
M_bar = M_bar;
K_bar = K_bar;

for i=1:1
W = diag(w^2);

    D = K - W.*M;

    D_pp = D(m+1:N,m+1:N);
    D_ps = D(m+1:N,1:m);
    D_ss = D(1:m,1:m);
    D_sp = D(1:m,m+1:N);


    D_bar = D_pp-D_ps*inv(D_ss)*D_sp;
    T_bar = -inv(D_ss)*D_sp;


end

D_bar = D_bar;                          % Eq.(l)              for mode 2
T_bar = T_bar;                          % Eq.(k)              for mode 2


M_bar = 1/W.*(K_bar-D_bar);             % Eq.(m)              for mode 2

[a_2, s_2] = eig(K_bar, M_bar);

s_22 = s_2(2,2)                         % Eq.(n)              for mode 2

a_21 = (a_2(:,2));                      % Eq.(n) for u2, u4   for mode 2
a_22 = T_bar*(a_21);                    % Eq.(o) for u1, u3   for mode 2

a_2 = [a_21; a_22]                      % Eq.(p) u2,u4,u1,u3 for mode 2
```

9.5 Summary

The reduction of secondary or dependent degrees of freedom is usually accomplished in practice by the Static Condensation Method. This method consists of determining, by a partial Gauss-Jordan elimination, the reduced stiffness matrix corresponding to the primary degrees of freedom and the transformation matrix relating the secondary and primary degrees of freedom. The same transformation matrix is used in an orthogonal transformation to reduce the mass and damping matrices of the system. Static condensation introduces errors when applied to the solution structural dynamics problems. However, as is shown in this chapter, the application of the Dynamic Condensation Method

substantially reduces or eliminates these errors. Furthermore, the Dynamic Condensation Method converges rapidly to the exact solution when iteration is applied.

9.6 Problems

Problem 9.1
The stiffness and mass matrices of a certain structure are given by

$$[K] = \begin{bmatrix} 10 & -2 & -1 & 0 \\ -2 & 6 & -3 & -2 \\ -1 & -3 & 12 & -1 \\ 0 & -2 & -1 & 8 \end{bmatrix}, \qquad [M] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

(a) Use the Static Condensation Method to determine the transformation matrix and the reduced stiffness and mass matrices corresponding to the elimination of the first two degrees of freedom (the massless degrees of freedom).
(b) Determine the natural frequencies and corresponding normal modes for reduced system.

Problem 9.2
Repeat (a) and (b) of Problem 9.1 for a structure having stiffness matrix as indicated in that problem, but mass matrix given by

$$[M] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

Problem 9.3
Determine the natural frequencies and modal shape of the system in Problem 9.2 in terms of its four original coordinates; find the errors in the two modes obtained in Part (b) of Problem 9.2.

Problem 9.4
Consider the shear building shown in Fig. P9.4.
(a) Determine the transformation matrix and the reduced stiffness and mass matrices corresponding to the static condensation of the coordinates u_1 u_3, and u_4 as indicated in the figure.

(b) Determine the natural frequencies and normal modes for the reduced system obtained in (a).
(c) Use the results of (b) to determine the modal shapes, described by the five original coordinates, corresponding to the two lowest frequencies.

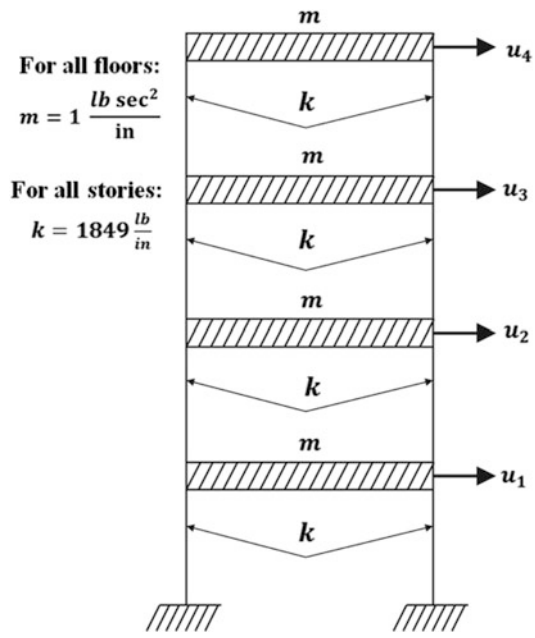

Fig. P9.4

Problem 9.5
Use the results obtained in Problem 9.4 to determine the maximum shear forces in the stories of the building in Fig. P9.4 when subjected to an earthquake for which the response spectrum is given in Fig. 5.10 of Sect. 5.4.

Problem 9.6
Use the results in Problem 9.4 to determine the maximum shear forces in the stories of the building in Fig. P9.4 when subjected to an earthquake the response spectrum is given in Fig. 5.10 of Sect. 5.4.

Problem 9.7
Repeat Problem 9.2 using the Dynamic Condensation Method.

Problem 9.8
Repeat Problem 9.4 using the Dynamic Condensation Method and compare results with the exact solution.

Problem 9.9
Consider the five-story shear building of Fig. P9.4 subjected at its foundation to the time-acceleration excitation depicted in Fig. P9.9. Use static condensation of the coordinates u_1, u_3, and u_4 and determine:

(a) The two natural frequencies and corresponding modal shapes of the reduced system.
(b) The displacements at the floor levels considering two modes.
(c) The shear forces in the columns of the structure also considering two modes.

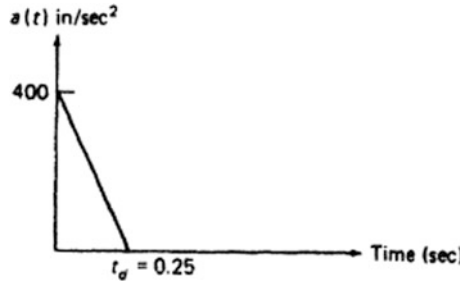

Fig. P9.9

Problem 9.10
Solve Problem 9.9 using the Dynamic Condensation Method. Problem 9.11.

Problem 9.11
The stiffness and mass matrices for a certain structure are

$$[K] = 10^6 \begin{bmatrix} 0.906 & 0.294 & 0.424 \\ 0.294 & 0.318 & 0.176 \\ 0.424 & 0.176 & 80.000 \end{bmatrix}$$

$$[M] = \begin{bmatrix} 288 & -8 & 1566 \\ -8 & 304 & 644 \\ 1566 & 644 & 80,000 \end{bmatrix}$$

Calculate the fundamental natural frequency of the system after reduction of the first coordinate by the following methods: (a) Static condensation; and (b) Dynamic condensation. Also obtain the natural frequencies as a three-degrees-of-freedom system and compare results for the fundamental frequency.

Problem 9.12
Repeat Problem 9.11 using the Modified Dynamic Condensation Method and compare results with the solution obtained with no condensation.

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Part III

Framed Structures Modeled as Discrete Multi-Degree-of-Freedom Systems

Dynamic Analysis of Beams

10

In this chapter, we shall study the dynamic behavior of structures designated as beams, that is, structures that carry loads that are mainly transverse to the longitudinal direction, thus producing flexural stresses and lateral displacements. We begin by establishing the static characteristics for a beam segment; and then introduce the dynamic effects produced by the inertial forces. Two approximate methods are presented to take into account the inertial effect in the structure: (1) the lumped mass method in which the distributed mass is assigned to point masses, and (2) the consistent mass method in which the assignment to point masses includes rotational effects. The latter method is consistent with the static deflections of the beam. In Part IV on Special Topics the exact theory for dynamics of beams considering the elastic and inertial distributed properties will be presented.

10.1 Shape Functions for a Beam Segment

Consider a uniform beam element of cross-sectional moment of inertia I, length L, and material modulus of elasticity E as shown in Fig. 10.1. We shall establish the relation between static forces and moments designated as P_1, P_2, P_3 and P_4 and the corresponding linear and angular displacements δ_1, δ_2, δ_3 and δ_4 at the ends of the beam element as indicated in Fig. 10.1. The relationship thus obtained is known as the stiffness matrix equation for a beam element. The forces P_t and the displacements δ_i are said to be at the nodal coordinates defined at the ends for the beam element.

The differential equation for small transverse displacements of a beam, which bis well known from elementary studies of strength and materials, is given by

$$EI\frac{d^2u}{dx^2} = M(x) \tag{10.1}$$

in which $M(x)$ is the bending moment at a section x of the beam and u is the transverse displacement.

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M. Paz, Y. H. Kim, *Structural Dynamics*, https://doi.org/10.1007/978-3-319-94743-3_10

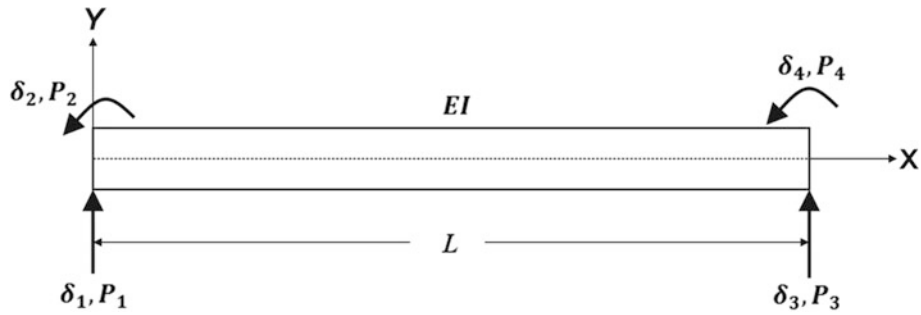

Fig. 10.1 Bean segment showing forces and displacement at the nodal coordinates

The differential Eq. (10.1) for a uniform beam element is equivalent to

$$EI\frac{d^4u}{dx^4} = p(x) \tag{10.2}$$

since

$$\frac{dM(x)}{dx} = V(x) \tag{10.3}$$

and

$$\frac{dV(x)}{dx} = p(x)$$

Where $p(x)$ is the beam load per unit length and $V(x)$ is the shear force.

We state first the general definition of the stiffness coefficient which is designated by k_{ij}, that is, k_{ij} is the force at nodal coordinate i due to a unit displacement at nodal coordinate j while all other nodal coordinates are maintained at zero displacement. Figure 10.2 shows the displacement curves and the corresponding stiffness coefficients due to a unit displacement at each one of the four nodal coordinates of the beam element. To determine the expressions for the stiffness coefficients k_{ij}, we begin by finding the equations for displaced curves shown in Fig. 10.2. We first consider the beam element in Fig. 10.1 free of loads $[p(x) = 0]$, except for the forces P_1, P_2, P_3 and P_4 applied at the nodal coordinates. In this case, Eq. (10.2) is reduced to

$$\frac{d^4u}{dx^4} = 0 \tag{10.4}$$

Successive integrations of Eq. (10.4) yields

$$\frac{d^3u}{dx^3} = C_1$$
$$\frac{d^2u}{dx^2} = C_1x + C_2$$

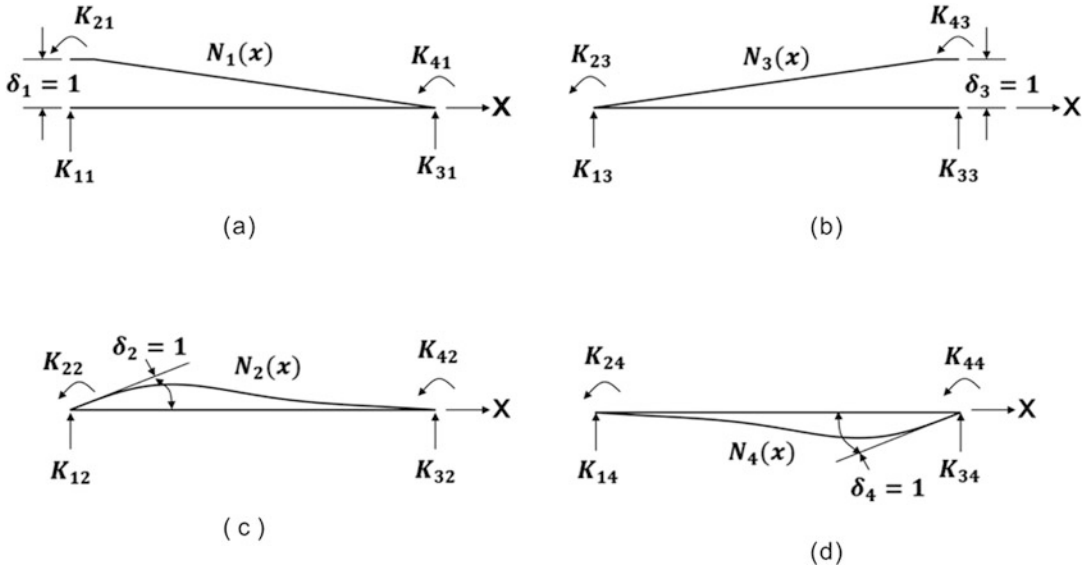

Fig. 10.2 Beam element showing static deflection curves due to a unit displacement at one of the nodal coordinates

$$\frac{du}{dx} = \frac{1}{2}C_1x^2 + C_2x + C_3 \tag{10.5}$$

$$u = \frac{1}{6}C_1x^3 + \frac{1}{2}C_2x^2 + C_3x + C_4 \tag{10.6}$$

in which C_1, C_2, C_3 and C_4 are constants of integration to be evaluated using boundary conditions. For example, to determine the function $N_1(x)$ for the curve shown in Fig. 10.2a, we make use of the following boundary conditions:

$$\text{at} \quad x = 0 \quad u(0) = 1 \quad \text{and} \quad \frac{du(0)}{dx} = 0 \tag{10.7}$$

$$\text{at} \quad x = L \quad u(L) = 0 \quad \text{and} \quad \frac{du(L)}{dx} = 0 \tag{10.8}$$

Use of these conditions in Eqs. (10.5) and (10.6), results in an algebraic system of four equations to determine the constants C_1, C_2, C_3 and C_4.

Then the subsequent substitution of these constants into Eq. (10.6) results in the equation of the deflected curve for the beam element in Fig. (10.2a) as

$$N_1(x) = 1 - 3\left(\frac{x}{L}\right)^2 + 2\left(\frac{x}{L}\right)^3 \tag{10.9a}$$

in which $N_I(x)$ is used instead of $u(x)$ to correspond to the condition $\delta_1 = 1$ imposed on the beam element. Proceeding in analogous fashion, we obtain for the equations of the deflected curves in the other cases depicted in Fig. 10.2 the following equations:

$$N_2(x) = x\left(1 - \frac{x}{L}\right)^2 \tag{10.9b}$$

$$N_3(x) = 3\left(\frac{x}{L}\right)^2 - 2\left(\frac{x}{L}\right)^3 \tag{10.9c}$$

$$N_4(x) = \frac{x^2}{L}\left(\frac{x}{L} - 1\right) \tag{10.9d}$$

Since N_1(x) is the deflection corresponding to a unit displacement $\delta_1 = 1$, the displacement resulting from an arbitrary displacement δ_1, is $N_1(x)\ \delta_1$. Analogously, the deflection resulting from nodal displacements δ_2, δ_3 and δ_4 are, respectively $N_2(x)\ \delta_2$, $N_3(x)\delta_3$ and $N_4(x)\delta_4$. Therefore, the total deflection $u(x)$ at coordinate x due to arbitrary displacements at the nodal coordinates of the beam element is given by superposition as

$$u(x) = N_1(x)\delta_1 + N_2(x)\delta_2 + N_3(x)\delta_3 + N_4(x)\delta_4 \tag{10.10}$$

The shape equations which are given by Eqs. (10.9a, b, c and d) and which correspond to unit displacements at the nodal coordinates of a beam element may be used to determine expressions for the stiffness coefficients. For example, consider the beam in Fig. 10.2b which is in equilibrium with the forces producing the displacement $\delta_2 = 1.0$. For this beam in the equilibrium position, we assume that a virtual displacement equal to the deflection curve shown in Fig. 10.2a takes place. We then apply the principle of virtual work, which states that, for an elastic system in equilibrium, the work done by the external forces is equal to the work of the internal forces during the virtual displacement. In order to apply this principle, we note that the external work W_E is just equal to the product of the force k_{12} displaced by $\delta_1 = 1$, that is

$$W_E = k_{12}\delta_1 \tag{10.11}$$

This work, as stated above, is equal to the work performed by the elastic forces during the virtual displacement. Considering the work performed by the bending moment, we obtain for the internal work

$$W_1 = \int_0^L M(x)d\theta \tag{10.12}$$

in which $M(x)$ is the bending moment at section x of the beam and $d\theta$ is the incremental angular displacement of this section of the element.

For the virtual displacement under consideration, the transverse deflection of the beam is given by Eq. (10.9b), which is related to the bending moment through the differential Eq. (10.1). Substitution of the second derivative $N_2''(x)$ of Eq. (10.9b) into Eq. (10.1) results in

$$EI\, N_2''(x) = M(x) \tag{10.13}$$

The angular deflections $d\theta$ produced during this virtual displacement is related to the resulting transverse deflection of the beam $N_1(x)$ by

$$\frac{d\theta}{dx} = \frac{d^2N_1(x)}{dx^2} = N_1''(x)$$

or

$$d\theta = N_1''(x)dx \tag{10.14}$$

Equating the external virtual work W_E from Eq. (10.11) with the internal virtual work W_1 from Eq. (10.12) after using $M(x)$ and $d\theta$, respectively, from Eqs. (10.13) and (10.14) finally gives the stiffness coefficient as

$$k_{12} = \int_0^L EI\, N_1''(x) N_2''(x) dx \tag{10.15}$$

In general, any stiffness coefficient k_{ij} associated with beam flexure, therefore, may be expressed as

$$k_{ij} = \int_0^L EIN_i''(x) N_j''(x) dx \tag{10.16}$$

It may be seen from Eq. (10.16) that $k_{ij} = k_{ji}$ since the interchange of indices requires only an interchange of the two factors, $N_i''(x)$ and $N_j''(x)$ in Eq. (10.16). This equivalence of $k_{ij} = k_{ji}$ is a particular case of Betti's theorem, but it is better known as *Maxwell's reciprocal theorem.*

It should be pointed out that although the shape functions, Eqs. (10.9a, b, c and d), were obtained for a uniform beam, in practice they are nevertheless also used in determining the stiffness coefficients for non-uniform beams.

Considering the case of a uniform beam element of length L and cross-sectional moment of inertia I, we may calculate any stiffness coefficient from Eqs. (10.16) and the use of Eqs. (10.9a, b, c and d). In particular, the stiffness coefficient k_{12} is calculated as follows:

From Eq.(10.9a), we obtain

$$N_1''(x) = -\frac{6}{L^2} + \frac{12x}{L^3}$$

and from Eq. (10.9b)

$$N_2''(x) = -\frac{4}{L} + \frac{6x}{L^2}$$

Substitution in Eq. (10.15 gives)

$$k_{12} = EI \int_0^L \left(\frac{-6}{L^2} + \frac{12x}{L^3}\right)\left(\frac{-4}{L} + \frac{6x}{L^2}\right) dx$$

and performing the integration results in

$$k_{12} = \frac{6EI}{L^2}$$

Since the stiffness coefficient k_{1j} is defined as the force at the nodal coordinate 1 due to unit displacement at the coordinate j, the forces at coordinate 1 due to successive displacement δ_1, δ_2, δ_3 and δ_4 at the four nodal coordinates of the beam element are given, respectively, by $k_{11}\ \delta_1$, $k_{12}\ \delta_2$, $k_{13}\ \delta_3$ and $k_{14}\ \delta_4$. Therefore, the total force P_1 at coordinate 1 resulting from these nodal displacements is obtained by the superposition of the resulting forces, that is,

$$P_1 = k_{11}\delta_1 + k_{12}\delta_2 + k_{13}\delta_3 + k_{14}\delta_4$$

Analogously, the forces at the other nodal coordinates resulting from the nodal displacement δ_1, δ_2, δ_3, δ_4 are

$$\begin{aligned} P_2 &= k_{12}\delta_1 + k_{22}\delta_2 + k_{23}\delta_3 + k_{24}\delta_4 \\ P_3 &= k_{31}\delta_1 + k_{32}\delta_2 + k_{33}\delta_3 + k_{34}\delta_4 \\ P_4 &= k_{41}\delta_1 + k_{42}\delta_2 + k_{43}\delta_3 + k_{44}\delta_4 \end{aligned} \tag{10.17}$$

The above equations are written conveniently in matrix notation as

$$\begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{bmatrix} = \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \\ k_{31} & k_{32} & k_{33} & k_{34} \\ k_{41} & k_{42} & k_{43} & k_{44} \end{bmatrix} \begin{bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \end{bmatrix} \tag{10.18}$$

or in condensed notation as

$$\{P\} = [k]\{\delta\} \tag{10.19}$$

in which $\{P\}$ and $\{\delta\}$ are, respectively, the force and the displacement vectors at the four nodal coordinates of the beam element and $[k]$ is the beam element stiffness matrix.

The use of Eq. (10.16) in the manner shown above to determine the coefficient k_{12} will result in the evaluation of all the coefficients of the stiffness matrix. This result for a uniform beam element is

$$\begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{bmatrix} = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \begin{bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \end{bmatrix} \tag{10.20}$$

or in condensed notation

$$\{P\} = [k]\{\delta\} \tag{10.21}$$

10.2 System Stiffness Matrix

Thus far we have established in Eq. (10.20) the stiffness equation for a uniform beam element, that is, we have obtained the relationship between nodal displacements (linear and angular) and nodal forces (forces and moments). Our next objective is to obtain the same type of relationship between the nodal displacements and the nodal forces, but now for the entire structure (system stiffness equation). Furthermore, our aim is to obtain the system stiffness matrix from the stiffness matrix of each element of the system. The procedure is perhaps better explained through a specific example such as the cantilever beam shown in Fig. 10.3.

The first step in obtaining the system stiffness matrix is to divide the structure into elements. The beam in Fig. 10.3 has been divided into three elements which are numbered sequentially for

identification. The second step is to identify the nodes or joints between elements and to number consecutively those nodal coordinates that are not constrained.

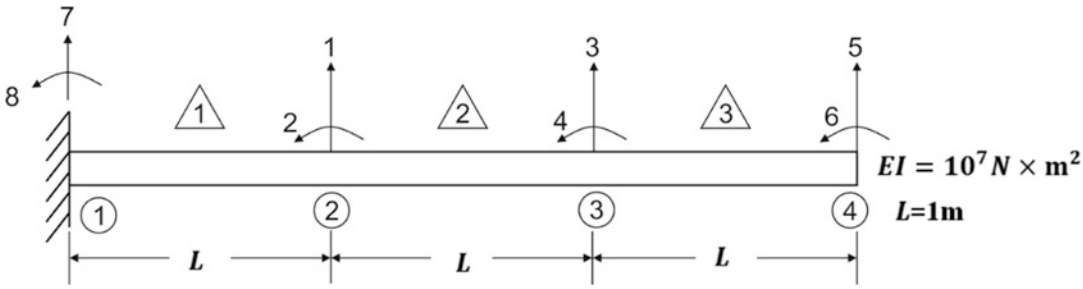

Fig. 10.3 Cantilever beam divided into three beam segments with numbered system nodal coordinates

The constrained or fixed nodal coordinates are the last to be labeled. For beams, we consider only two possible displacements at each node, a vertical deflection and an angular displacement. The cantilever beam in Fig. 10.3 with its three elements results in a total of six free nodal coordinates and two fixed nodal coordinates. The third step is to obtain systematically the stiffness matrix for each element in the system and to add the element stiffness coefficients appropriately to obtain the system stiffness matrix. This method of assembling the system stiffness matrix is called the direct method. In effect, any stiffness coefficient k_{ij} of the system may be obtained by adding together the corresponding stiffness coefficients associated with those nodal coordinates. Thus, for example, to obtain the system stiffness coefficient k_{33}, it is necessary to add the stiffness coefficients of beam elements 2 and 3 corresponding to node three. These coefficients are designated $k_{33}^{(2)}$ and $k_{11}^{(3)}$, respectively. The upper indices serve to identify the beam element, and the lower indices to locate the appropriate stiffness coefficients in the corresponding element stiffness matrices.

Proceeding with the example in Fig. 10.3 and using Eq. (10.20), we obtain the following expression for the stiffness matrix of beam element 2, namely

$$\left[k^{(2)}\right] = 10^7 \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 12 & 6 & -12 & 6 \\ 6 & 4 & -6 & 2 \\ -12 & -6 & 12 & -6 \\ 6 & 2 & -6 & 4 \end{bmatrix} \end{array} \tag{10.22}$$

For the beam element 2, the nodal coordinates numbered one to four coincide with the assignment of system nodal coordinates also numbered 1–4, as may be seen in Fig. 10.3. However, for the beam elements 1 and 3 of this beam, the assignment of element nodal coordinates numbered 1–4 does not coincide with the assigned system coordinates. For example, for element 1 the assigned system coordinates as seen in Fig. 10.3 are 7, 8, 1, 2; for element 3, 3, 4, 5, 6. In the process of assembling the system stiffness, coefficients for element 2 will be correctly allocated to coordinates 1, 2, 3, 4; for element 1 to coordinates 7, 8, 1, 2; and for element 3 to coordinates 3, 4, 5, 6. A simple way to indicate this allocation of coordinates, when working by hand, is to write at the top and on the right of the element stiffness matrix the coordinate numbers corresponding to the system nodal coordinates

for the element as it is indicated in Eq. (10.22) for element 2. The stiffness matrices for elements 1 and 3 with the corresponding indication of system nodal coordinates are, respectively,

$$\begin{matrix} & 7 & 8 & 1 & 2 \\ \left[k^{(1)}\right] = 10^7 \begin{bmatrix} 12 & 6 & -12 & 6 \\ 6 & 4 & -6 & 2 \\ -12 & -6 & 12 & -6 \\ 6 & 2 & -6 & 4 \end{bmatrix} & \begin{matrix} 7 \\ 8 \\ 1 \\ 2 \end{matrix} \end{matrix} \tag{10.23}$$

and

$$\begin{matrix} & 3 & 4 & 5 & 6 \\ \left[k^{(3)}\right] = 10^7 \begin{bmatrix} 12 & 6 & -12 & 6 \\ 6 & 4 & -6 & 2 \\ -12 & -6 & 12 & -6 \\ 6 & 2 & -6 & 4 \end{bmatrix} & \begin{matrix} 3 \\ 4 \\ 5 \\ 6 \end{matrix} \end{matrix} \tag{10.24}$$

Proceeding systematically to assemble the system stiffness matrix, we transfer each entry in the element stiffness matrices, Eqs. (10.22, 10.23, and 10.24), to the appropriate location in the system stiffness matrix. For instance, the stiffness coefficient for element, 3, $k^{(3)}{}_{13} = -12 \times 10^7$ should be transferred to location at row 3 and column 5 since these are the nodal coordinates indicated at right and top of matrix Eq. (10.24) for this entry. Every element stiffness coefficient transferred to its appropriate location in the system stiffness matrix is added to the other coefficients accumulated at that location. The stiffness coefficients corresponding to columns or rows carrying a label of a fixed system nodal coordinate (seven and eight in the present example) are simply disregarded since the constrained nodal coordinates are not unknown quantities. The assemblage of the system matrix in the manner described results for this example in a 6×6 matrix, namely

$$[k] = 10^7 \begin{bmatrix} 24 & 0 & -12 & 6 & 0 & 0 \\ 0 & 8 & -6 & 2 & 0 & 0 \\ -12 & -6 & 24 & 0 & -12 & 6 \\ 6 & 2 & 0 & 8 & -6 & 2 \\ 0 & 0 & -12 & -6 & 12 & -6 \\ 0 & 0 & 6 & 2 & -6 & 4 \end{bmatrix} \tag{10.25}$$

Equation (10.25) is thus the system stiffness matrix for the cantilever beam shown in Fig. 10.3 which has been segmented into three elements. As such, the system stiffness matrix relates the forces and the displacements at the nodal system coordinates in the same manner as the element stiffness matrix relates forces and displacements at the element nodal coordinates.

10.3 Inertial Properties-Lumped Mass

The simplest method for considering the inertial properties for a dynamic system is to assume that the mass of the structure is lumped at the nodal coordinates where translational displacements are defined, hence the name lumped mass method. The usual procedure is to distribute the mass of each element to the nodes of the element. This distribution of the mass is determined by statics. Figure 10.4 shows, for beam segments of length L and distributed mass $\bar{m}$ (x) per unit of length, the nodal allocation for uniform, triangular, and general mass distribution along the beam segment. The assemblage of the mass matrix for the entire structure will be a simple matter of adding the contributions of lumped masses at the nodal coordinates defined as translations.

In this method, the inertial effect associated with any rotational degree of freedom is usually assumed to be zero, although a finite value may be associated with rotational degrees of freedom by calculating the mass moment of inertia of a fraction of the beam segment about the nodal points.

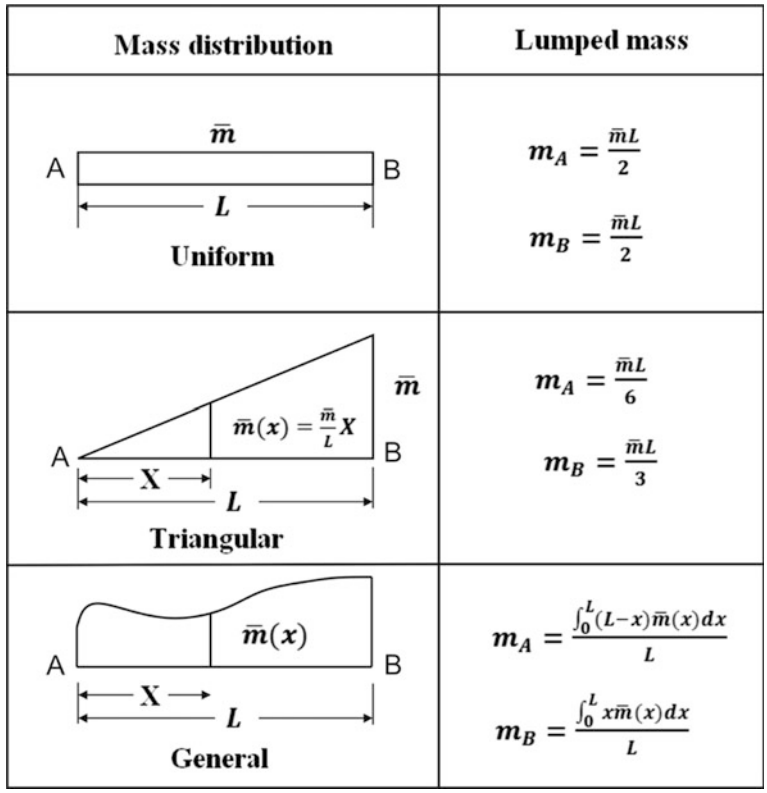

Fig. 10.4 Lumped masses for beam segments with distributed mass

For example, for a uniform beam, this calculation would result in determining the mass moment of inertia of half of the beam segment about each node, that is

$$I_A = I_B = \frac{1}{3}\left(\frac{\bar{m}L}{2}\right)\left(\frac{L}{2}\right)^2$$

where $\bar{m}$ is the mass per unit length along the beam. For the cantilever beam shown in Fig. 10.3 in which only translation mass effects are considered, the mass matrix of the system would be the diagonal matrix, namely

$$[M] = \begin{matrix} \\ \\ \\ \\ \\ \\ \end{matrix} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \begin{bmatrix} m_1 & & & & & \\ & 0 & & & & \\ & & m_3 & & & \\ & & & 0 & & \\ & & & & m_5 & \\ & & & & & 0 \end{bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{matrix} \tag{10.26}$$

in which

$$m_1 = \frac{\bar{m}L_1}{2} + \frac{\bar{m}L_2}{2}$$
$$m_3 = \frac{\bar{m}L_2}{2} + \frac{\bar{m}L_3}{2}$$
$$m_5 = \frac{\bar{m}L_3}{2}$$

Using a special symbol $\lceil\ \rfloor$ for diagonal matrices, we may write Eq. (10.26) as

$$[M] = \lceil m_1 \quad 0 \quad m_3 \quad 0 \quad m_5 \rfloor \tag{10.27}$$

10.4 Inertial Properties—Consistent Mass

It is possible to evaluate the mass coefficients corresponding to the nodal coordinates of a beam element by a procedure similar to the determination of element stiffness coefficients. First, we define the mass coefficient m_{ij} as the force at nodal coordinate i due to a unit acceleration at nodal coordinate j while all other nodal coordinates are maintained at zero acceleration.

Consider the beam segment shown in Fig. 10.5a which has distributed mass $\bar{m}(x)$ per unit of length. In the consistent mass method, it is assumed that the deflections resulting from unit dynamic displacements at the nodal coordinates of the beam element are given by the same functions $N_1(x)$, $N_2(x)$, $N_3(x)$, and $N_4(x)$ of Eqs. (10.9a, b, c and d) which were obtained from static considerations. If the beam segment is subjected to a unit nodal acceleration at one of the nodal coordinates, say $\ddot{\delta}_2$, the transverse acceleration developed along the length of the beam is given by the second derivative with respect to time of Eq. (10.10). In this case, with $\ddot{\delta}_1 = \ddot{\delta}_3 = \ddot{\delta}_4 = 0$, we obtain

$$\ddot{u}_2(x) = N_2(x)\ddot{\delta}_2 \tag{10.28}$$

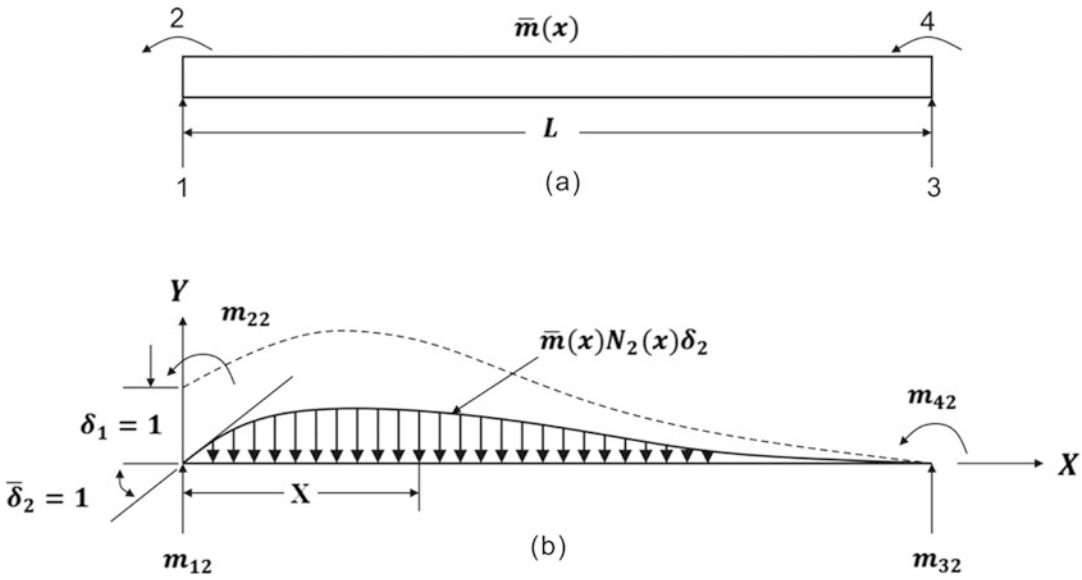

Fig. 10.5 (**a**) Beam element with distributed mass showing nodal coordinates. (**b**) element supporting inertial load due to acceleration $\ddot{\delta}_2 = 1$, undergoing virtual displacement $\delta_1 = 1$

The inertial force $f_1(x)$ per unit of length along the beam due to this acceleration is then given by

$$f_1(x) = \bar{m}(x)\ddot{u}_2(x)$$

or using Eq. (10.28) by

$$f_1(x) = \bar{m}(x)N_2(x)\ddot{\delta}_2$$

or, since $\ddot{\delta}_2 = 1$, gb

$$f_1(x) = \bar{m}(x)N_2(x) \tag{10.29}$$

Now to determine the mass coefficient m_{12}, we give to the beam in Fig. 10.5b, which is in equilibrium with the forces resulting from a unit acceleration $\ddot{\delta}_2 = 1$, a virtual displacement corresponding to a unit displacement at coordinate 1, $\delta_1 = 1$, and proceed to apply the Principle of Virtual Work for an elastic system (external work equal to internal virtual work). The virtual work of the external force is simply

$$W_E = m_{12}\delta_1 = m_{12} \tag{10.30}$$

since the only external force undergoing virtual displacement is the inertial force reaction $m_{12}\delta_1$ with $\delta_1 = 1$. The virtual work of the internal forces per unit of length along the beam segment is

$$\delta W_I = f_1(x)N_1(x)$$

or by Eq. (10.29)

$$\delta W_I = \bar{m}(x)N_2(x)N_1(x)$$

and for the entire beam

$$W_I = \int_0^L \bar{m}(x) N_2(x) N_1(x) dx \tag{10.31}$$

Equating the external and internal virtual work given, respectively, by Eqs. (10.30) and (10.31) results in

$$m_{12} = \int_0^L \bar{m}(x) N_2(x) N_1(x) dx \tag{10.32}$$

which is the expression for the consistent mass coefficient m_{12}. In general, a consistent mass coefficient may be calculated from

$$m_{ij} = \int_0^L \bar{m}(x) N_i(x) N_j(x) dx \tag{10.33}$$

It may be seen from Eq. (10.33) that $m_{ij} = m_{ji}$ since the interchange of the subindices only results in an interchange of the order of the factors N_i (x) and N_j (x) under the integral.

In practice, the cubic Eqs. (10.9a, b, c and d) are used in calculating the mass coefficients of any straight beam element. For the special case of the beam with uniformly distributed mass, the use of Eq. (10.33) gives the following relation between inertial forces and acceleration at the nodal coordinates:

$$\begin{Bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{Bmatrix} = \frac{\bar{m}L}{420} \begin{bmatrix} 156 & 22L & 54 & -13L \\ 22L & 4L^2 & 13L & -3L^2 \\ 54 & 13L & 156 & -22L \\ -13L & -3L^2 & -22L & 4L^2 \end{bmatrix} \begin{Bmatrix} \ddot{\delta}_1 \\ \ddot{\delta}_2 \\ \ddot{\delta}_3 \\ \ddot{\delta}_4 \end{Bmatrix} \tag{10.34}$$

When the mass matrix, Eq. (10.34), has been evaluated for each beam element of the structure, the mass matrix for the entire system is assembled by exactly the same procedure (direct method) as described in developing the stiffness matrix for the system. The resulting mass matrix will in general have the same arrangement of nonzero terms as the stiffness matrix.

The dynamic analysis using the lumped mass matrix requires considerably less computational effort than the analysis using the consistent mass method for the following reasons. The lumped mass matrix for the system results in a diagonal mass matrix whereas the consistent mass matrix has many off diagonal terms which are called mass coupling. Also, the lumped mass matrix contains zeros in its main diagonal due to assumed zero rotational inertial forces. This fact permits the elimination by static condensation (Chap. 9) of the rotational degrees of freedom, thus reducing the dimension of the dynamic problem. Nevertheless, the dynamic analysis using the consistent mass matrix gives results that are better approximations to the exact solution compared to the lumped mass method for the same element discretization.

Illustrative Example 10.1

Determine the lumped mass and the consistent mass matrices for the cantilever beam in Fig. 10.6. Assume uniform mass, $\bar{m} = 420$ kg/m.

Solution:

(a) Lumped Mass Matrix.
The lumped mass at each node of any of the three beam segments, into which the cantilever beam has been divided, is simply half of the mass of the segment. In the present case, the lumped mass at each node is 210 kg as shown in Fig. 10.6. The lumped mass matrix $[M_L]$ for this structure is a diagonal matrix of dimension 6×6, namely,

$$[M_L] = \lceil 420 \;\; 0 \;\; 420 \;\; 0 \;\; 210 \;\; 0 \rfloor$$

(b) Consistent Mass Matrix.
The consistent mass matrix for a uniform beam segment is given by Eq. (10.34). The substitution of numerical values for this example $L = 1\ m$,

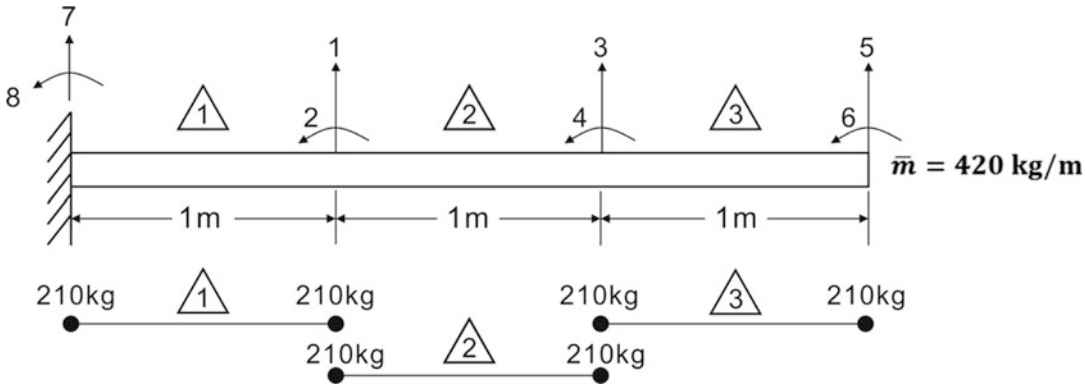

Fig. 10.6 Lumped masses for Illustrative Example 10.1

$\bar{m} = 420$ kg/m into Eq. (10.34) gives the consistent mass matrix $[m]_c$ for any of the three beam segments as

$$[m]_c = \begin{matrix} \\ 1 \\ 2 \\ 3 \\ 4 \end{matrix} \begin{matrix} \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{bmatrix} 156 & 22 & 54 & -13 \\ 22 & 4 & 13 & -3 \\ 54 & 13 & 156 & -22 \\ -13 & -3 & -22 & 4 \end{bmatrix} \end{matrix} \qquad \text{(a)}$$

The assemblage of the system mass matrix from the element mass matrices is carried out in exactly the same manner as the assemblage of the system stiffness matrix from the element stiffness matrices, that is, the element mass matrices are allocated to appropriate entries in the system mass matrix. For the second beam segment, this allocation corresponds to the first four coordinates as indicated above and on the right of Eq. (a). For the beam segment $\triangle\!\!\!

segments as indicated in Fig. 10.6. The consistent mass matrix $[M]_c$ for this example obtained in this manner is given by

$$[M]_c = \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \\ \begin{bmatrix} 312 & 0 & 54 & -13 & 0 & 0 \\ 0 & 8 & 13 & -3 & 0 & 0 \\ 54 & 13 & 312 & 0 & 54 & -13 \\ -13 & -3 & 0 & 8 & 13 & -3 \\ 0 & 0 & 54 & 13 & 156 & -22 \\ 0 & 0 & -13 & -3 & -22 & 4 \end{bmatrix} \end{array} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{matrix} \tag{b}$$

We note that the mass matrix $[M]_c$ is symmetric and also banded as in the case of the stiffness matrix for this system. These facts are of great importance in developing computer programs for structural analysis, since it is possible to perform the necessary calculations storing in the computer only the diagonal elements and the elements to one of the sides of the main diagonal. The maximum number of nonzero elements in any row which are required to be stored is referred to as the bandwidth of the matrix. For the matrix Eq. (b) the bandwidth is equal to four (NBW = 4). In this case, it is necessary to store a total of $6 \times 4 = 24$ coefficients, whereas if the square matrix were to be stored, it would require $6 \times 6 = 36$ storage spaces. This economy in storing spaces becomes more dramatic for structures with a large number of nodal coordinates. The dimension of the bandwidth is directly related to the largest difference of the nodal coordinate labels assigned to any of the elements of the structure. Therefore, it is important to number the system nodal coordinates so as to minimize this difference.

10.5 Damping Properties

Damping coefficients are defined in a manner entirely parallel to the definition of the stiffness coefficient or the mass coefficient. Specifically, the damping coefficient c_{ij} is defined as the force developed at coordinate i due to a unit velocity at j. If the damping forces distributed in the structure could be determined, the damping coefficients of the various structural elements would then be used in obtaining the damping coefficient corresponding to the system. For example, the damping coefficient c_{ij} for an element might be of the form

$$c_{ij} = \int_0^L c(x) N_i(x) N_j(x) dx \tag{10.35}$$

where $c(x)$ represents the distributed damping coefficient per unit length. If the element damping matrix could be calculated, the damping matrix for the entire structure could be assembled by a superposition process equivalent to the direct stiffness matrix. In practice, the evaluation of the damping property $c(x)$ is impracticable. For this reason, the damping is generally expressed in terms of damping ratios obtained experimentally rather than by a direct evaluation of the damping matrix using Eq. (10.35). These damping ratios are evaluated or estimated for each natural mode of vibration. If the explicit expression of the damping matrix $[C]$ is needed, it may be computed from the specified relative damping coefficients by any of the methods described in Chap. 22.

10.6 External Loads

When the dynamic loads acting on the structure consist of concentrated forces and moments applied at defined nodal coordinates, the load vector can be written directly.

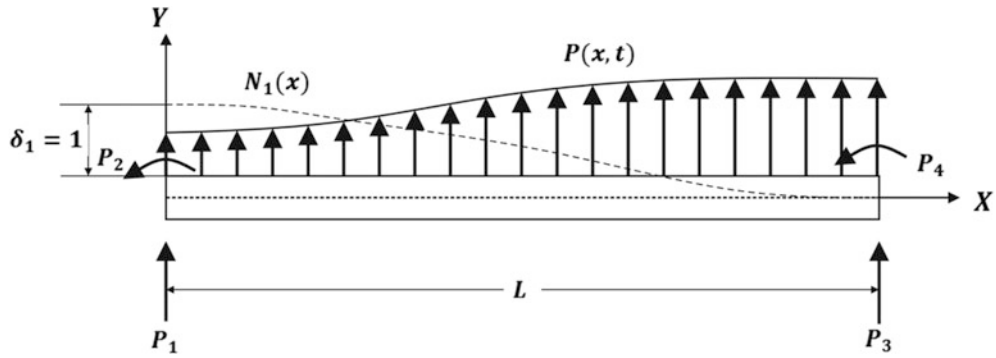

Fig. 10.7 Beam element supporting arbitrary distributed load undergoing virtual displacement $\delta_1 = 1$

In general, however, loads are applied at points other than nodal coordinates. In addition, the external load may include the action of distributed forces. In this case, the load vector corresponding to the nodal coordinates consists of the equivalent nodal forces. The procedure to determine the equivalent nodal forces which is consistent with the derivation of the stiffness matrix and the consistent mass matrix is to assume the validity of the static deflection functions, Eqs. (10.9a, b, c and d), for the dynamic problem and to use the principle of virtual work.

Consider the beam element in Fig. 10.7 when subjected to an arbitrary distributed force p (x, t) which is a function of position along the beam as well as a function of time. The equivalent force P_1 at the nodal coordinate 1 may be found by giving a virtual displacement $\delta_1 = 1$ at this coordinate and equating the resulting external work and internal work during this virtual displacement. In this case, the external work is

$$W_E = P_1\delta_1 = P_1 \tag{10.36}$$

since $\delta_1 = 1$. The internal work per unit of length along the beam is $p(x,t)$ times N_1 (x). Then the total internal work is

$$W_I = \int_0^L p(x,t)N_1(x)dx \tag{10.37}$$

Equating external work, Eq. (10.36), and internal work, Eq. (10.37), gives the equivalent nodal force as

$$P_1(t) = \int_0^L p(x,t)N_1(x)dx \tag{10.38}$$

Thus the element equivalent nodal forces can be expressed in general as

$$P_i(t) = \int_0^L p(x,t)N_1(x)dx \tag{10.39}$$

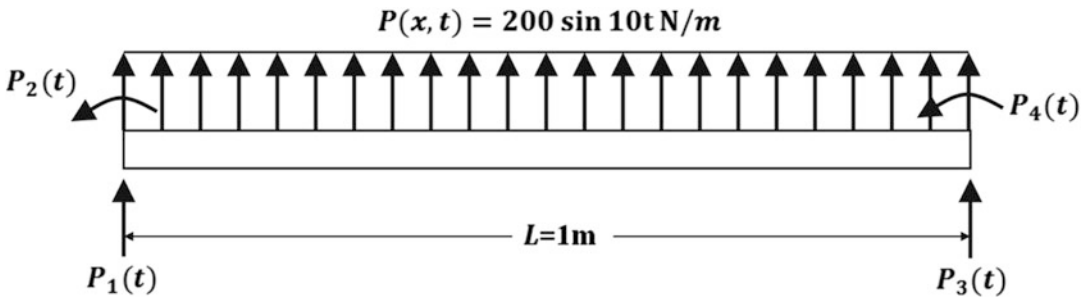

Fig. 10.8 Beam segment with uniformly distributed load, showing the equivalent nodal forces

Illustrative Example 10.2
Consider the beam segment in Fig. 10.8 and determine the equivalent nodal forces for a uniform distributed force along the length of the beam given by

$$p(x,t) = 200 \sin 10t \, N/m$$

Solution:
Introduction of numerical values into the displacements functions, Eqs. (10.9a, b, c and d), and substitution in Eq. (10.39) yield

$$P_1(t) = 200 \int_0^1 (1 - 3x^2 + 2x^3) \sin 10t dx = 100 \sin 10t$$
$$P_2(t) = 200 \int_0^1 x(1-x)^2 \sin 10t dx = 16.67 \sin 10t$$
$$P_3(t) = 200 \int_0^1 (3x^2 - 2x^3) \sin 10t dx = 100 \sin 10t$$
$$P_4(t) = 200 \int_0^1 x^2(x-1) \sin 10t dx = -16.67 \sin 10t$$

10.7 Geometric Stiffness

When a beam element is subjected to an axial force in addition to flexural loading, the stiffness coefficients are modified by the presence of the axial force. The modification corresponding to the stiffness coefficient k_{ij} is known as the geometric stiffness coefficient k_{Gij}, which is defined as the force corresponding to the nodal coordinate i due to a unit displacement at coordinate j and resulting from the axial forces in the structure. These coefficients may be evaluated by application of the principle of virtual work. Consider a beam element as used previously but now subjected to a distributed axial force per unit of length N (x), as depicted in Fig. 10.9a. In the sketch in Fig. 10.9b, the beam segment is subjected to a unit rotation of the left end, $\delta_2 = 1$. By definition, the nodal forces due to this displacement are the corresponding geometric stiffness coefficients; for example, k_{G12} is the vertical force at the left end. If we now give to this deformed beam a unit displacement $\delta_1 = 1$, the resulting external work is

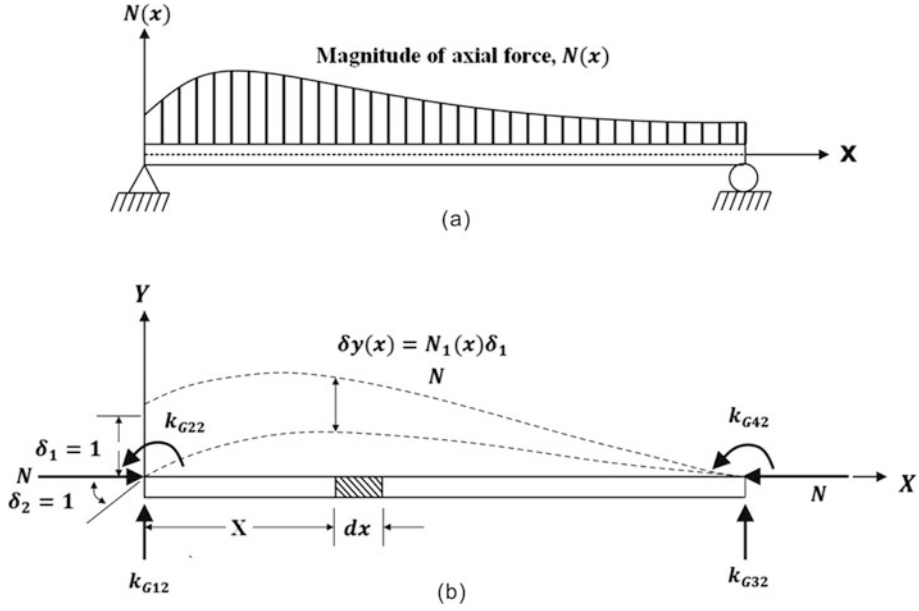

Fig. 10.9 (**a**) Beam element loaded with arbitrary distributed axial force, (**b**) Beam element acted on by nodal forces resulting form displacement $\delta_2 = 1$ undergoing a virtual displacement $\delta_1 = 1$

$$W_e = k_{G12}\delta_1$$

or

$$W_e = k_{G12} \tag{10.40}$$

since $\delta_1 = 1$.

The internal work during this virtual displacement is found by considering a differential element of length dx taken from the beam in Fig. 10.9b and shown enlarged in Fig. 10.10. The work done by the axial force $N(x)$ during the virtual displacement is

$$dW_I = N(x)\delta_e \tag{10.41}$$

where δ_e represents the relative displacement experienced by the normal force N(x) acting on the differential element during the virtual displacement. From Fig. 10.10, by similar triangles (triangles I and II), we have

$$\frac{\delta_e}{dN_1(x)} = \frac{dN_2(x)}{dx}$$

or

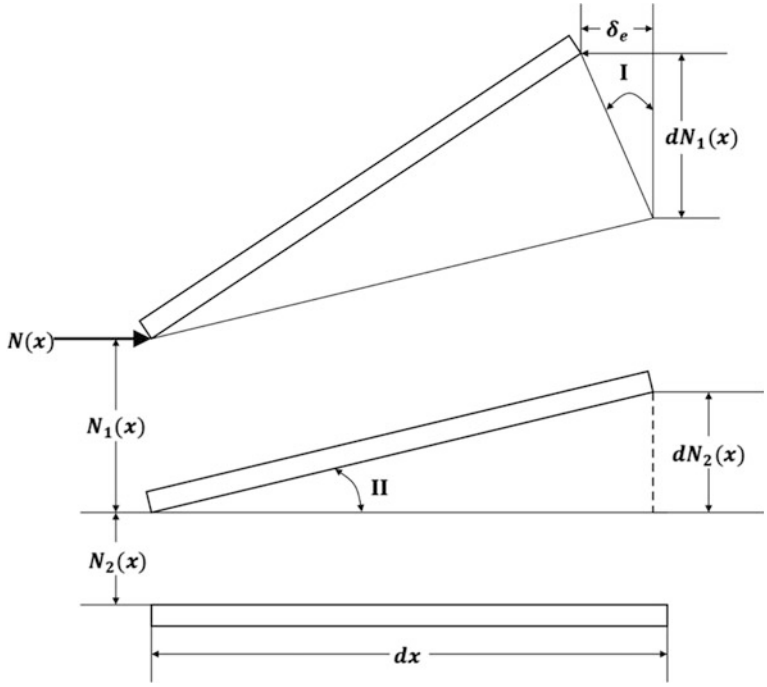

Fig. 10.10 Differential segment of deflected beam in Fig. 10.9

$$\delta_e = \frac{dN_1(x)}{dx} \cdot \frac{dN_2(x)}{dx} dx$$
$$\delta_e = N_1^{'}(x) N_2^{'}(x) dx$$

in which $N_1^{'}(x)$ and $N_2^{'}(x)$ are the derivatives with respect to x of the corresponding displacement functions defined in Eqs. (10.9a, b, c and d).

Now, substituting δ_e into Eq. (10.41), results in

$$dW_I = N(x) N_1^{'}(x) N_2^{'}(x) dx \tag{10.42}$$

Then integrating this expression and equating the result to the external work, Eq. (10.40), finally give

$$k_{G12} = \int_0^L N(x) N_1^{'}(x) N_2^{'}(x) dx \tag{10.43}$$

In general, any geometric stiffness coefficient may be expressed as

$$k_{Gij} = \int_0^L N(x) N_i^{'}(x) N_j^{'}(x) dx \tag{10.44}$$

In the derivation of Eq. (10.44), it is assumed that the normal force $N(x)$ is independent of time. When the displacement functions, Eqs. (10.9a, b, c and d), are used in Eq. (10.44) to calculate the geometric stiffness coefficients, the result is called the consistent geometric stiffness matrix. In the special case where the axial force is constant along the length of the beam, use of Eqs. (10.44) and (10.9a, b, c and d) gives the geometric stiffness matrix equation as

$$\begin{Bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{Bmatrix} = \frac{N}{30L} \begin{bmatrix} 36 & 3L & -36 & 3L \\ 3L & 4L^2 & -3L & -L^2 \\ -36 & -3L & 36 & -3L \\ 3L & -L^2 & -3L & 4L^2 \end{bmatrix} \begin{Bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \end{Bmatrix} \tag{10.45}$$

The assemblage of the system geometric stiffness matrix can be carried out exactly in the same manner as for the assemblage of the elastic stiffness matrix. The resulting geometric stiffness matrix will have the same configuration as the elastic stiffness matrix. It is customary to define the geometric stiffness matrix for a compressive axial force. In this case, the combined stiffness matrix $[K_c]$ for the structure is given by

$$[K_c] = [K] - [K_G] \tag{10.46}$$

in which $[K]$ is the assembled elastic stiffness matrix for the structure and $[K_G]$ the corresponding geometric stiffness matrix.

Illustrative Example 10.3

For the cantilever beam in Fig. 10.11, determine the system geometric matrix when an axial force of magnitude 30 Newtons is applied at the free end as shown in this figure.

Solution:

The substitution of numerical values into Eq. (10.45) for any of the three beam segments in which the cantilever beam has been divided gives the element geometric matrix

$$[K_G] = \begin{bmatrix} 36 & 3 & -36 & 3 \\ 3 & 4 & -3 & -1 \\ -36 & -3 & 36 & -3 \\ 3 & -1 & -3 & 4 \end{bmatrix}$$

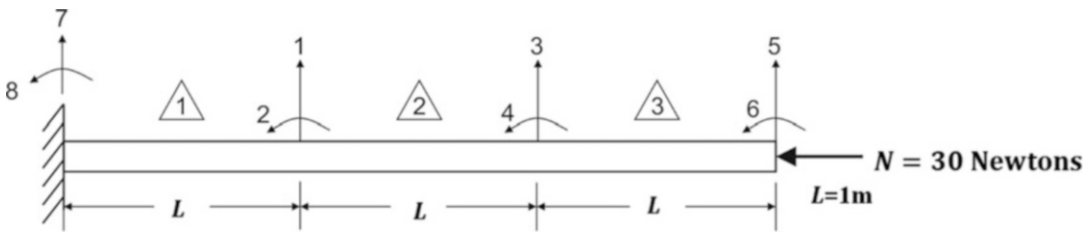

Fig. 10.11 Cantilever beam subjected to constant axial force (Illustrative Example 10.3)

since in this example $L = 1$ m and $N = 30$ Newtons. Use of the direct method gives the assembled system geometric matrix as

$$[K_G] = \begin{bmatrix} 72 & 0 & -36 & 3 & 0 & 0 \\ 0 & 8 & -3 & -1 & 0 & 0 \\ -36 & -3 & 72 & 0 & -36 & 3 \\ 3 & -1 & 0 & 8 & -3 & -1 \\ 0 & 0 & -36 & -3 & 36 & -3 \\ 0 & 0 & 3 & -1 & -3 & 4 \end{bmatrix}$$

10.8 Equations of Motion

In the previous sections of this chapter, the distributed properties of a beam and its load were expressed in terms of discrete quantities at the nodal coordinates. The equations of motion as functions of these coordinates may then be established by imposing conditions of dynamic equilibrium between the inertial forces $\{F_I(t)\}$, damping forces $\{F_D(t)\}$, elastic forces $\{F_s(t)\}$, and the external forces $\{F(t)\}$, that is,

$$\{F_I(t)\} + \{F_d(t)\} + \{F_s(t)\} = \{F(t)\} \tag{10.47}$$

For a linear system the forces on the left-hand side of Eq. (10.47) are expressed in terms of the system mass matrix, the system damping matrix, and the system stiffness matrix as

$$\{F_I(t)\} = [M]\{\ddot{u}\} \tag{10.48}$$

$$\{F_d(t)\} = [C]\{\dot{u}\} \tag{10.49}$$

$$\{F_s(t)\} = [K]\{u\} \tag{10.50}$$

Substitution of these equations into Eq. (10.47) gives the differential equation of motion for a linear system as

$$[M]\{\ddot{u}\} + [C]\{\dot{u}\} + [K]\{u\} = \{F(t)\} \tag{10.51}$$

In addition, if the effect of axial forces is considered in the analysis, Eq. (10.51) is modified so that

$$[M]\{\ddot{u}\} + [C]\{\dot{u}\} + [K_c]\{u\} = \{F(t)\} \tag{10.52}$$

in which

$$[K_c] = [K] - [K_G] \tag{10.53}$$

In practice, the solution of Eq. (10.51) or (10.52) is accomplished by standard methods of analysis and the assistance of appropriate computer programs as those described in this and the following chapters. We illustrate these methods by presenting here some simple problems for hand calculation.

Illustrative Example 10.4
Consider in Fig. 10.12 a uniform beam with the ends fixed against translation or rotation. In preparation for analysis, the beam has been divided into four equal segments. Determine the first three natural frequencies and corresponding modal shapes. Use the lumped mass method in order to simplify the numerical calculations.

Solution:
We begin by numbering sequentially the nodal coordinates starting with the rotational coordinates which have to be condensed in the lumped mass method (no inertial effect associated with rotational coordinates), continuing to number the coordinates associated with translation, and assigning the last numbers 7 through 10 to the fixed nodal coordinates as shown in Fig. 10.12. The stiffness matrix for any of the beam segments for this example is obtained from Eq. (10.20) as

$$[K]=\begin{matrix}4\\7\end{matrix}\begin{matrix}1\\8\end{matrix}\begin{matrix}5\\4\end{matrix}\begin{matrix}2\\1\end{matrix}\begin{bmatrix}12 & 6 & -12 & 6\\ 6 & 4 & -6 & 2\\ -12 & -6 & 12 & -6\\ 6 & 2 & -6 & 4\end{bmatrix}\begin{matrix}74\\81\\45\\12\end{matrix} \tag{10.54}$$

With the aid of the system nodal coordinates for each beam segment written at the top and on the right of the stiffness matrix, Eq. (10.54), we proceed to assemble the system stiffness matrix using the direct method. For the beam segment $\overset{1}{\triangle}$, the corresponding labels are 7, 8, 4, 1, we need to transfer only the lowest 2×2 sub-matrix that correspond to labels of rows and columns with indices 4, 1. For the beam segment $\overset{2}{\triangle}$, we translate the 4×4 elements of matrix Eq. (10.54) to the system stiffness matrix to rows and columns designated by combination of indices 4, 1, 5, 2 as labeled for this element and so forth for the other two beam segments. The assembled system stiffness matrix obtained in this manner results in the following system stiffness matrix:

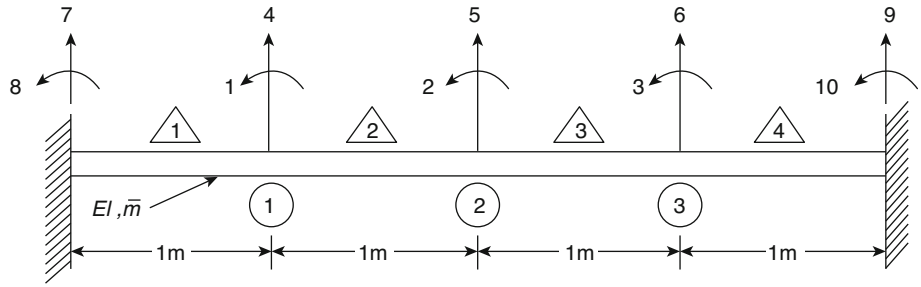

Fig. 10.12 Fixed beam divided in four element with indication of system nodal coordinates (Illustrative Example 10.4)

$$[K] = EI\begin{bmatrix} 8 & 2 & 0 & 0 & -6 & 0\\ 2 & 8 & 2 & 6 & 0 & -6\\ 0 & 2 & 8 & 0 & 6 & 0\\ 0 & 6 & 0 & 24 & -12 & 0\\ -6 & 0 & 6 & -12 & 24 & -12\\ 0 & -6 & 0 & 0 & -12 & 24\end{bmatrix} \tag{10.55}$$

The reduction or condensation of Eq. (10.55) is accomplished as explained in Chap. 9 by simply performing the Gauss-Jordan elimination of the first three rows since, in this case, we should condense these first three coordinates. This elimination reduces Eq. (10.55) to the following matrix:

$$[A] = \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & -0.214 & -0.750 & 0.214 \\ 0 & 1 & 0 & 0.858 & 0 & -0.858 \\ 0 & 0 & 1 & -0.214 & 0.750 & 0.214 \\ \hdashline 0 & 0 & 0 & 18.86EI & -12.00EI & 5.14EI \\ 0 & 0 & 0 & -12.00EI & 15.00EI & -12.00EI \\ 0 & 0 & 0 & 5.14EI & -12.00EI & 18.86EI \end{array}\right] \tag{10.56}$$

Comparison of Eq. (10.56) in partition form with Eq. (9.11) permits the identification of the reduced stiffness matrix $[\bar{k}]$ and the transformation matrix $[\bar{T}]$, so that

$$[\bar{K}] = EI \begin{bmatrix} 18.86 & -12.00 & 5.14 \\ -12.00 & 15.00 & -12.00 \\ 5.14 & -12.00 & 18.86 \end{bmatrix} \tag{10.57}$$

and

$$[\bar{T}] = EI \begin{bmatrix} 0.214 & 0.750 & -0.214 \\ -0.858 & 0 & 0.858 \\ 0.214 & -0.750 & -0.214 \end{bmatrix} \tag{10.58}$$

The general transformation matrix, Eq. (9.9), is then

$$[T] = \begin{bmatrix} 0.214 & 0.750 & -0.214 \\ -0.858 & 0 & 0.858 \\ 0.214 & -0.750 & -0.214 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{10.59}$$

As an exercise, the reader may check Eq. (9.10) for this example by simply performing the matrix multiplications

$$[\bar{K}] = [T]^T [K][T] \tag{10.60a}$$

The lumped mass method applied to this example gives three equal masses of magnitude m at each of the three translatory coordinates as indicated in Fig. 10.13. Therefore, the reduced lumped mass matrix is

$$[\bar{M}] = \bar{m} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{10.60b}$$

The natural frequencies and modal shapes are found by solving the undamped free vibration problem, that is,

$$\left[\bar{M}\right]\{\ddot{u}\} + \left[\bar{K}\right]\{u\} = \{0\} \tag{10.61}$$

Assuming the harmonic solution ($\{u\} = \{a\} \sin \omega t$), we obtain

$$\left(\left[\bar{K}\right] - \omega^2\left[\bar{M}\right]\right)\{a\} = \{0\} \tag{10.62}$$

requiring for a nontrivial solution that the determinant

$$\left|\left[\bar{K}\right] - \omega^2\left[\bar{M}\right]\right| = 0 \tag{10.63}$$

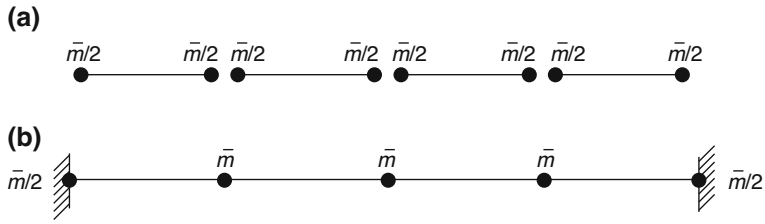

Fig. 10.13 (**a**) Lumped masses for uniform beam segment, (**b**) Lumped masses at system nodal coordinates

Substitution in this last equation $[\bar{k}]$ and $[\bar{M}]$, respectively, from Eqs. (10.57) and (10.60a and b) yields

$$\begin{vmatrix} 18.86 - \lambda & -12.00 & 5.14 \\ -12.00 & 15.00 - \lambda & -12.00 \\ 5.14 & -12.00 & 18.86 - \lambda \end{vmatrix} = 0 \tag{10.64}$$

in which

$$\lambda = \frac{\bar{m}\omega^2}{EI} \tag{10.65}$$

The roots of the cubic equation resulting from the expansion of the determinant in Eq. (10.64) are found to be

$$\lambda_1 = 1.943, \lambda_2 = 13.720, \lambda_3 = 37.057 \tag{10.66}$$

Then from Eq. (10.65)

$$\begin{aligned} \omega_1 &= 1.393\sqrt{EI/\bar{m}} \\ \omega_2 &= 3.704\sqrt{EI/\bar{m}} \\ \omega_3 &= 6.087\sqrt{EI/\bar{m}} \end{aligned} \tag{10.67}$$

The first three natural frequencies for a uniform fixed beam of length $L = 4\,m$ determined by the exact analysis (Chap. 17) are

$$\begin{aligned}\omega_1(exact) &= 1.398\sqrt{EI/\bar{m}}\\ \omega_2(exact) &= 3.854\sqrt{EI/\bar{m}}\\ \omega_3(exact) &= 7.556\sqrt{EI/\bar{m}}\end{aligned} \tag{10.68}$$

The first two natural frequencies determined using the three-degrees-of-freedom reduced system compare very well with the exact values. A practical rule in condensing degrees of freedom is to condense those nodal coordinates which have the least inertial effect; in this problem these are the rotational coordinates.

The modal shapes are determined by solving two of the equations in Eq. (10.62) after substituting successively values of ω_1, ω_2, ω_3 from Eq. (10.67) and conveniently setting the first coordinate for each modal shape be equal to one. The resulting modal shapes are

$$\{a\}_1 = \begin{bmatrix}1.00\\ 1.84\\ 1.00\end{bmatrix}, \{a\}_2 \begin{bmatrix}1.00\\ 0\\ -1.00\end{bmatrix}, \{a\}_3 = \begin{bmatrix}1.00\\ -1.08\\ 1.00\end{bmatrix} \tag{10.69}$$

The normalized modal shapes which are obtained by division of the elements of Eq. (10.69) by corresponding values of $\sqrt{\Sigma m_i a_{ij}^2}$ ($\mathrm{m_i} = 100$ kg for this example) are arranged in the columns of the modal matrix are

$$[\Phi]_p = \begin{bmatrix}0.0431 & 0.0707 & 0.0562\\ 0.0793 & 0 & -0.0607\\ 0.0431 & -0.0707 & 0.0562\end{bmatrix} \tag{10.70a}$$

The modal shapes in terms of the six original coordinates are then obtained by Eq. (9.8) as

$$[\Phi] = [T][\Phi]_p$$

$$[\Phi] = \begin{bmatrix}0.0594 & 0.0301 & -0.0457\\ 0 & -0.1212 & 0\\ -0.0594 & 0.0301 & 0.0457\\ 0.0431 & 0.0707 & 0.0562\\ 0.0793 & 0 & -0.0607\\ 0.0431 & 0.0707 & 0.0562\end{bmatrix} \tag{10.70b}$$

Illustrative Example 10.5

Determine the steady-state response for the beam of Illustrative Example 10.4 when subjected to the harmonic forces

$$\begin{aligned}F_1 &= F_{01}\sin \varpi t\\ F_2 &= F_{02}\sin \varpi t\\ F_3 &= F_{03}\sin \varpi t\end{aligned}$$

applied, respectively, at joints, 1, 2 and 3 of the beam in Fig. 10.12. Neglect damping and let $EI = 10^8$ $(N.m^2)$, $\bar{m} = 100$ kg/m, $\varpi = 3000$ rad/sec, $F_{01} = 2000\ N$, $F_{02} = 3000\ N$, and $F_{03} = 1000\ N$.

Solution:
The modal equations (uncoupled equations) can readily be written using the results of Illustrative Example 10.4. In general the nth normal equation is given by

$$\ddot{z}_n + \omega_n^2 z_n = P_n \sin \varpi t \tag{10.71}$$

in which

$$P_n = \sum_{i=1}^{N} \phi_{in} F_{0i}$$

The steady-state solution of Eq. (10.71) is given by Eq. (3.4) as

$$z_n = Z_n \sin \varpi t = \frac{P_n \sin \varpi t}{\omega_n^2 - \varpi^2} \tag{10.72}$$

The calculations required in Eq. (10.72) are conveniently arranged in Table 10.1. The deflections at the modal coordinates are found from the transformation

$$\begin{aligned} \{u\} &= [\Phi]\{z\} \\ \{U\} &= [\Phi]\{Z\} \end{aligned} \tag{10.73}$$

in which [Φ] is the modal matrix and

$$\begin{aligned} \{u\} &= \{U\} \sin \varpi t \\ \{z\} &= \{Z\} \sin \varpi t \end{aligned}$$

The substitution into Eq. (10.73) of the modal matrix from Eq. (10.70b) and the values of {Z} from the last column of Table 10.1 gives the amplitudes at nodal coordinates as

$$\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \\ U_6 \end{Bmatrix} = \begin{bmatrix} 0.0594 & 0.0301 & -0.0457 \\ 0 & -0.1212 & 0 \\ -0.0594 & 0.0301 & 0.0457 \\ 0.0431 & 0.0707 & 0.0562 \\ 0.0793 & 0 & -0.0607 \\ 0.0431 & -0.0707 & 0.0562 \end{bmatrix} \begin{Bmatrix} -5.200 \\ 1.500 \\ -0.048 \end{Bmatrix} 10^{-5}$$

or

$$\begin{aligned} U_1 &= -2.616 \times 10^{-6} \text{rad} \\ U_2 &= -1.818 \times 10^{-6} \text{rad} \\ U_3 &= 3.524 \times 10^{-6} \text{rad} \\ U_4 &= -1.207 \times 10^{-6} m \\ U_5 &= -4.094 \times 10^{-6} m \\ U_6 &= -3.329 \times 10^{-6} m \end{aligned}$$

Table 10.1 Modal Response for Illustrative Example 10.5

Mode (n)	ω_n^2	$P_n = \sum_i \phi_{in} F_{oi}$	$Z_n = \dfrac{P_n}{\omega_n^2 - \varpi^2}$
1	1.943×10^6	367.2	-5.200×10^{-5}
2	13.720×10^6	70.7	1.5000×10^{-5}
3	37.057×10^6	−13.5	-0.048×10^{-5}

Therefore, the motion at these nodal coordinates is given by

$$\begin{aligned} u_1(t). &= -2.616 \times 10^{-6} \sin 3000t \text{ rad} \\ u_2(t). &= -1.818 \times 10^{-6} \sin 3000t \text{ rad} \\ u_3(t). &= -3.524 \times 10^{-6} \sin 3000t \text{ rad} \\ u_4(t). &= -1.207 \times 10^{-6} \sin 3000t \, m \\ u_5(t). &= -4.094 \times 10^{-6} \sin 3000t \, m \\ u_6(t). &= -3.329 \times 10^{-6} \sin 3000t \, m \end{aligned} \tag{10.74}$$

The minus sign in the resulting amplitudes of motion simply indicates that the motion is 180° out of phase with the applied harmonic forces.

10.9 Element Forces At Nodal Coordinates

The central problem to be solved using the dynamic stiffness method is to determine the displacements at the nodal coordinates. Once these displacements have been determined, it is a simple matter of substituting the appropriate displacements in the condition of dynamic equilibrium for each element to calculate the forces at the nodal coordinates. The nodal element forces $\{P\}$ may be obtained by adding the inertial force $\{P_I\}$, the damping force $\{P_D\}$, the elastic force $\{P_s\}$, and subtracting the nodal equivalent forces $\{P_E\}$.
Therefore, we may write:

$$\{P\} = \{P_I\} + \{P_D\} + \{P_S\} - \{P_E\}$$

or

$$\{P\} = [m]\{\ddot{\delta}\} + [c]\{\dot{\delta}\} + [k]\{\delta\} - \{P_E\} \tag{10.75}$$

In Eq. (10.75) the inertial force, the damping force, and the elastic force are, respectively,

$$\begin{aligned} \{P_I\} &= [m]\{\ddot{\delta}\} \\ \{P_D\} &= [c]\{\dot{\delta}\} \\ \{P_S\} &= [k]\{\delta\} \end{aligned} \tag{10.76}$$

where $[m]$ is the element mass matrix; $[c]$ the element damping matrix; $[k]$ the element stiffness matrix; and $\{\ddot{\delta}\}$, $\{\dot{\delta}\}$, $\{\delta\}$ represents, respectively, the displacement, velocity, and acceleration vectors at the nodal coordinates of the element.

Although, the determination of element end-forces at nodal coordinates is given by Eq. (10.75), commercial computer programs only use the forces due to the elastic displacements, that is, the element end forces are calculated as

$$\{P\} = [k]\{\delta\}$$

This simplified approach may be justified by the mathematical model in which the inertial forces, damping forces, and external forces are assumed to be acting directly, at the nodes. Therefore, it is important in the process of discretization to select beam elements relatively short as to approximate the distributed properties with discrete concentrated values at the nodes between the elements. The determination of the element nodal forces is illustrated in the following example.

Illustrative Example 10.6

Determine the element nodal forces and moments for the four beam elements of Illustrative Example 10.5.

Solution:

Since in this example damping is neglected and there are no external forces applied on the beam elements except those at the nodal coordinates, Eq. (10.75) reduces to

$$\{P\} = [m]\{\ddot{\delta}\} + [k]\{\delta\} \tag{10.77}$$

The displacement functions for the six nodal coordinates of the beam in Fig. 10.12 are given by Eq. (10.74). These displacements are certainly also the displacements of the element nodal coordinates. The identification for this example of corresponding nodal coordinates between beam segments and system nodal coordinates is

$$\{\delta\}_1 = \begin{bmatrix} 0 \\ 0 \\ u_4 \\ u_1 \end{bmatrix}, \{\delta\}_2 = \begin{bmatrix} u_4 \\ u_1 \\ u_5 \\ u_2 \end{bmatrix}, \{\delta\}_3 = \begin{bmatrix} u_5 \\ u_2 \\ u_6 \\ u_3 \end{bmatrix}, \{\delta\}_4 = \begin{bmatrix} u_6 \\ u_3 \\ 0 \\ 0 \end{bmatrix} \tag{10.78}$$

where $\{\delta\}_i$ is the vector of nodal displacement for i beam segment. The substitution of appropriate quantities into Eq. (10.77) for the first beam element results in

$$\begin{Bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{Bmatrix} = \frac{\bar{m}}{2} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ u_4 \\ u_1 \end{Bmatrix} (-\omega^2) \sin \varpi t + EI \begin{bmatrix} 12 & 6 & -12 & 6 \\ 6 & 4 & -6 & 2 \\ -12 & -6 & 12 & -6 \\ 6 & 2 & -6 & 4 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ u_4 \\ u_1 \end{Bmatrix} \sin \varpi t$$

To complete this example, we substitute the numerical values of $\varpi = 3000$ rad/sec, m = 100 kg/m, and EI = 10^8 (N. m^2) and obtain

$$\begin{Bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{Bmatrix} = \begin{bmatrix} 7.5 & 6 & -12 & 6 \\ 6 & 4 & -6 & 2 \\ -12 & -6 & 7.5 & -6 \\ 6 & 2 & -6 & 4 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ -120.7 \\ -261.6 \end{Bmatrix} \sin 3000t$$

which then gives

$$P_1 = -121.2 \sin 3000t\, N$$
$$P_2 = 201.0 \sin 3000t\, N.m$$
$$P_3 = 664.3 \sin 3000t\, N$$
$$P_4 = -322.2 \sin 3000t\, N.m$$

The nodal element forces found in this manner for all of the four beam segments in this example are given in Table 10.2.

The results in Table 10.2 may be used to check that the dynamic conditions of equilibrium are satisfied in each beam segment. The free body diagrams of the four elements of this beam are shown in Fig. 10.14 with inclusion of nodal inertial forces. These forces are computed by multiplying the nodal mass by the corresponding nodal acceleration.

Table 10.2 Element Nodal Forces (Amplitudes) for Illustrative Example 10.6

	Beam segment				
Force	1	2	3	4	Units
P_1	−121.2	1347.2	1947.9	−382.3	*N*
P_2	201.0	322.2	−481.4	−587.0	*N.m*
P_3	664.3	1038.3	1392.4	1880.4	*N*
P_4	−322.2	481.4	587.0	−1292.6	*N.m*

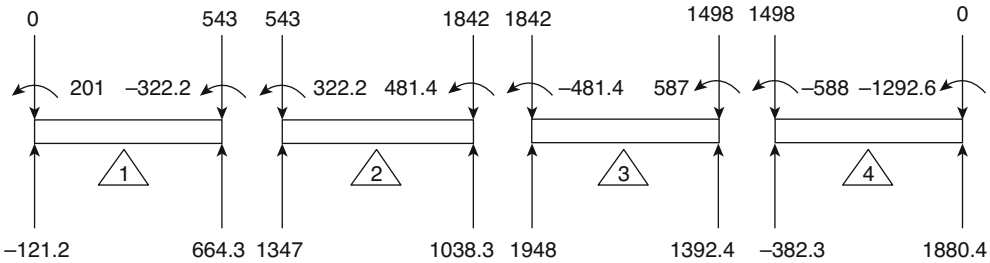

Fig. 10.14 Dynamic equilibrium for beam element of Illustrative Example 10.6

10.10 Program 13—Modeling Structures as Beams

The computer program presented in this section calculates the stiffness and mass matrices for a beam and stores the coefficients of these matrices in a file, for future use. Since the stiffness and mass matrices are symmetric, only the upper triangular portion of these matrices needs to be stored. The program also stores in another file, named by the user, the general information on the beam. The information stored in these files is needed by programs which perform dynamic analysis such as

calculation of natural frequencies or determination of the response of the structure subjected to external excitation.

Illustrative Example 10.7
Determine the stiffness and mass matrices for the uniform fixed-ended beam shown in Fig. 10.15. The following are the properties of the beam:

Length: $L = 200$ in
Cross-section moment of inertia: $I = 100$ in^4
Modulus of elasticity: $E = 6.58\ E6$ lb/in^2
Mass per unit length: $m = 0.10$ (lb. s^2/in/in)

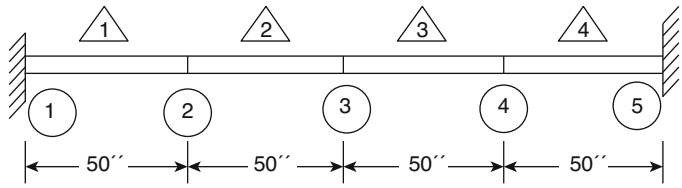

Fig. 10.15 Fixed beam modeled for Illustrative Example 10.7

Determine: (a) the stiffness and mass matrices, (b) the natural frequencies and modal shapes, (c) the response to concentrated force of 10,000 lb suddenly applied at the center of the beam for 0.1 s and removed linearly as shown in Fig. 10.16. Use time step of integration $\Delta t = 0.01$ s.

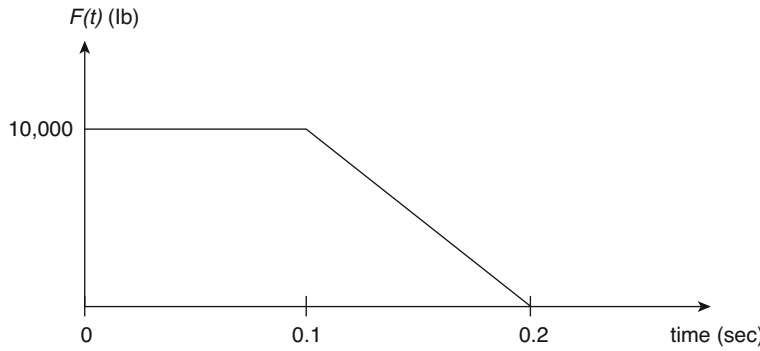

Fig. 10.16 Force applied to the beam of Illustrative Example 10.8

Solution:
Step 1 print out the output of (a) stiffness and mass matrices and (b) natural frequencies and modal shapes.

The following MATLAB file is used to compute stiffness and mass matrices using the system matrix.

```
close all
clear all
%_____________________________________________________________________
% Determine System Matrices/Determine Force
%_____________________________________________________________________

%%%%-GIVEN VALUES-%%%%

E=6.58*10^6; L=50;                  %E(psi), L(in.)
inertia = 100;                      %Second Moment of Inertia (in^4)
EI = E*inertia;                     %Beam stiffness


nodes =[0:L:4*L];                   %Coordinates of nodes
debc=[1,2,9,10];                    %Dofs are eliminated at supports

m_bar = 0.1;                        %Mass per unit length (lb-sec^2/in/in)

dof = 2*length(nodes);              %Total No. dofs

K= zeros(dof);
M= zeros(dof);

%%%Generate equations for each element and assemble them.
for i=1:4
    lm=[2*(i-1)+1,2*(i-1)+2,2*(i-1)+3,2*(i-1)+4];
    ke = BeamElement(EI, nodes([i:i+1]));
    K(lm, lm) = K(lm, lm) + ke;
end

%%%Generate mass matrix for each element and assemble them.
for i=1:4
    lm=[2*(i-1)+1,2*(i-1)+2,2*(i-1)+3,2*(i-1)+4];
    m=BeamConsMass(m_bar, nodes([i:i+1]));
    M(lm, lm) = M(lm, lm) + m;
end

K;

M;

%%%Define the load vector
F = zeros(dof,1); F(5) = 10000;          %Applied force at specific dofs

%%%System Matrices
[Kf, Mf, Rf] = System(K, M, F, [1,2,9,10]);

Kf

Mf

Rf
%_____________________________________________________________________
% Solve the eigenvalue problem and normalized eigenvectors
%_____________________________________________________________________

%%%Solve for eigenvalues (D) and eigenvectors (a)
[a, D] = eig(Kf, Mf);

[omegas,ii] = sort(sqrt(diag(D)));         %Natural Frequencies

omegas

a = a(:,ii)                                %Mode Shapes

T = 2*pi./omegas;                          %Natural Periods

f = 1./T;                                  %Natural Frequency (cps)

save ('temp0.mat', 'Mf', 'Kf' ,'Rf');
```

The function file of MATLAB is used to assemble the stiffness matrix of each beam element for global stiffness matrix.

```
function ke = BeamElement(EI, coord)
% ke = BeamElement(EI, w, coord)
% Generates equations for a beam element
% EI = beam stiffness
% coord = coordinates at the element ends

L=coord(2)-coord(1);
ke = [(12*EI)/L^3, (6*EI)/L^2, -((12*EI)/L^3), (6*EI)/L^2;
    (6*EI)/L^2, (4*EI)/L, -((6*EI)/L^2), (2*EI)/L;
    -((12*EI)/L^3), -((6*EI)/L^2), (12*EI)/L^3, -((6*EI)/L^2);
    (6*EI)/L^2, (2*EI)/L, -((6*EI)/L^2), (4*EI)/L];
```

The function file of MATLAB is used to assemble the mass matrix of each beam element for global mass matrix.

```
function m = BeamConsMass(m_bar, coord)
% BeamConsMass(m_bar, nodes(con,:))
% Generates mass matrix for a beam element
% m = Mass per unit length (lb-sec^2/in/in)
% L = length
% coord = coordinates at the element ends

L=coord(2)-coord(1);
m = m_bar*L/420*[156   22*L  54     -13*L;
                 22*L  4*L^2 13*L   -3*L^2;
                 54    13*L  156    -22*L;
                 -13*L -3*L^2 -22*L  4*L^2];
```

The function file of MATLAB is used to construct the system matrix of the given problem. This file will be used for other element types including frame element (Chap. 11), grid frame element (Chap. 12), three-dimensional frame element (Chap. 13), and truss element (Chap. 14).

```
function [Kf, Mf, Rf] = System(K, M, R, debc)
%%% Computes System Matrix
%%% K = global stiffness matrix
%%% M = global mass matrix
%%% R = global right hand side vector
%%% debc = list of degrees of freedom to delete rows and columns

dof = length(R);
df = setdiff(1:dof, debc);
Kf = K(df, df);
Mf = M(df, df);
Rf = R(df);
```

The function file of MATLAB is used to determine the function of the applied force and solve the uncoupled equation.

```
function  q = MDOFP(t, q)
load ('temp2.mat', 'omega', 'P', 'm' ,'zeta')

%_____________________________________
%illustrative Example 10.8
%_____________________________________

if t <= 0.1
    P = P;

elseif t <0.2

   P = 2*P*(1-5*t);

else
   P = 0;

end

q = [q(2); -omega*omega*q(1)-2*zeta*omega*q(2)+P];
```

Step2: The response can be plotted using MATLAB file after running the Step 1 (Fig. 10.17):

```
clc
clear all
close all

%_____________________________________________________________________
% Inputs:
%   M, K
%   F = forcing function
%   t = Time period
%   u0 = initial displacement
%   v0 = initial velocity
%_____________________________________________________________________
 t = 0:0.01:0.5;

load ('temp0.mat', 'Mf', 'Kf', 'Rf')

M = Mf

% Deifne Stiffness Matrix

K = Kf

[n,n]= size(M);

F = Rf;

nstep = size(t');

u0 = zeros(n,1); u0(1) =0;
v0 = zeros(n,1); v0(1) =0;
[n,n]= size(M);
```

```
%_____________________________________________________________________
% Solve the eigenvalue problem and normalized the eigenvectors
%_____________________________________________________________________

[a, D] = eig(K, M)                    % Solve for eigenvalues (D) and eigenvectors (a)

[omegas,k] = sort(sqrt(diag(D)));  % Natural Frequencies
a = a(:,k)

T = 2*pi./omegas;                     % Natural Periods

aMa = diag(a'*M*a)                    % aMa = {a}'*[M]*(a}

nom_phi = (a)*inv(sqrt(diag(aMa))) % Normalized modal matrix

orth_M = nom_phi'*M*nom_phi         % Check the orthogonality condition for Mass Matrix

orth_K = nom_phi'*K*nom_phi         % Check the orthogonality condition for Stiffness
Matrix
%_____________________________________________________________________
% Initial conditions
%_____________________________________________________________________

P = nom_phi'*F;                       % Normalized force, P = nom_F
q0 = nom_phi'*M*u0
dq0 = nom_phi'*M*v0

%_____________________________________________________________________
% Damping matrix using the proportional damping matrix
% [C] = a[M]+b[K]
% zetas = damping ratios
%_____________________________________________________________________
a = 0;
b = 0;
nom_C = nom_phi'*(a*M+b*K)*nom_phi;
zetas = diag((1/2)*nom_C*inv(diag(omegas)));
save ('temp1.mat', 'omegas', 'P' ,'zetas');
q = [];
r = [];
for i=1:n
q0_i = q0(i,:);
dq0_i = dq0(i,:);

load temp1.mat
omega = omegas(i,:);
P = P(i,:);
m = M(i,i);
zeta = zetas(i,:);

save ('temp2.mat', 'omega', 'P', 'm', 'zeta');

[t,q] = ode45(@MDOFP, t, [q0_i dq0_i]',[]);

r(:,i) = q(:,1);
save ('temp3.mat', 'r')

end


load ('temp3.mat', 'r');

yim = nom_phi*[r'];

save ('response.mat','yim');

figure
subplot(3,2,1);
plot(t, yim(1,:))       % Node 2: y dipl (in.).
title ('Node 2'); xlabel ('Time (sec)'); ylabel ('u_3(in.)'); grid on
```

```
subplot(3,2,2);
plot(t, yim(2,:))       % Node 2: rotation (radian).
title ('Node 2'); xlabel ('Time (sec)'); ylabel ('u_4(radian)'); grid on

subplot(3,2,3);
plot(t, yim(3,:))       % Node 3: y displ (in.).
title ('Node 3'); xlabel ('Time (sec)'); ylabel ('u_5(in.)'); grid on

subplot(3,2,4);
plot(t, yim(4,:))       % Node 3: rotation (radian).
title ('Node 3'); xlabel ('Time (sec)'); ylabel ('u_6(radian)'); grid on

subplot(3,2,5);
plot(t, yim(5,:))       % Node 4: y displ (in.).
title ('Node 4'); xlabel ('Time (sec)'); ylabel ('u_7(in.)'); grid on

subplot(3,2,6);
plot(t, yim(6,:))       % Node 4: rotation (radian).
title ('Node 4'); xlabel ('Time (sec)'); ylabel ('u_8(radian)'); grid on
```

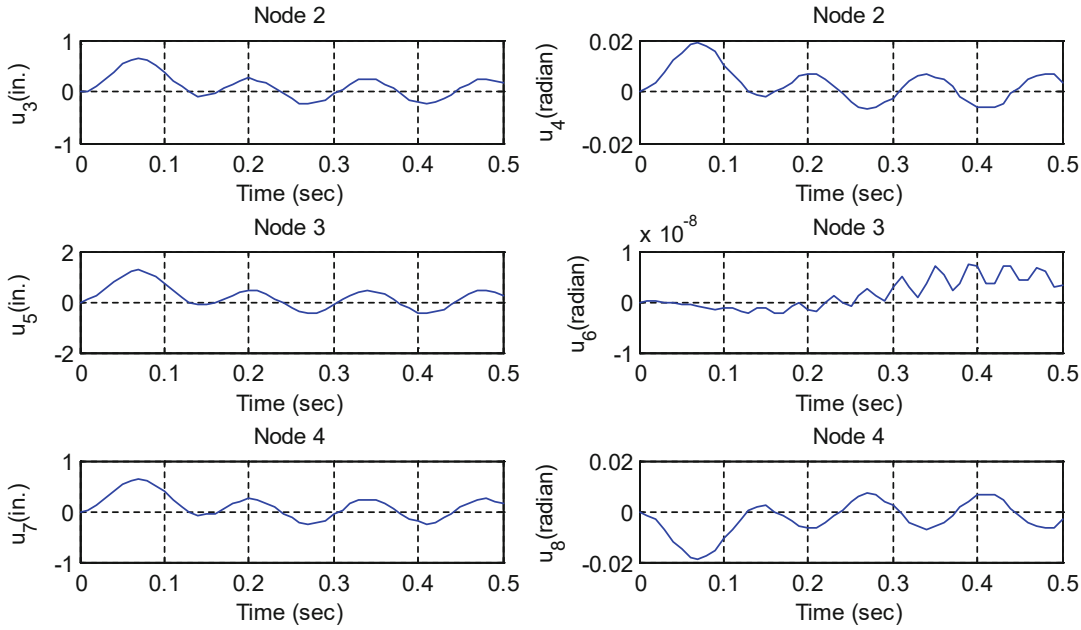

Fig. 10.17 Response of Illustrative Example 10.7

10.11 Summary

In this chapter, we have formulated the dynamic equations for beams in reference to a discrete number of nodal coordinates. These coordinates are translation and rotational displacements defined at joints between structural elements of the beam (beam segments). The dynamic equations for a linear system are conveniently written in matrix notation as

$$|M|\{\ddot{u}\} + |C|\{\dot{u}\} + |K|\{u\} = \{F(t)\}$$

where $F(t)$ is the force vector and $[M]$, $[C]$, and $[K]$ are, respectively, the mass, damping, and stiffness matrices of the structure. These matrices are assembled by the appropriate superposition (direct method) of the matrices determined for each beam segment of the structure.

The solution of the dynamic equations {i.e., the response) of a linear system may be found by the modal superposition method. This method requires the determination of the natural frequencies ω_n (n = 1, 2, 3, …, N) and the corresponding normal modes which are conveniently written as the columns of the modal matrix [Φ]. The linear transformation $\{u\} = [\Phi]\ \{z\}$ applied to the dynamic equations reduces them to a set of independent equations (uncoupled equations) of the form

$$\ddot{z}_n + 2\omega_n \xi_n \dot{z}_n + \omega_n^2 z_n = P_n(t)$$

where ξ_n is the modal damping ratio and P_n (t) = $\Phi_{in} F_i(t)$ is the modal force.

An alternate method for determining the response of linear systems (also valid for nonlinear systems) is the numerical integration of the dynamic equations. Chapter 20 presents the step-by-step linear acceleration method (with a modification introduced by Wilson) which is an efficient method for solving the dynamic equations.

An instructional computer program is also described in this chapter for the dynamic analysis of beams. This program performs the task of assembling and storing in a file the stiffness and mass matrices of the system. These matrices are subsequently used by other programs to calculate natural frequencies or the response of the beam to external excitation.

10.12 Problems

Problem 10.1

A uniform beam of flexural stiffness EI = 10^9 (lb. in^2) and length 300 in has one end fixed and the other simply supported. Determine the system stiffness matrix considering three beam segments and the free nodal coordinates indicated in Fig. P10.1.

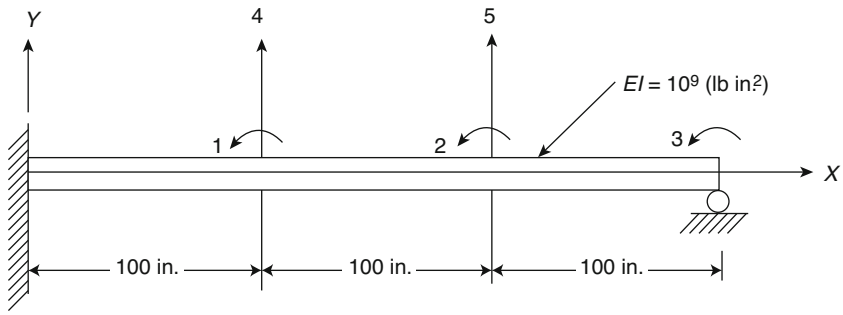

Fig. P10.1

Problem 10.2
Assuming that the beam shown in Fig. P10.1 carries a uniform weight per unit length q = 3.86 lb/in, determine the system mass matrix corresponding to the lumped mass formulation.

Problem 10.3
Determine the system mass matrix for Problem 10.2 using the consistent mass method.

Problem 10.4
For the beam in Problems 10.1 and 10.2, use static condensation to eliminate the massless degrees of freedom and determine the transformation matrix and the reduced stiffness and mass matrices.

Problem 10.5
For the beam in Problems 10.1 and 10.3 use static condensation to eliminate the rotational degrees of freedom. Find the transformation matrix and the reduced tiffness and mass matrices.

Problem 10.6
Determine the natural frequencies and corresponding normal modes using the reduced stiffness and mass matrices obtained in Problem 10.4.

Problem 10.7
Determine the natural frequencies and corresponding normal modes using the reduced stiffness and mass matrices obtained in Problem 10.3.

Problem 10.8
Determine the geometric stiffness matrix for the beam of Problem 10.1 when it is subjected to a constant tensile force of 10,000 lb as shown in Fig. P10.8.

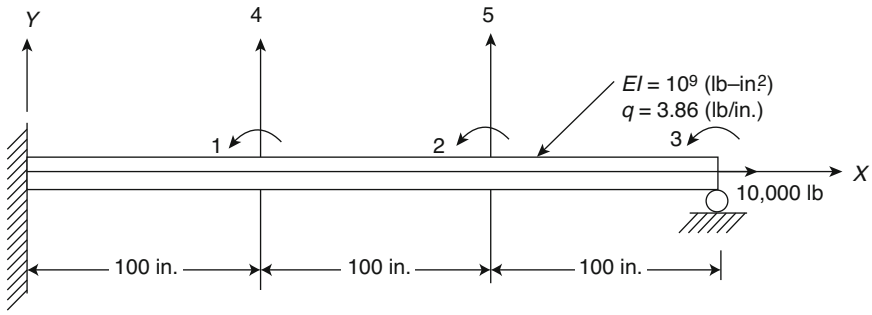

Fig. P10.8

Problem 10.9
Perform static condensation to reduce the geometric stiffness matrix obtained in Problem 10.8. Eliminate the rotational coordinates.

Problem 10.10
Use results from Problems 10.4 and 10.9 and determine the natural frequencies and corresponding normal modes for the beam shown in Fig. P10.8.

Problem 10.11
Use the results of Problems 10.5 and 10.9 and determine the natural frequencies and corresponding normal modes for the beam shown in Fig. P10.8.

Problem 10.12
Determine the stiffness matrix for a beam segment in which the flexural stiffness has a linear variation as shown in Fig. P10.12.

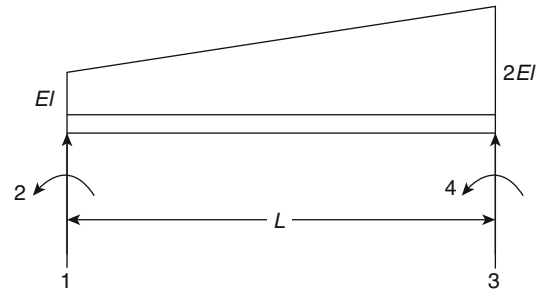

Fig. P10.12

Problem 10.13
Determine the lumped mass matrix for a beam segment in which the mass has a linear distribution as shown in Fig. P10.13.

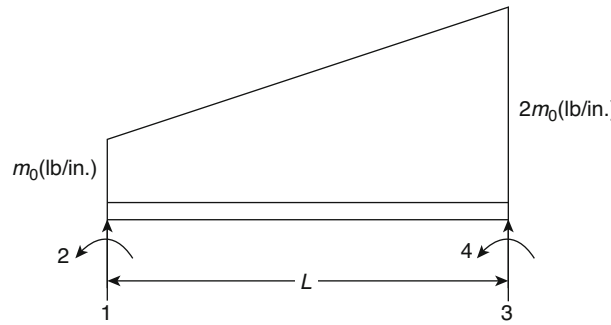

Fig. P10.13

Problem 10.14
Determine the consistent mass matrix for the beam segment shown in Fig. P10.13.

Problem 10.15
The uniform beam shown in Fig. P10.15 is subjected to a constant force of 5000 lb suddenly applied along the nodal coordinate 4. Use the results obtained in Problem 10.6 to determine the response by the modal superposition method. (Use only the two modes left by the static condensation.)

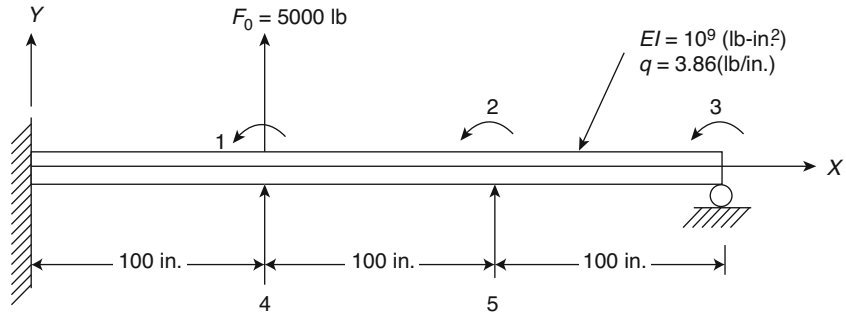

Fig. P10.15

Problem 10.16

Solve Problem 10.15 using the results obtained in Problem 10.7 which are based on the consistent mass formulation.

Problem 10.17

Solve Problem 10.15 using the results obtained in Problem 10.9 which includes the effect of the axial force in the stiffness of the system.

Problem 10.18

Determine the steady-state response for the beam shown in Fig. P10.18 which is acted upon by a harmonic force F(t) = 5000 sin 30 t (lb) as shown in the figure. Eliminate the rotational coordinates by static condensation (Problem 10.5).

Neglect damping in the system.

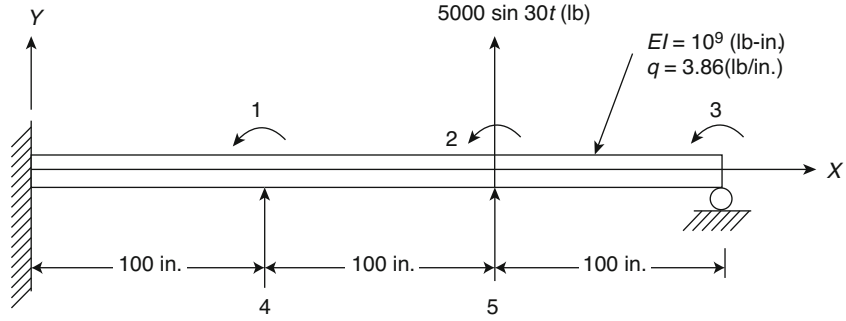

Fig. P10.18

Problem 10.19

Determine the natural frequencies and corresponding normal modes for the beam shown in Fig. P10.1: (a) condensing the three rotational nodal coordinates; (b) no condensing coordinates. (Use the consistent mass method.)

Problem 10.20

Determine the response for the beam shown in Fig. P10.15. Neglect damping. Do not condense coordinates; (b) condense the three rotational coordinates.

Problem 10.21
Repeat Problem 10.20 assuming 10% damping in all the modes.

Problem 10.22
Determine the steady-state response for the beam shown in Fig. P10.18 when subjected to a harmonic force as shown in the figure. Do not condense coordinates, and neglect damping in the system.

Problem 10.23
Repeat Problem 10.22 assuming that the damping is proportional to stiffness of the system where the constant of proportionality $a_0 = 0.2$.

Problem 10.24
Solve Problem 10.22 after condensing the three rotational coordinates.

Problem 10.25
Repeat Problem 10.24 assuming 15% damping in all the modes.

Problem 10.26
Determine the steady-state response for the beam shown in Fig. P10.15. Do not condense coordinates, and neglect damping in the system.

Check for updates

Dynamic Analysis of Plane Frames 11

The dynamic analysis using the stiffness matrix method for structures modeled as beams was presented in Chap. 10. This method of analysis when applied to beams requires the calculation of element matrices (stiffness, mass, and damping matrices), the assemblage from these matrices of the corresponding system matrices, the formation of the force vector, and the solution of the resultant equations of motion. These equations, as we have seen, may be solved in general by the modal superposition method or by numerical integration of the differential equations of motion. In this chapter and in the following chapters, the dynamic analysis of structures modeled as frames is presented.

We begin in the present chapter with the analysis of structures modeled as plane frames and with the loads acting in the plane of the frame. The dynamic analysis of such structures requires the inclusion of the axial effects in the stiffness and mass matrices. It also requires a coordinate transformation of the nodal coordinates from element or local coordinates to system or global coordinates. Except for the consideration of axial effects and the need to transform these coordinates, the dynamic analysis by the stiffness method when applied to frames is identical to the analysis of beams as discussed in Chap. 10.

11.1 Element Stiffness Matrix for Axial Effects

The inclusion of axial forces in the stiffness matrix of a flexural beam element requires the determination of the stiffness coefficients for axial loads. To derive the stiffness matrix for an axially loaded member, consider in Fig. 11.1 a beam segment acted on by the axial forces P_1 and P_2 producing axial displacements δ_1 and δ_2 at the nodes of the element. For a prismatic and uniform beam segment of length L and cross-sectional A, it is relatively simple to obtain the stiffness relation for axial effects by the application of Hooke's law. In relation to the beam shown in Fig. 11.1, the displacements δ_1 produced by the force P_1 acting at node 1 while node 2 is maintained fixed ($\delta_2 = 0$) is given by

$$\delta_1 = \frac{P_1 L}{AE} \tag{11.1}$$

From Eq. (11.1) and the definition of the stiffness coefficient k_{11} (force at node 1 to produce a unit displacement, $\delta_1 = 1$), we obtain

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M. Paz, Y. H. Kim, *Structural Dynamics*, https://doi.org/10.1007/978-3-319-94743-3_11

$$k_{11} = \frac{P_1}{\delta_1} = \frac{AE}{L} \tag{11.2a}$$

The equilibrium of the beam segment acted upon by the force k_{11} requires a force k_{21} at the other end of equal magnitude but in opposite direction, namely

$$k_{21} = -k_{11} = -\frac{AE}{L} \tag{11.2b}$$

Analogously, the other stiffness coefficients due to a unit displacement at node 2 ($\delta_2 = 1$) are:

$$k_{22} = \frac{AE}{L} \tag{11.2c}$$

and

$$k_{12} = -\frac{AE}{L} \tag{11.2d}$$

The stiffness coefficients as given by Eq. (11.2a) are the elements of the stiffness matrix relating axial forces and displacements for a prismatic beam segment, that is,

$$\begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix} = \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} \delta_1 \\ \delta_2 \end{Bmatrix} \tag{11.3}$$

The stiffness matrix corresponding to the nodal coordinates for the beam segment shown in Fig. 11.2 is obtained by combining in a single matrix the stiffness matrix for axial effects, Eq. (11.3), and the stiffness matrix for flexural effects, Eq. (10.20). The matrix resulting from this combination relates the forces P_i and the displacements δ_1 at the coordinates indicated in Fig. 11.2 as

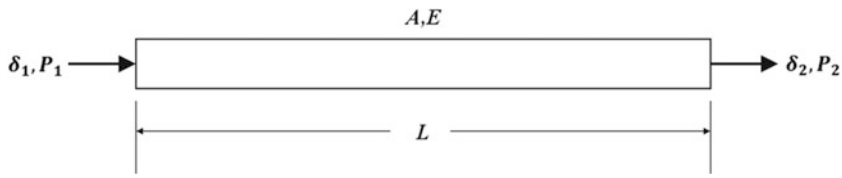

Fig. 11.1 Beam element showing nodal axial loads P_1, P_2, and corresponding nodal displacements δ_1, δ_2

$$\begin{Bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_6 \end{Bmatrix} = \frac{EI}{L^3} \begin{bmatrix} AL^2/I & & & & & \\ 0 & 12 & & & & \\ 0 & 6L & 4L^2 & & & \\ -AL^2/I & 0 & 0 & AL^2/I & & \\ 0 & -12 & -6L & 0 & 12 & \\ 0 & 6L & 2L^2 & 0 & -6L & 4L^2 \end{bmatrix} \begin{Bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \\ \delta_5 \\ \delta_6 \end{Bmatrix} \tag{11.4}$$

or, in concise notation,

$$\{P\} = [K]\{\delta\} \tag{11.5}$$

11.2 Element Mass Matrix for Axial Effects

The determination of mass influence coefficients for axial effects of a beam element may be carried out by any of two methods indicated previously for the flexural effects: (1) the lumped mass method and (2) the consistent mass method. In the lumped mass method, the mass allocation to the nodes of the beam element is found from static considerations, which for a uniform beam gives half of the total mass of the beam segment allocated at each node. Then for a prismatic beam segment, the relation between modal axial forces and modal accelerations is given by

$$\begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix} = \frac{\bar{m}L}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \ddot{\delta}_1 \\ \ddot{\delta}_2 \end{Bmatrix} \tag{11.6}$$

where $\bar{m}$ is the mass per unit of length. The combination of the flexural lumped mass coefficient and axial mass coefficients gives, in reference to the modal coordinates in Fig. 11.2, the following diagonal matrix:

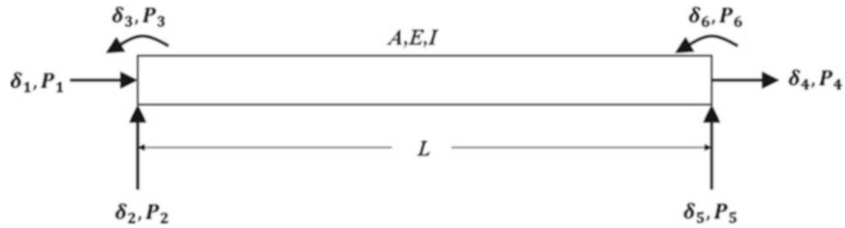

Fig. 11.2 Beam element showing flexural and axial nodal forces and displacements

$$\begin{Bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_6 \end{Bmatrix} = \frac{\bar{m}L}{2} \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 0 & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 0 \end{bmatrix} \begin{Bmatrix} \ddot{\delta}_1 \\ \ddot{\delta}_2 \\ \ddot{\delta}_3 \\ \ddot{\delta}_4 \\ \ddot{\delta}_5 \\ \ddot{\delta}_6 \end{Bmatrix} \tag{11.7}$$

To calculate the coefficients for the consistent mass matrix, it is necessary first to determine the displacement functions corresponding to a unit axial displacement at one of the modal coordinates. Consider in Fig. 11.3 an axial unit displacement $\delta_1 = 1$ of node 1 while the other node 2 is kept fixed so that $\delta_2 = 0$. If $u = u\,(x)$ is the displacement at section x, the displacement at section $x + dx$ will be $u + du$. It is evident then that the element dx in the new position has changed in length by an amount du, and thus, the strain is du/dx. Since from Hooke's law, the ratio of stress to strain is equal to the modulus of elasticity E, we can write

$$\frac{du}{dx} = \frac{P(x)}{AE} \tag{11.8}$$

Integration with respect to x yields

$$u = \frac{P(x)}{AE}x + C \tag{11.9}$$

in which C is a constant of integration. Introducing the boundary conditions, $u = 1$ at $x = 0$ and $u = 0$ at $x = L$, we obtain the displacement function $u_1(x)$ corresponding to a unit displacement $\delta_1 = 1$ as

$$u_1(x) = 1 - \frac{x}{L} \tag{11.10}$$

Analogously, the displacement function $u_2(x)$ corresponding to a unit displacement $\delta_2 = 1$ is found to be:

$$u_2(x) = \frac{x}{L} \tag{11.11}$$

The application of the principle of virtual work results in a general expression for the calculation of the stiffness coefficients. For example, consider the beam in Fig. 11.3, which is in equilibrium with the forces $P_1 = k_{11}$ and $P_2 = k_{21}$ at its two ends.

Assume that a virtual displacement $\delta_2 = 1$ takes place. Then, according to the principle of virtual work, during this virtual displacement, the work of the external and internal forces are equal. The external force k_{21} performs the work
or

$$W_E = k_{21} \tag{11.12}$$

since $\delta_2 = 1$. The internal force $P(x)$ at any section x is obtained from Eq. (11.8) as

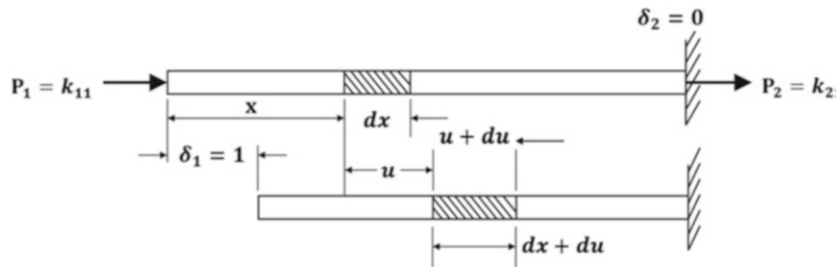

Fig. 11.3 Displacement at node 1 ($\delta_1 = 1$) of a beam element

$$P(x) = AEu_1'(x) \tag{11.13}$$

in which $u_1'(x) = du_1/dx$.

The incremental displacement δu_2 of element dx during this virtual displacement may be expressed as

$$du_2 = \frac{du_2}{dx}dx \tag{11.14}$$

Hence the internal work for element dx is obtained from Eqs. (11.13) and (11.14) as

$$dW_I = AEu_1'(x)u_2'(x)dx$$

and for the beam segment of length L

$$W_I = \int_0^L AEu_1'(x)u_2'(x)dx \tag{11.15}$$

Finally, equating $W_E = W_I$ from Eqs. (11.12) and (11.15) gives the stiffness coefficient

$$k_{21} = \int_0^L AEu_1'(x)u_2'(x)dx \tag{11.16}$$

In general, the stiffness coefficient k_{ij} for axial effects may be obtained from

$$k_{ij} = \int_0^L AEu_i'(x)u_j'(x)dx \tag{11.17}$$

Using Eq. (11.17), the reader may check the results obtained in Eq. (11.3) for a uniform beam. However, Eq. (11.17) could as well be used for nonuniform beams in which the cross-sectional area A would in general be a function of x. In practice, the same displacement $u_1(x)$ and $u_2(x)$ obtained for a uniform beam, are also used in Eq. (11.17) for a non uniform member. The displacement u(x, t) at any section x of a beam element due to dynamic nodal displacements, $\delta_1(t)$ and $\delta_2(t)$ is obtained by superposition. Hence

$$u(x,t) = u_1(x)\delta_1(t) + u_2(x)\delta_2(t) \tag{11.18}$$

in which $u_1(x)$ and $u_2(x)$ are given by Eqs. (11.10) and (11.11) (Fig. 11.4).

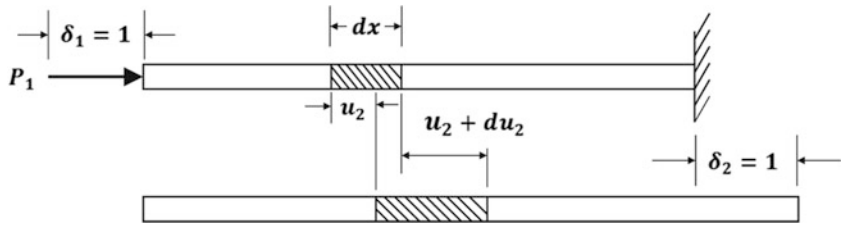

Fig. 11.4 Displacement along of a beam element subjected to axial loading that give a unit displacement ($\delta_2 = 1$)

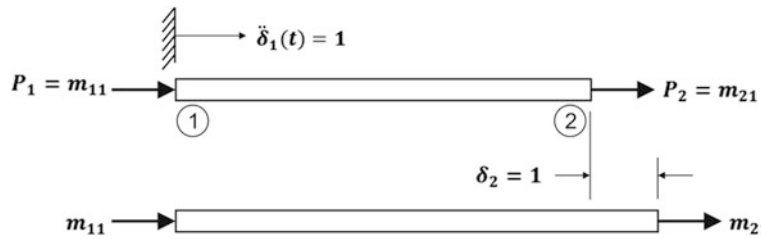

Fig. 11.5 Beam element with unit displacement at node 2($\delta_2 = 1$) undergoing a unit axial acceleration at node 1 $\left[\ddot{\delta}_1(t) = 1\right]$

Now consider the beam of Fig. 11.5 while undergoing a unit acceleration, $\ddot{\delta}_1(t) = 1$ which by Eq. (11.18) results in an acceleration at x given by

$$\ddot{u}_1(x,t) = u_1(x)\ddot{\delta}_1(t)$$

or

$$\ddot{u}_1(x,t) = u_1(x)$$

since $\ddot{\delta}_1(t) = 1$. The inertial force per unit length along the beam resulting from this unit acceleration is

$$f_I = \bar{m}(x)u_1(x) \tag{11.19}$$

where $\bar{m}(x)$ is the mass per unit length along the beam. Now, to determine the mass coefficient m_{21}, we give to the beam shown in Fig. 11.5 a virtual displacement $\delta_2 = 1$. The only external force doing work during this virtual displacement is the reaction m_{21}. This work is then

$$W_E = m_{21}\delta_2$$

or

$$W_E = m_{21} \tag{11.20}$$

since $\delta_2 = 1$. The internal work per unit length along the beam performed by the inertial force f_I during this virtual displacement is

$$\delta W_I = f_I u_2(x)$$

or, from Eq. (11.19)

$$\delta W_I = \bar{m}(x)\, u_1(x)\, u_2(x)$$

Hence the total internal work is

$$W_I = \int_0^L \bar{m}(x)u_1(x)u_2(x)dx \tag{11.21}$$

Finally, equating Eqs. (11.20) and (11.21) yields

$$m_{21} = \int_0^L \bar{m}(x)u_1(x)u_2(x)dx \tag{11.22}$$

or, in general,

$$m_{ij} = \int_0^L \bar{m}(x)u_i(x)u_j(x)dx \tag{11.23}$$

The application of Eq. (11.23) to the special case of a uniform beam results in

$$m_{11} = \int_0^L \bar{m}\left(1 - \frac{x}{L}\right)^2 dx = \frac{\bar{m}L}{3} \tag{11.24}$$

Similarly,

$$m_{22} = \frac{\bar{m}L}{3}$$

and

$$m_{12} = m_{21} = \int_0^L \bar{m}\left(1 - \frac{x}{L}\right)\left(\frac{x}{L}\right)dx = \frac{\bar{m}L}{6} \tag{11.25}$$

In matrix form, the axial inertial force relationship for a uniform beam may be written as

$$\begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix} = \frac{\bar{m}L}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{Bmatrix} \ddot{\delta}_1 \\ \ddot{\delta}_2 \end{Bmatrix} \tag{11.26}$$

Finally, combining the mass matrix Eq. (10.34) for flexural effects with Eq. (11.26) for the axial effects, we obtain the consistent mass matrix for a uniform element of a plane frame in reference to the modal coordinates shown in Fig. 11.2 as

$$\begin{Bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_6 \end{Bmatrix} = \frac{\bar{m}L}{420}\begin{bmatrix} 140 & & & & & \\ 0 & 156 & & & & \\ 0 & 22L & 4L^2 & & & \\ 70 & 0 & 0 & 140 & & \\ 0 & 54 & 13L & 0 & 156 & \\ 0 & -13L & -3L^2 & 0 & -22L & 4L^2 \end{bmatrix}\begin{Bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \\ \delta_5 \\ \delta_6 \end{Bmatrix} \tag{11.27}$$

or, in condensed notation,

$$\{P\} = [M_c]\{\ddot{\delta}\}$$

in which $[M_c]$ is the consistent mass matrix for an element of a plane frame.

11.3 Coordinate Transformation

The stiffness matrix for an element of a plane frame in Eq. (11.4) as well as the mass matrix in Eq. (11.27) are in reference to nodal coordinates defined by coordinate axes fixed on the beam element. These axes are called *local* or *element coordinate axes* while the coordinate axes for the whole structure are known as *global* or *system coordinate axes*. Figure 11.6 shows a beam element with nodal forces P_1, P_2, …, P_6 referred to the local coordinate axes x, y, z, and $\bar{P}_1, \bar{P}_2, \ldots, \bar{P}_6$ referred to global coordinate set of axes X, Y, Z. The objective is to transform the element matrices (stiffness, mass, etc.) from the reference of local coordinate axes to the global coordinate axes. This transformation is required in order that the matrices for all the elements refer to the same set of coordinates; hence, the matrices become compatible for assemblage into the system matrices for the structure. We begin by expressing the forces (P_1, P_2, P_3) in terms of the forces $(\bar{P}_1, \bar{P}_2, \bar{P}_3)$. Since these two sets of forces are equivalent, we obtain from Fig. 11.6 the following relationships:

$$\begin{aligned} P_1 &= \bar{P}_1 \cos\theta + \bar{P}_2 \sin\theta \\ P_2 &= -\bar{P}_1 \sin\theta + \bar{P}_2 \cos\theta \\ P_3 &= \bar{P}_3 \end{aligned} \tag{11.28}$$

The equations of Eq. (11.28) may be written in matrix notation as

$$\begin{Bmatrix} P_1 \\ P_2 \\ P_3 \end{Bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \bar{P}_1 \\ \bar{P}_2 \\ \bar{P}_3 \end{Bmatrix} \tag{11.29}$$

Analogously, we obtain for the forces on the other node the relationships:

$$\begin{aligned} P_4 &= \bar{P}_4 \cos\theta + \bar{P}_5 \sin\theta \\ P_5 &= -\bar{P}_4 \sin\theta + \bar{P}_5 \cos\theta \\ P_6 &= \bar{P}_6 \end{aligned} \tag{11.30}$$

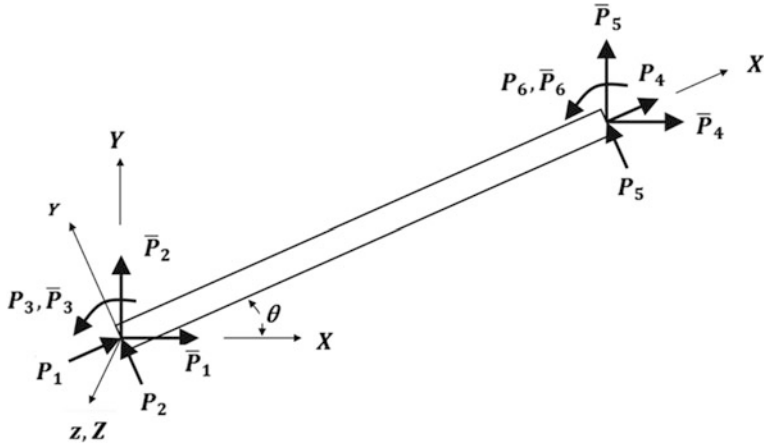

Fig. 11.6 Beam element showing nodal forces P_i in local (x, y, z) and nodal forces $\bar{P}_1$, in global coordinate axes (X, Y, Z)

Equations (11.28) and (11.30) may conveniently be arranged in matrix form as

$$\begin{Bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_6 \end{Bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\theta & \sin\theta & 0 \\ 0 & 0 & 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \bar{P}_1 \\ \bar{P}_2 \\ \bar{P}_3 \\ \bar{P}_4 \\ \bar{P}_5 \\ \bar{P}_6 \end{Bmatrix} \tag{11.31}$$

or in condensed notation

$$\{P\} = [T]\{\bar{P}\} \tag{11.32}$$

in which $\{P\}$ and $\{\bar{P}\}$ are, respectively, the vectors of the element nodal forces in local and global coordinates and $[T]$ is the transformation matrix given by the square matrix in Eq. (11.31).

Repeating the same procedure, we obtain the relation between nodal displacements $(\delta_1, \delta_2, \ldots, \delta_6)$ in local coordinates and the components of the nodal displacements in global coordinates $(\bar{\delta}_1, \bar{\delta}_2, \ldots, \bar{\delta}_6)$, namely

$$\begin{Bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \\ \delta_5 \\ \delta_6 \end{Bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\theta & \sin\theta & 0 \\ 0 & 0 & 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \overline{\delta}_1 \\ \overline{\delta}_2 \\ \overline{\delta}_3 \\ \overline{\delta}_4 \\ \overline{\delta}_5 \\ \overline{\delta}_6 \end{Bmatrix} \tag{11.33}$$

or

$$\{\delta\} = [T]\{\overline{\delta}\} \tag{11.34}$$

Now, the substitution of $\{P\}$ from Eq. (11.32) and $\{\delta\}$ from Eq. (11.34) into the stiffness equation referred to local axes $\{P\} = [\mathrm{K}]\ \{\delta\}$ results in

$$[T]\{\bar{P}\} = [K][T]\{\overline{\delta}\}$$

or

$$\{\bar{P}\} = [T]^{-1}[K][T]\{\overline{\delta}\} \tag{11.35}$$

where $[T]^{-1}$ is the inverse of matrix $[T]$. However, as the reader may verify, the transformation matrix $[T]$ in Eq. (11.31) is an orthogonal matrix, that is, $[T]^{-1} = [T]^T$. Hence

$$\{\bar{P}\} = [T]^{\mathrm{T}}[K][T]\{\overline{\delta}\} \tag{11.36}$$

or, in a more convenient notation,

$$\{\bar{P}\} = [\bar{K}]\{\overline{\delta}\} \tag{11.37}$$

in which

$$\{\bar{K}\} = [T]^{\mathrm{T}}[K][T] \tag{11.38}$$

is the stiffness matrix for an element of a plane frame in reference to the global system of coordinates.

Repeating the procedure of transformation as applied to the stiffness matrix for the lumped mass, Eq. (11.7), or the consistent mass matrix, Eq. (11.27), we obtain in a similar manner

$$\{\bar{P}\} = [\bar{M}]\{\overline{\delta}\}$$

in which

$$\{\bar{M}\} = [T]^{\mathrm{T}}[K][T] \tag{11.39}$$

is the mass matrix for an element of a plane frame in reference to the global system of coordinates and $[T]$ is the transformation matrix given by the square matrix in Eq. (11.33).

Illustrative Example 11.1

Consider in Fig. 11.7 a plane frame having two prismatic beam elements and three degrees of freedom as indicated in the figure. Using the consistent mass formulation, determine the three natural frequencies and corresponding normal modes for this discrete model of the frame.

Solution:
The stiffness matrix for element 1 or 2 in local coordinates by Eq. (11.4) is.

$$[K_1] = [K_2] = \begin{bmatrix} 600 & & & & \text{Symmetric} & \\ 0 & 12 & & & & \\ 0 & 600 & 40,000 & & & \\ -600 & 0 & 0 & 600 & & \\ 0 & -12 & -600 & 0 & 12 & \\ 0 & 600 & 20,000 & 0 & -600 & 40,000 \end{bmatrix}$$

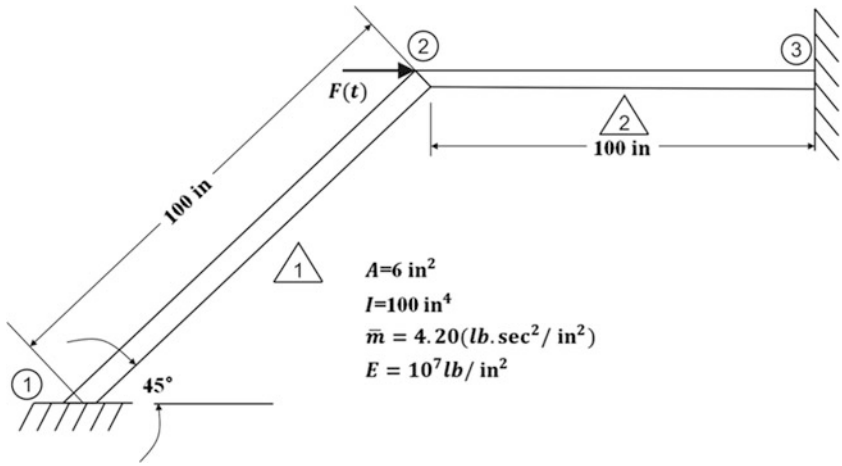

Fig. 11.7 Plane Frame of Illustrative Example 11.1

The transformation matrix for element 1 by Eq. (11.31) with $\theta = 45^\circ$ is

$$[T_1] \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{2} \end{bmatrix}$$

and for element 2 ith $\theta = 0^\circ$ is the identity matrix:

$$[T_2] = [I]$$

The mass matrix in local coordinates for either of the two elements of this frame from Eq. (11.27) is

$$[M_1] = [M_2] = \begin{bmatrix} 140 & & & & \text{Symmetric} & \\ 0 & 156 & & & & \\ 0 & 2200 & 40,000 & & & \\ 70 & 0 & 0 & 140 & & \\ 0 & 54 & 1300 & 0 & 156 & \\ 0 & -1300 & -30,000 & 0 & -2200 & 40,000 \end{bmatrix}$$

The element stiffness and mass matrices in reference to the global system of coordinates, are, respectively, calculated by Eqs. (11.38) and (11.39). For element 1 the stiffness matrix is

$$[\bar{K}_1] = 10^6 \begin{array}{c} \begin{matrix} 4 & 5 & 6 & 1 & 2 & 3 \end{matrix} \\ \begin{bmatrix} 0.306 & & & & \text{Symmetric} & \\ 0.294 & 0.306 & & & & \\ -0.424 & 0.424 & 40,000 & & & \\ -0.306 & -0.294 & 0.424 & 0.306 & & \\ -0.294 & -0.306 & -0.424 & 0.294 & 0.306 & \\ -0.424 & 0.424 & 20,000 & 0.424 & -0.424 & 40,000 \end{bmatrix} \end{array} \begin{matrix} 4 \\ 5 \\ 6 \\ 1 \\ 2 \\ 3 \end{matrix}$$

and the mass matrix

$$[\bar{M}_1] = \begin{array}{c} \begin{matrix} 4 & 5 & 6 & 1 & 2 & 3 \end{matrix} \\ \begin{bmatrix} 148 & & & & \text{Symmetric} & \\ -8 & 148 & & & & \\ -1556 & 1556 & 40,000 & & & \\ 62 & 8 & -919 & 148 & & \\ 8 & 62 & 919 & -8 & 148 & \\ 919 & -919 & -30,000 & 1556 & -1556 & 40,000 \end{bmatrix} \end{array} \begin{matrix} 4 \\ 5 \\ 6 \\ 1 \\ 2 \\ 3 \end{matrix}$$

For element 2:

$$[\bar{K}_2] = 10^6 \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \\ \begin{bmatrix} 0.600 & & & & \text{Symmetric} & \\ 0 & 0.012 & & & & \\ 0 & 0.600 & 40,000 & & & \\ -0.600 & 0 & 0 & 0.600 & & \\ 0 & -0.012 & -0.600 & 0 & 0.012 & \\ 0 & 0.600 & 20,000 & 0 & -0.600 & 40,000 \end{bmatrix} \end{array} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{matrix}$$

and

$$
\begin{array}{c}
\\
\left[\bar{M}_2\right] = \\
\\
\end{array}
\begin{array}{c}
\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\
\begin{bmatrix}
140 & & & & \text{Symmetric} & \\
0 & 156 & & & & \\
0 & 2200 & 40,000 & & & \\
70 & 0 & 0 & 140 & & \\
0 & 54 & 1300 & 0 & 156 & \\
0 & -1300 & -30,000 & 0 & -2200 & 40,000
\end{bmatrix}
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}
\end{array}
$$

The system stiffness and mass matrices are assembled by the direct method. As was mentioned before, it is expedient for hand calculation of these matrices to indicate the corresponding system nodal coordinates at the top and right of the element matrices. We thus obtain, considering only the free coordinates, the system stiffness matrix:

$$
\left[\bar{K}\right] = 10^6 \begin{bmatrix} 0.906 & 0.294 & 0.424 \\ 0.294 & 0.318 & 0.176 \\ 0.424 & 0.176 & 80.000 \end{bmatrix}
$$

and the system mass matrix as

$$
\left[\bar{M}\right] = \begin{bmatrix} 288 & -8 & 1556 \\ -8 & 304 & 644 \\ 1556 & 644 & 80,000 \end{bmatrix}
$$

The natural frequencies are found as the roots of the characteristic equation

$$
\left|[K] - \omega^2[M]\right| = 0
$$

which, upon substituting the numerical values given for this example, yields

$$
10^3 \begin{vmatrix} 906 - 0.288\omega^2 & 294 + 0.008\omega^2 & 424 - 1.556\omega^2 \\ 294 + 0.008\omega^2 & 318 - 0.304\omega^2 & 176 - 0.644\omega^2 \\ 424 - 1.556\omega^2 & 176 - 0.644\omega^2 & 80,000 - 80\omega^2 \end{vmatrix} = 0
$$

The roots then are found to be:

$$
{\omega_1}^2 = 638.5, {\omega_2}^2 = 976.6, {\omega_3}^2 = 4211.6,
$$

and the natural frequencies are

$$
\omega_1 = 25.26 \text{ rad/sec}, \ \omega_2 = 31.24 \text{ rad/sec}, \ \text{and} \ \omega_3 = 64.90 \text{ rad/sec},
$$

or

$$
f_1 = 4.02 \text{ cps}, \ f_2 = 4.97 \text{ cps}, \ \text{and} \ f_3 = 10.33 \text{ cps},
$$

The normal modes are given as the nontrivial solution of the eigenproblem

$$
\left([K] - \omega^2[M]\right)\{a\} = \{0\}
$$

Substituting ${\omega_1}^2 = 638.5$ and setting $a_{11} = 1.0$, we obtain the first mode shape as

$$\{a_1\} = \begin{Bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{Bmatrix} = \begin{Bmatrix} 1.00 \\ -2.38 \\ 0 \end{Bmatrix}$$

which is normalized with the factor

$$\sqrt{\{a_1\}^T [M]\{a_1\}} = 45.81$$

The normalized eigenvector is then

$$\{\phi_1\} = \begin{Bmatrix} \phi_{11} \\ \phi_{21} \\ \phi_{31} \end{Bmatrix} = \begin{Bmatrix} 0.0218 \\ -0.0527 \\ 0 \end{Bmatrix}$$

Analogously, for the other two modes, we obtain:

$$\{\phi_2\} = \begin{Bmatrix} \phi_{12} \\ \phi_{22} \\ \phi_{32} \end{Bmatrix} = \begin{Bmatrix} 0.00498 \\ 0.00206 \\ 0.00341 \end{Bmatrix} \quad \text{and} \quad \{\phi_3\} = \begin{Bmatrix} \phi_{13} \\ \phi_{23} \\ \phi_{33} \end{Bmatrix} = \begin{Bmatrix} 0.0583 \\ 0.0241 \\ -0.0016 \end{Bmatrix}$$

arranging these modal vectors into columns of the modal matrix, we obtain:

$$[\Phi] = \begin{bmatrix} 0.0218 & 0.00498 & 0.0583 \\ -0.0527 & 0.00206 & 0.0241 \\ 0 & 0.00341 & -0.0016 \end{bmatrix}$$

Illustrative Example 11.2

Determine the maximum displacement at the nodal coordinates of the frame in Fig. 11.7 when a force of magnitude 100,000 lb. is suddenly applied at nodal coordinate 1. Neglect damping.

Solution:

From Illustrative Example 11.1, the natural frequencies are $\omega_1 = 25.26$ rad/sec, $\omega_2 = 31.24$ rad/sec, and $\omega_3 = 64.90$ rad/sec; and the modal matrix is.

$$[\Phi] = \begin{bmatrix} 0.0218 & 0.00498 & 0.0583 \\ -0.0527 & 0.00206 & 0.0241 \\ 0 & 0.00341 & -0.0016 \end{bmatrix}$$

The modal equations have the form of

$$\ddot{z}_i + \omega_i^2 z_i = P_i \tag{a}$$

where

$$P_i = \sum_j \phi_{ji} F_j \tag{b}$$

In this example, the nodal applied forces are

$$F_1 = 100,000 \text{ lb}, \quad F_2 = 0, \quad F_3 = 0$$

We thus obtain, after substituting numerical values into Eqs. (a) and (b), the modal equations as

$$\begin{aligned} \ddot{z}_1 + 638.5 z_1 &= 2180 \\ \ddot{z}_2 + 976.6 z_2 &= 498 \\ \ddot{z}_3 + 4211.6 z_3 &= 5830 \end{aligned} \tag{c}$$

The solutions of these equations by (4.5) are of the form

$$z_i = \frac{P_i}{\omega_i^2}(1 - \cos \omega_i t)$$

Substitution for P_i and ω_i yields

$$\begin{aligned} z_1 &= 3.414(1 - \cos 25.2685t) \\ z_2 &= 0.510(1 - \cos 31.2506t) \\ z_3 &= 1.384(1 - \cos 64.8970t) \end{aligned} \tag{d}$$

The nodal displacements are obtained from

$$\{u\} = [\Phi]\{z\}$$

which results in

$$\begin{aligned} u_1 &= 0.1577 - 0.0744 \cos 25.26t - 0.00254 \cos 31.25t - 0.0807 \cos 64.9t \text{ (in)} \\ u_2 &= -0.1455 + 0.1800 \cos 25.26t - 0.00105 \cos 31.25t - 0.0333 \cos 64.9t \text{ (in)} \\ u_3 &= -0.000475 + 0 \cos 25.26t - 0.00174 \cos 31.25t + 0.0022 \cos 64.9t \text{ (radian)} \end{aligned} \tag{e}$$

The maximum possible displacements at the nodal coordinates may then be estimated as the summation of the absolute values of the coefficients in the above expressions. Hence

$$u_{1\max} = 0.3177 \text{ in} \quad u_{2\max} = 0.3589 \text{ in} \quad u_{3\max} = 0.0044 \text{ radian}$$

11.4 Modeling Structures as Plane Frames Using MATLAB

MATLAB program is used to determine the stiffness and the mass matrices for a plane frame.

Illustrative Example 11.3
Use MATLAB to determine the stiffness and mass matrices for the plane frame shown in Fig. 11.7.

Solution:
The following MATLAB file is used to compute stiffness and mass matrices using the system matrix. Two function files, FrameElement.m and FrameConMass.m are needed. After assembling matrices, the system matrix can be found using System.m file (Chap.10).

```
clc
close all
clear all

%_____________________________________________________________________
%  Determine System Matrices/Determine Force
%_____________________________________________________________________

%%%%-GIVEN VALUES-%%%%

E=10^7; A=6;                % E (psi), A (in.^2)
inertia = 100;              %Second Moment of Inertia (in^4)

%%%Create frame model (ith row of nodes is ith node)
nodes = [0, 0; 70.71,70.71; 170.71,70.71];
%%%Element number (ith row = ith element with two nodes)
conn=[1,2; 2,3];
%%%Dofs for ith element (ith row)
lmm=[1:3,4:6; 4:6,7:9];
%%%Dofs are elimniated at supports for system matrix
debc=[1:3,7:9];

m_bar = 4.2;                %Mass per unit length (lb-sec^2/in/in)

dof = 3*length(nodes);      % Total No. dofs

K= zeros(dof);
M= zeros(dof);

%%%Generate equations for each element and assemble them.
for i=1:2
    lm=lmm(i,:);
    con=conn(i,:);
    ke = PlaneFrameElement(E, inertia, A, nodes(con,:));
    K(lm, lm) = K(lm, lm) + ke
end

%%%Generate mass matrix for each element and assemble them.
for i=1:2
    lm=lmm(i,:);
    con=conn(i,:);
    m=FrameConsMass(m_bar, nodes(con,:));
    M(lm, lm) = M(lm, lm) + m;
end

K;

M;

%%%Define the load vector
F = zeros(dof,1);              %Applied force at specific dofs

%%%System Matrices
[Kf, Mf, Rf] = System(K, M, F, [1:3,7:9]);

Kf

Mf

Rf

save ('temp0.mat', 'Mf', 'Kf' ,'Rf');
```

The function file of MATLAB is used to assemble the stiffness matrix of each beam element for global stiffness matrix.

```
function ke = PlaneFrameElement(modulus, inertia, A, coord)
% ke = PlaneFrameElement(modulus, inertia, A, coord)
% Generates equations for a plane frame element
% modulus = modulus of elasticity
% inertia = moment of inertia
% A = area of cross-section
% coord = coordinates at the element ends

EI=modulus*inertia; EA = modulus*A;
x1=coord(1,1); y1=coord(1,2);
x2=coord(2,1); y2=coord(2,2);
L=sqrt((x2-x1)^2+(y2-y1)^2);
C=(x2-x1)/L; S=(y2-y1)/L;

ke = [(EA*L^2*C^2 + 12*EI*S^2)/L^3, ((-12*EI + EA*L^2)*C*S)/L^3, ...
        (-6*EI*S)/L^2, -((EA*L^2*C^2 + 12*EI*S^2)/L^3), ...
        ((12*EI - EA*L^2)*C*S)/L^3, (-6*EI*S)/L^2;
    ((-12*EI + EA*L^2)*C*S)/L^3, (12*EI*C^2 + EA*L^2*S^2)/L^3, ...
        (6*EI*C)/L^2, ((12*EI - EA*L^2)*C*S)/L^3, ...
        -((12*EI*C^2 + EA*L^2*S^2)/L^3), (6*EI*C)/L^2;
    (-6*EI*S)/L^2, (6*EI*C)/L^2, (4*EI)/L, ...
        (6*EI*S)/L^2, (-6*EI*C)/L^2,(2*EI)/L;
    -((EA*L^2*C^2 + 12*EI*S^2)/L^3), ((12*EI -EA*L^2)*C*S)/L^3, ...
        (6*EI*S)/L^2, (EA*L^2*C^2 + 12*EI*S^2)/L^3, ...
        ((-12*EI + EA*L^2)*C*S)/L^3, (6*EI*S)/L^2;
    ((12*EI - EA*L^2)*C*S)/L^3, -((12*EI*C^2 + EA*L^2*S^2)/L^3),...
        (-6*EI*C)/L^2, ((-12*EI + EA*L^2)*C*S)/L^3, ...
        (12*EI*C^2 + EA*L^2*S^2)/L^3, (-6*EI*C)/L^2;
    (-6*EI*S)/L^2, (6*EI*C)/L^2, (2*EI)/L, (6*EI*S)/L^2, ...
        (-6*EI*C)/L^2, (4*EI)/L];
```

The function file of MATLAB is used to assemble the mass matrix of each beam element for global mass matrix.

```
function m = FrameConsMass(m_bar, coord)
% FrameConsMass(m_bar, nodes(con,:))
% Generates mass matrix for a plane frame element
% m = mass (lb.sec^2/in.^2)
% L = length
% coord = coordinates at the element ends

x1=coord(1,1); y1=coord(1,2);
x2=coord(2,1); y2=coord(2,2);
L=sqrt((x2-x1)^2+(y2-y1)^2);
ls=(x2-x1)/L; ms=(y2-y1)/L;
T = [ls ms 0 0 0 0;
    -ms ls 0 0 0 0;
       0 0 1 0 0 0;
     0 0 0 ls ms 0;
     0 0 0 -ms ls 0;
     0 0 0 0 0 1];

m = m_bar*L/420*T.'*[140  0    0     70   0       0  ;
                     0  156   22*L   0   54     -13*L;
                     0  22*L 4*L^2   0   13*L   -3*L^2;
                     70  0      0   140   0       0  ;
                      0  54    13*L  0  156     -22*L;
                      0 -13*L -3*L^2 0  -22*L   4*L^2]*T;
```

We obtain:

$$[\Phi] = \begin{bmatrix} 0.0218 & 0.0050 & -0.0583 \\ -0.0527 & 0.0021 & -0.0242 \\ 0 & 0.0034 & 0.0016 \end{bmatrix}$$

This is identical to modal matrix in Illustrative Example 11.2.

11.5 Dynamic Analysis of Plane Frames Using MATLAB

Illustrative Example 11.4

For the structure shown in Fig. 11.7 which was modeled in Illustrative Example 11.3, determine: (a) natural frequencies and modal shapes; (b) the response to a force of magnitude 100,000 lb. suddenly applied at nodal coordinate 2.

Solution:

This MATLAB file is to yield the results of section (a). Two function files, FrameElement.m and FrameConMass.m are needed. After assembling matrices, the system matrix can be found using System.m file (Chap.10). Using system matrices, the natural frequencies and mode shapes will be found.

```
clc
close all
clear all

%_____________________________________________________________________
%  Determine System Matrices/Determine Force
%_____________________________________________________________________

%%%%-GIVEN VALUES-%%%%

E=10^7; A=6;                % E (psi), A (in.^2)
inertia = 100;              %Second Moment of Inertia (in^4)

%%%Create frame model (ith row of nodes is ith node)
nodes = [0, 0; 70.71,70.71; 170.71,70.71];
%%%Element number (ith row = ith element with two nodes)
conn=[1,2; 2,3];
%%%Dofs for ith element (ith row)
lmm=[1:3,4:6; 4:6,7:9];
%%%Dofs are elimniated at supports for system matrix
debc=[1:3,7:9];

m_bar = 4.2;                %Mass per unit length (lb-sec^2/in/in)

dof = 3*length(nodes);      % Total No. dofs

K= zeros(dof);
M= zeros(dof);

%%%Generate equations for each element and assemble them.
for i=1:2
    lm=lmm(i,:);
    con=conn(i,:);
    ke = PlaneFrameElement(E, inertia, A, nodes(con,:));
    K(lm, lm) = K(lm, lm) + ke
end

%%%Generate mass matrix for each element and assemble them.
for i=1:2
    lm=lmm(i,:);
    con=conn(i,:);
    m=FrameConsMass(m_bar, nodes(con,:));
    M(lm, lm) = M(lm, lm) + m;
end

K;

M;
```

```
%%%Define the load vector
F = zeros(dof,1); F(4) = 100000;              %Applied force at specific dofs

%%%System Matrices
[Kf, Mf, Rf] = System(K, M, F, [1:3,7:9]);

Kf

Mf

Rf
%_____________________________________________________________________
% Solve the eigenvalue problem and normalized eigenvectors
%_____________________________________________________________________

%%%Solve for eigenvalues (D) and eigenvectors (a)
[a, D] = eig(Kf, Mf);

[omegas,ii] = sort(sqrt(diag(D)));        %Natural Frequencies

omegas

a = a(:,ii)                               %Mode Shapes

T = 2*pi./omegas;                         %Natural Periods

save ('temp0.mat', 'Mf', 'Kf' ,'Rf');
```

This MATLAB file is to yield the results of section (b).
After running the program above, the following MATLAB program will yield the response. The duration of response is from 0 to 5 seconds with the interval of 0.01 second.

```
clear all
close all
%_____________________________________________________________________
% Inputs:
%   M, K
%   F = forcing function
%   t = Time period
%   u0 = initial displacement
%   v0 = initial velocity
%_____________________________________________________________________
 t = 0:0.01:0.5;

load ('temp0.mat', 'Mf', 'Kf', 'Rf')

%%%Deifne Mass Matrix

M = Mf

%%%Deifne Stiffness Matrix

K = Kf

[n,n]= size(M);

F = Rf;

u0 = zeros(n,1); u0(1) =0;
v0 = zeros(n,1); v0(1) =0;
[n,n]= size(M);
```

```
%_____________________________________________________________________
% Solve the eignevalue problem and normalized the eigenvectors
%_____________________________________________________________________
[a, D] = eig(K, M)                  % Solve for eigenvalues (D) and eigenvectors (a)

[omegas,k] = sort(sqrt(diag(D)));   % Natrual Frequencies
a = a(:,k)

T = 2*pi./omegas;                   % Natural Periods

aMa = diag(a'*M*a)                  % aMa = {a}'*[M]*(a}

nom_phi = (a)*inv(sqrt(diag(aMa))) % Normalized modal matrix

%_____________________________________________________________________
% Initial conditions
%_____________________________________________________________________

P = nom_phi'*F;                     % Normalized force, P = nom_F
q0 = nom_phi'*M*u0
dq0 = nom_phi'*M*v0
%_____________________________________________________________________
% Damping matrix using the proportional damping matrix
% [C] = a[M]+b[K]
% zetas = damping ratios
%_____________________________________________________________________
a = 0;
b = 0;
nom_C = nom_phi'*(a*M+b*K)*nom_phi;
zetas = diag((1/2)*nom_C*inv(diag(omegas)));

save ('temp1.mat', 'omegas', 'P' ,'zetas');
q = [];
r = [];
for i=1:n
q0_i = q0(i,:);
dq0_i = dq0(i,:);

load temp1.mat
omega = omegas(i,:);
P = P(i,:);
m = M(i,i);
zeta = zetas(i,:);

save ('temp2.mat', 'omega', 'P', 'm', 'zeta');

[t,q] = ode45(@MDOFP, t, [q0_i dq0_i]',[]);

r(:,i) = q(:,1);
save ('temp3.mat', 'r')

end

load ('temp3.mat', 'r');

yim = nom_phi*[r'];

save ('response.mat','yim');

figure
subplot(3,1,1);          % Node 2: x displ (in.).

xlabel ('Time (sec)'); ylabel ('u_2(in.)'); grid on


subplot(3,1,2);          % Node 2: y displ (in.).
plot(t, yim(2,:))
xlabel ('Time (sec)'); ylabel ('v_2(in.)'); grid on

subplot(3,1,3);          % Node 2: rotation (radian).
plot(t, yim(3,:))
xlabel ('Time (sec)'); ylabel ('\phi_2(radian)'); grid on

umax_1=max(abs(yim(1,:)))
umax_2=max(abs(yim(2,:)))
umax_3=max(abs(yim(3,:)))
```

The function file of MATLAB is used to determine the function of the applied force and solve the uncoupled equation (Fig. 11.8).

```
function  q = MDOFP(t, q)
load ('temp2.mat', 'omega', 'P', 'm' ,'zeta')

    P = P;

q = [q(2); -omega*omega*q(1)-2*zeta*omega*q(2)+P];
```

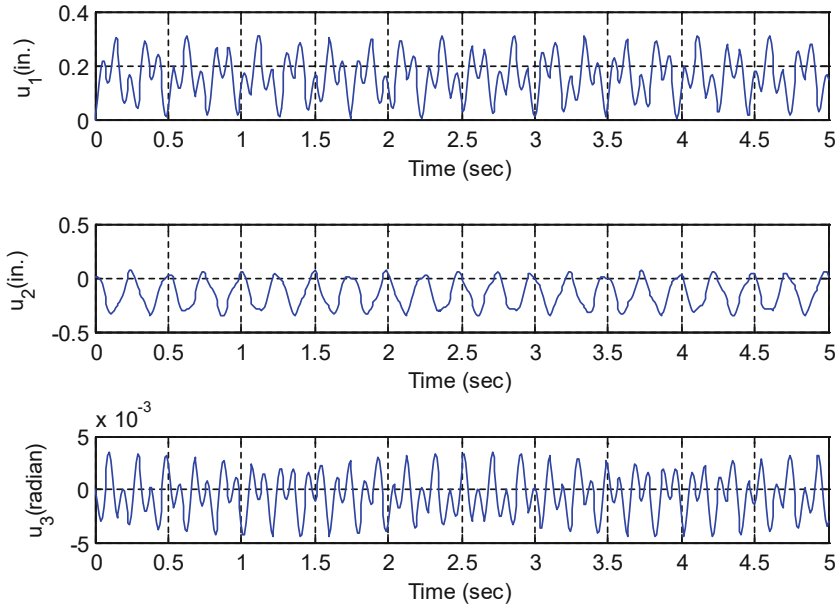

Fig. 11.8 Response of Illustrative Example 11.4

The maximum displacements at the nodal coordinates were estimated using MATLAB. These are very close to the estimation of the summation of the absolute values obtained from Illustrative Example 11.2.

$$u_{1\max} = 0.3119 \text{ in} \quad u_{2\max} = 0.3590 \text{ in} \quad u_{3\max} = 0.0045 \text{ radian}$$

Illustrative Example 11.5

Use MATLAB to model the plane frame shown in Fig. 11.7 using a total of two beam elements and to calculate the response due to the force $F(t)$ shown in Fig. 11.9.

Solution:

The same files will be used in illustrative example 11.3 and 11.4 with the change of MDOFP function as bellow. The plot is generated from the range of t = 0 to 0.5 seconds.

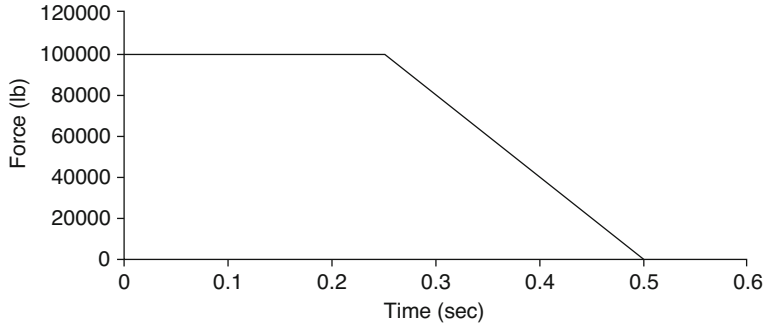

Fig. 11.9 Forcing function applied at joint 2 of the plane frame of Example 11.5

The function file of MATLAB is used to determine the function of the applied force and solve the uncoupled equation (Fig. 11.10).

```
function  q = MDOFP(t, q)
load ('temp2.mat', 'omega', 'P', 'm' ,'zeta')

if t <= 0.25
    P = P;

elseif t <0.5

   P = 2*P*(1-2*t);

else
   P = 0;

end

q = [q(2); -omega*omega*q(1)-2*zeta*omega*q(2)+P];
```

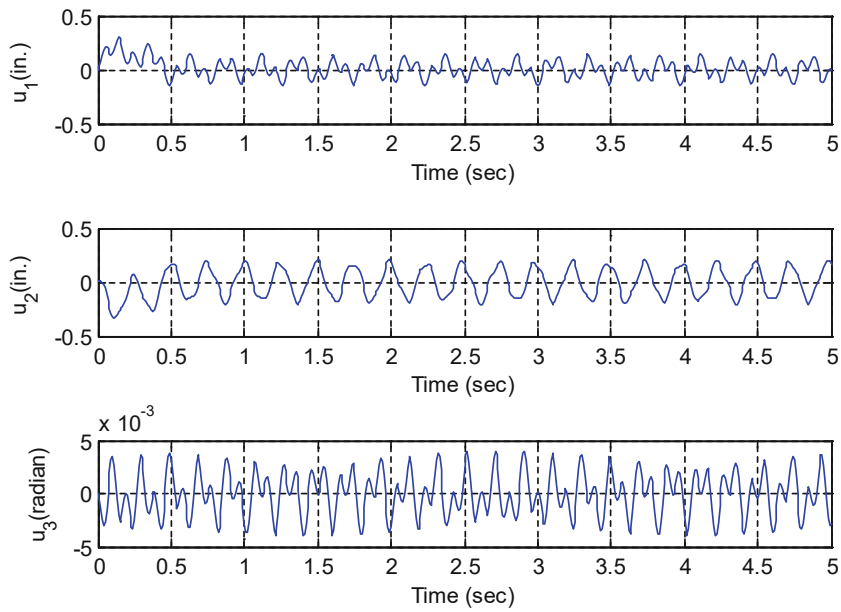

Fig. 11.10 The maximum displacements at the nodal coordinates were estimated using MATLAB

$u_{1\max} = 0.3037$ in. $u_{2\max} = 0.3352$ in. $u_{3\max} = 0.0040$ radian

11.6 Summary

The dynamic analysis of plane frames by the stiffness method requires the inclusion of the axial effects in the system matrices (stiffness, mass, etc.). It also requires a transformation of coordinates in order to refer all the element matrices to the same coordinate system, so that the appropriate superposition can be applied to assemble the system matrices.

The required matrices for consideration of axial effects as well as the matrix required for the transformation of coordinates are developed in this chapter. A computer program for modeling structures as plane frames is also presented. This program is organized following the pattern of the BEAM program of the preceding chapter.

11.7 Problems

The following problems are intended for hand calculation, though it is recommended that whenever possible solutions should also be obtained using Program 14 to model the structure as a plane frame, Program 8 to determine natural frequencies and modal shapes, and Program 9 to calculate the response using modal superposition method.

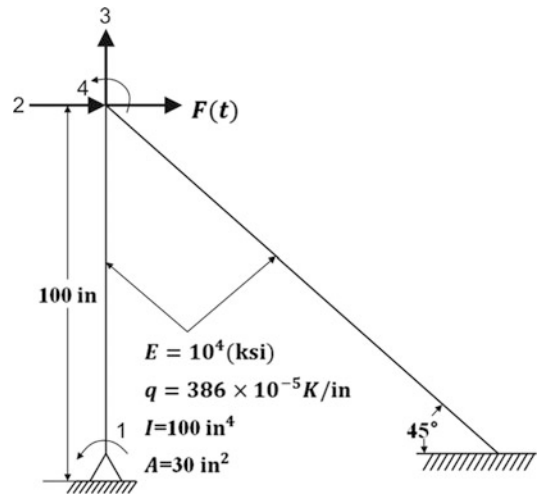

Fig. P11.1

Problem 11.1

For the plane frame shown in Fig. P11.1 determine the system stiffness and mass matrices. Base the analysis on the four free nodal coordinates indicated in the figure. Use consistent mass method.

Problem 11.2
Use the results obtained in Problem 11.1 in performing the static condensation to eliminate the rotational degree of freedom at the support to determine the transformation matrix and the reduced stiffness and mass matrices.

Problem 11.3
Determine the natural frequencies and corresponding normal modes for the reduced system in Problem 11.2.

Problem 11.4
Determine the response of the frame shown in Fig. P11.1 when it is acted upon by a force $F(t) = 1.0$ Kip suddenly applied at nodal coordinate 2 for 0.05 sec. Use results of Problem 11.3 to obtain the modal equations. Neglect damping in the system.

Problem 11.5

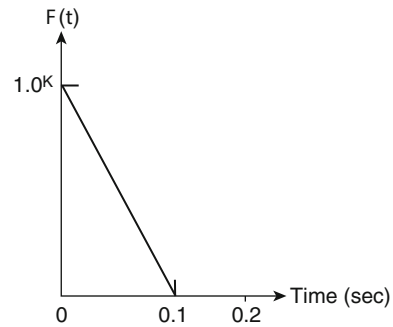

Fig. P11.5

Determine the maximum response of the frame shown in Fig. P11.1 when subjected to the triangular impulsive load (Fig. P11.5) along the nodal coordinate 2. Use results of Problem 11.3 to obtain the modal equations and use the appropriate response spectrum to find maximum modal response (Fig. 4.5). Neglect damping in the system.
Pit).

Problem 11.6
Determine the steady-state response of the frame shown in Fig. P11.1 when subjected to harmonic force $F(t) = 10 \sin 30$ t (Kip) along nodal coordinate 2. Neglect damping in the system.

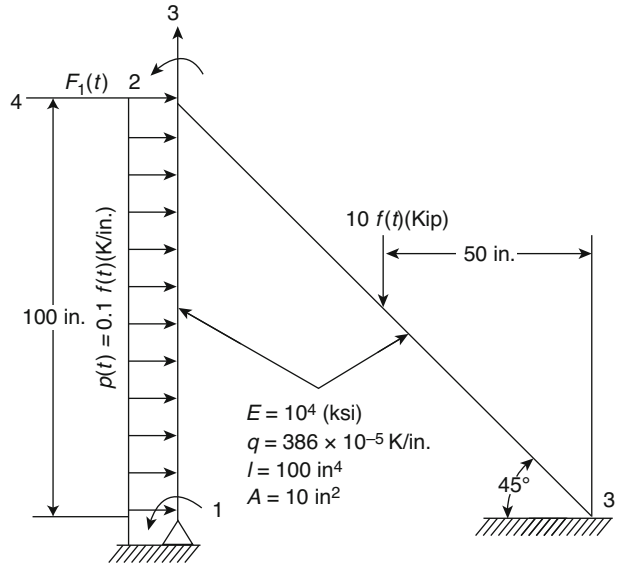

Fig. P11.8

Problem 11.7

Repeat Problem 11.6 assuming that the damping is proportional to the stiffness matrix of the system, $[C] = a_0\,[K]$, where $a_0 = 0.2$.

Problem 11.8

The frame shown in Fig. P11.8 is acted upon by the dynamic forces shown in he figure. Determine the equivalent nodal forces corresponding to each member of the frame.

Problem 11.9

Assemble the system equivalent nodal forces $\{F_e\}$ from equivalent member nodal forces which were calculated in Problem 11.8.

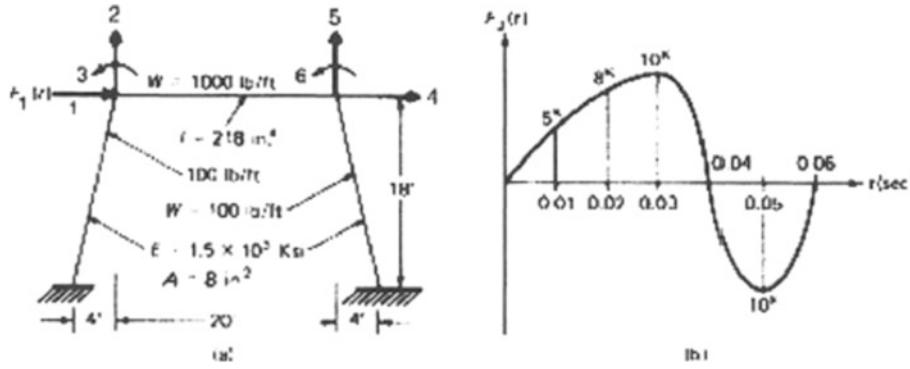

Fig. P11.11

Problem 11.10
Determine the natural frequencies and corresponding normal modes for the frame shown in Fig. P11.8.

Problem 11.11
Determine the response for the frame shown in Fig. P11.11 (a) when subjected to the force $F_3(t)$ [Fig. 11.11(b)] acting along nodal coordinate 1. Assume 5% damping in all the modes.

Problem 11.12
Determine the steady-state response of the frame in Fig. P11.11 acted upon harmonic force $F_1(t) = 10\cos 50\ t$ (Kip) as indicated in the figure. Neglect damping in the system.

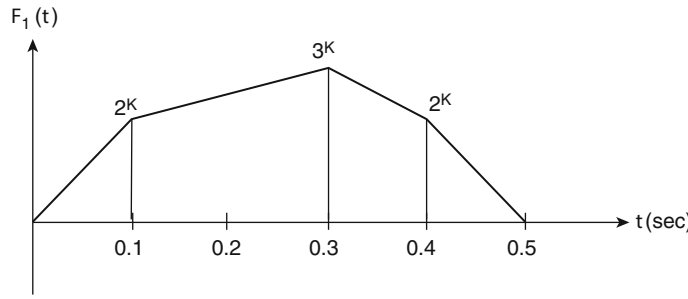

Fig. P11.14

Problem 11.13
Solve Problem 11.11 using step-by-step linear acceleration method (Program 19). Neglect damping.

Problem 11.14
Determine the response of the frame shown in Fig. P11.1 when acted upon by the force $F_1(t)$ (depicted in Fig. P11.14 applied at nodal coordinate 1. Assume 10% damping in all the modes. Use modal superposition method.

Problem 11.15
Find the response in Problem 11.14 using step-by step linear acceleration method (Program 19). Assume damping proportional to stiffness by a factor $a_0 = 0.01$.

Check for updates

12 Dynamic Analysis of Grid Frames

In Chap. 11 consideration was given to the dynamic analysis of the plane frame when subjected to forces acting on the plane of the structure. When the planar structural system is subjected to loads applied normally to its plane, the structure is referred to as a grid frame. This structure can also be treated as a special case of the three-dimensional frame to be presented in Chap. 13. The reason for considering the planar frame, whether loaded in its plane or normal to its plane, as a special case, is the immediate reduction of unknown nodal coordinates for an element of these special structures, hence a considerable reduction in the number of unknown displacements for the structural system.

When analyzing the planar frame under action of loads in the plane, the possible components of joint displacements that had to be considered were translations in the X and Y directions and rotation about the Z axis. However, if a plane frame is loaded normal to the plane of the structure, the components of joint displacements required to describe the displacements of a joint are a translation in the Z direction and rotations about the X and Y axes. Thus treating the planar grid structure as a special case, it will be necessary to consider only three components of nodal displacements at each end of a typical element of a grid frame.

12.1 Local and Global Coordinate Systems

For an element of a grid frame, the local orthogonal axes will be established such that the x defines the longitudinal centroidal axis of the member and the x-y plane will coincide with the plane of the structural system. In this case, the z axis will define the minorprincipal axis of the cross section while the y axis will define the major axis of the cross section. It will be assumed that the shear center of the cross section coincides with the centroid of the cross section. The grid member may have either a variable or constant cross section along its length.

The possible nodal displacements with respect to the local or to the global systems of coordinates are identified in Fig. 12.1. It can be seen that the linear displacements along the z direction for local axes and along the Z direction for the global system are identical since the two axes coincide. However, in general, rotational components at the nodal coordinates differ for these two coordinate systems. Hence, a transformation of coordinates will be required to transform the element matrices from the local to the global coordinates.

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M. Paz, Y. H. Kim, *Structural Dynamics*, https://doi.org/10.1007/978-3-319-94743-3_12

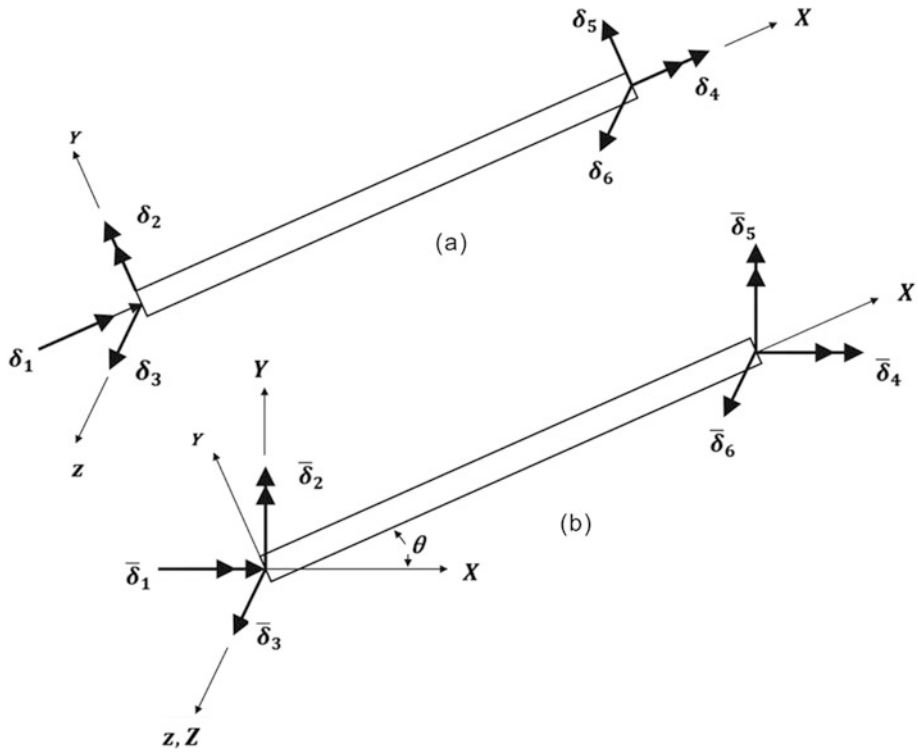

Fig. 12.1 Components of nodal displacements for a grid member. (**a**) Local coordinate system. (**b**) Global coordinate system

12.2 Torsional Effects

The dynamic analysis by the stiffness method for grid frames, that is, for plane frames subjected to normal loads, requires the determination of the torsional stiffness and mass coefficients for a typical element of the grid frame. The derivation of these coefficients is essentially identical to the derivation of the stiffness and mass coefficients for axial effects on a beam element. Similarity between these two derivations occurs because the differential equations for both problems have the same mathematical form. For the axial problem, the differential equation for the displacement function is given by Eq. (11.8) as

$$\frac{du}{dx} = \frac{P}{AE} \text{ (repeated)} \tag{11.8}$$

Likewise, the differential equation for torsional displacement is

$$\frac{d\theta}{dx} = \frac{T}{JG} \tag{12.1}$$

in which θ is the angular displacement, T is the torsional moment, G is the modulus of elasticity in shear, and J is the torsional constant of the cross section (polar moment of inertia for circular sections).

As a consequence of the analogy between Eqs. (11.8) and (12.1), we can write the following results already obtained for axial effects. The displacement functions for the torsional effects are the same as the corresponding functions giving the displacements for axial effects; hence by analogy to Eqs. (11.10) and (11.11) and in reference to the nodal coordinates of Fig. 12.2, we obtain

$$\theta_1(x) = \left(1 - \frac{x}{L}\right) \tag{12.2}$$

and

$$\theta_2(x) = \frac{x}{L} \tag{12.3}$$

in which the angular displacement function $\theta_1(x)$ corresponds to a unit angular displacement $\delta_1 = 1$ at nodal coordinate 1 and $\theta_2\ (x)$ corresponds to the displacement.

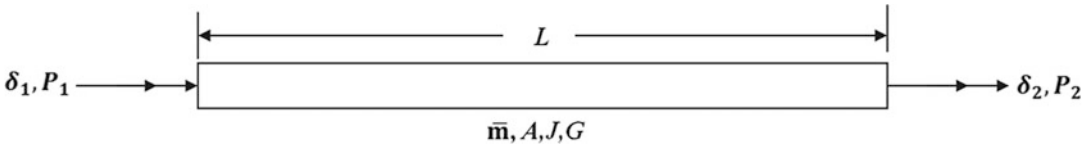

Fig. 12.2 Nodal torsional coordinates for a beam element

function resulting from a unit angular displacement $\delta_2 = 1$ at nodal coordinate 2. Analogous to Eq. (11.17), the stiffness coefficients for torsional effects may be calculated from

$$k_{ij} = \int_0^L JG\theta_i'(x)\theta_j'(x)dx \tag{12.4}$$

in which θ_1'(x) and θ_2'(x) are the derivatives with respect to x of the displacement functions $\theta_1(x)$ and $\theta_2(x)$. Also analogous to Eq. (11.23), the consistent mass matrix coefficients for torsional effects are given by

$$m_{ij} = \int_0^L I_{\bar{m}}\theta_i(x)\theta_j'(x)dx \tag{12.5}$$

in which $I_{\bar{m}}$ is the polar mass moment of inertia, per unit length along the beam element. This moment of inertia may conveniently be expressed as the product of the mass $\bar{m}$ per unit length times the radius of gyration squared, k^2. The radius of gyration may, in turn, be calculated as the ratio I_0/A. Therefore, the mass polar moment of inertia per unit length $I_{\bar{m}}$ is given by

$$I_{\bar{m}} = \bar{m}\frac{I_0}{A} \tag{12.6}$$

in which I_0 is the polar moment of inertia of the cross-sectional area and A the cross-sectional area.

The application of Eqs. (12.4) and (12.5) for a uniform beam yields the stiffness and mass matrices for torsional effects as

$$\begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} = \frac{JG}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} \delta_1 \\ \delta_2 \end{Bmatrix} \tag{12.7}$$

and

$$\begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} = \frac{I_{\bar{m}} L}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{Bmatrix} \ddot{\delta}_1 \\ \ddot{\delta}_2 \end{Bmatrix} \tag{12.8}$$

in which $I_{\bar{m}}$ is given by Eq. (12.6), and T_1, T_2 are torsional moments at the ends of the beam element labeled in Fig. 12.2 as P_1 and P_2.

12.3 Stiffness Matrix for a Grid Element

The torsional stiffness matrix, Eq. (12.7), is combined with the flexural stiffness matrix, Eq. (10.20), to obtain the stiffness matrix for a typical element of a grid frame. In reference to the local coordinate system indicated in Fig. 12.1a, the stillness equation for a uniform element is then

$$\begin{Bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_6 \end{Bmatrix} = \frac{EI}{L^3} \begin{Bmatrix} JGL^2/EI & & & & & \\ 0 & 4L^2 & & & & \\ 0 & -6L & 12 & & & \\ -JGL^2/EI & 0 & 0 & JGL^2/EI & & \\ 0 & 2L^2 & -6L & 0 & 4L^2 & \\ 0 & 6L & -12 & 0 & 6L & 12 \end{Bmatrix} \begin{Bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \\ \delta_5 \\ \delta_6 \end{Bmatrix} \tag{12.9}$$

or in condensed form

$$\{P\} = [K]\{\delta\} \tag{12.10}$$

12.4 Consistent Mass Matrix for a Grid Element

The combination of the consistent mass matrix for flexural effects (10.34) with the consistent mass matrix for torsional effects (12.8) results in the consistent mass matrix for a typical member of a grid, namely

$$\begin{Bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_6 \end{Bmatrix} = \frac{\bar{m}L}{420} \begin{bmatrix} 140I_0/A & & & \textit{Symmetric} & & \\ & 4L^2 & & & & \\ 0 & 22L & 156 & & & \\ 70I_0/A & 0 & 0 & 140I_0/A & & \\ 0 & -3L^2 & -13L & 0 & 4L^2 & \\ 0 & 13L & 54 & 0 & -22L & 156 \end{bmatrix} \begin{Bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \\ \delta_5 \\ \delta_6 \end{Bmatrix} \tag{12.11}$$

or in concise notation

$$\{P\} = [M_c]\{\delta\} \tag{12.12}$$

in which $[M_c]$ is the mass matrix for a typical uniform member of a grid structure.

12.5 Lumped Mass Matrix for a Grid Element

The lumped mass allocation to the nodal coordinates of a typical grid member is obtained from static considerations. For a uniform member having a uniform distributed mass along its length, the nodal mass is simply one-half of the total rotational mass $I_{\bar{m}}L$. The matrix equation for the lumped mass matrix corresponding to the torsional effects is then

$$\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \frac{I_{\bar{m}}L}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \ddot{\delta}_1 \\ \ddot{\delta}_2 \end{bmatrix} \tag{12.13}$$

in which $I_{\bar{m}}$ is given by Eq. (12.6). The combination of the lumped torsional mass and the lumped translatory mass results in the diagonal matrix which is the lumped mass matrix for the grid element. This matrix, relating forces and accelerations at nodal coordinates, is given by the following equation:

$$\begin{Bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_6 \end{Bmatrix} = \frac{\bar{m}L}{2} \begin{bmatrix} I_0/A & & & & & \\ & 0 & & & & \\ & & 1 & & & \\ & & & I_0/A & & \\ & & & & 0 & \\ & & & & & 1 \end{bmatrix} \begin{Bmatrix} \ddot{\delta}_1 \\ \ddot{\delta}_2 \\ \ddot{\delta}_3 \\ \ddot{\delta}_4 \\ \ddot{\delta}_5 \\ \ddot{\delta}_6 \end{Bmatrix} \tag{12.14}$$

or briefly

$$\{P\} = \lceil M_L \rfloor \{\delta\} \tag{12.15}$$

in which $\lceil \mathrm{M}_L \rfloor$ is, in this case, the diagonal lumped mass matrix for a grid element.

12.6 Transformation of Coordinates

The stiffness matrix, Eq. (12.9), as well as the consistent and the lumped mass matrices in Eqs. (12.11) and (12.14), respectively, are in reference to the local system of coordinates. Therefore, it is necessary to transform the reference of these matrices to the global system of coordinates before their assemblage in the corresponding matrices for the structure. As has been indicated, the z axis for the local coordinate system coincides with the Z axis for the global system. Therefore, the only step left to perform is a rotation of the coordinates in the x-y plane. The corresponding matrix for this transformation may be obtained by establishing the relationship between components of the moment at the nodes expressed in these two systems of coordinates. In reference to Fig. 12.3, these relations when written for node ① are

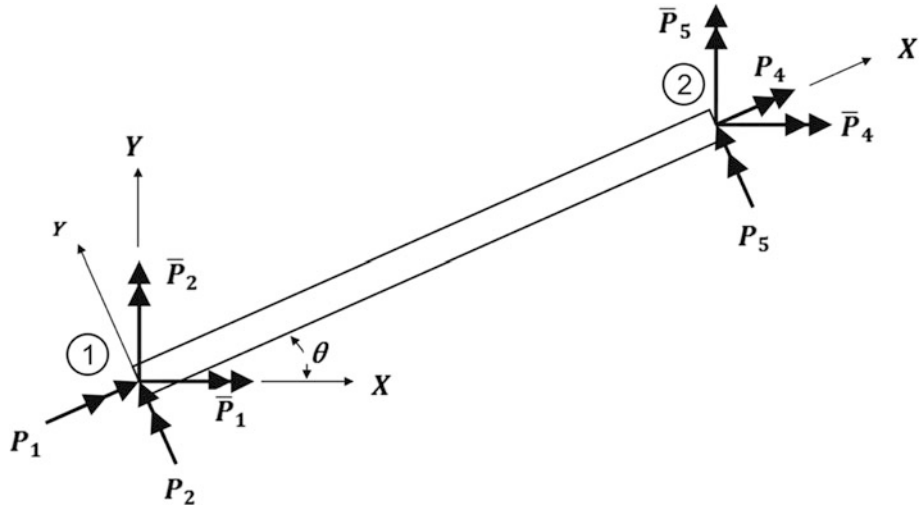

Fig. 12.3 Components of the nodal moments in local and global coordinates

$$\begin{aligned} P_1 &= \bar{P}_1 \cos\theta + \bar{P}_2 \sin\theta \\ P_2 &= -\bar{P}_1 \sin\theta + \bar{P}_2 \sin\theta \\ P_3 &= \bar{P}_3 \end{aligned} \tag{12.16a}$$

and for node ②

$$\begin{aligned} P_4 &= \bar{P}_4 \cos\theta + \bar{P}_5 \sin\theta \\ P_5 &= -\bar{P}_4 \sin\theta + \bar{P}_5 \sin\theta \\ P_6 &= \bar{P}_6 \end{aligned} \tag{12.16b}$$

The identical form of these equations with those derived for the transformation of coordinates for nodal forces of an element of a plane frame, Eqs. (11.28) and (11.30), should be noted. Equations (12.16) may be written in matrix notation as

$$\begin{Bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_6 \end{Bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos\theta & \sin\theta & 0 \\ 0 & 0 & 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \bar{P}_1 \\ \bar{P}_2 \\ \bar{P}_3 \\ \bar{P}_4 \\ \bar{P}_5 \\ \bar{P}_6 \end{Bmatrix} \tag{12.17}$$

or in short notation

$$\{P\} = [T]\{\bar{P}\} \tag{12.18}$$

in which $\{P\}$ and $\{\bar{P}\}$ are, respectively, the vectors of the nodal forces of a typical grid member in local and global coordinates and $[T]$ the transformation matrix. The same transformation matrix $[T]$

serves also to transform the nodal components of the displacements from a global to a local system of coordinates. In condensed notation, this relation is given by

$$\{\delta\} = [T]\{\bar{\delta}\} \tag{12.19}$$

where $\{\delta\}$ and $\{\bar{\delta}\}$ are, respectively, the components of nodal displacements in local and global coordinates. The substitution of Eqs. (12.18) and (12.19) in the stiffness relation Eq. (12.10) yields the element stiffness matrix in reference to the global coordinate system, that is,

$$[T]\{\bar{P}\} = [K][T]\{\bar{\delta}\}$$

or, since $[T]$ is an orthogonal matrix $[(T)^{-1} = (T)^T]$, it follows that

$$\{\bar{P}\} = [T]^T[K][T]\{\bar{\delta}\}$$

or

$$\{\bar{P}\} = [\bar{K}]\{\bar{\delta}\} \tag{12.20}$$

in which

$$\{\bar{K}\} = [T]^T[K][T] \tag{12.21}$$

is the stiffness matrix of an element of a grid frame in reference to the global system of coordinates.

Analogously, for the mass matrix, we find

$$\{\bar{P}\} = [\bar{M}]\{\ddot{\bar{\delta}}\} \tag{12.22}$$

in which

$$\{\bar{M}\} = [T]^T[M][T] \tag{12.23}$$

is the transformed mass matrix.

Illustrative Example 12.1

Figure 12.4 shows a grid frame in a horizontal plane consisting of two prismatic beam elements with a total of three degrees of freedom as indicated. Determine the natural frequencies and corresponding mode shapes. Use the consistent mass formulation.

Solution:

The stiffness matrix for elements 1 or 2 of the grid frame in reference to the local system of coordinates, by Eq. (12.9), is

$$[K_1] = 10^6 \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \\ \begin{bmatrix} 40 & 0 & 0 & -40 & 0 & 0 \\ 0 & 200 & -5 & 0 & 100 & 5 \\ 0 & -5 & 0.167 & 0 & -5 & -0.167 \\ -40 & 0 & 0 & 40 & 0 & 0 \\ 0 & 100 & -5 & 0 & 200 & 5 \\ 0 & 5 & -0.167 & 0 & 5 & 0.167 \end{bmatrix} \end{array} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{matrix}$$

The transformation matrix for element ⚠ with $\theta = 0^0$ is simply the unit matrix $[T_1] = [I]$. Hence

$$\left[\bar{K}_1\right] = [T_1]^T[K_1][T_1] = [K_1]$$

and for element (2) with $\theta = 90°$.

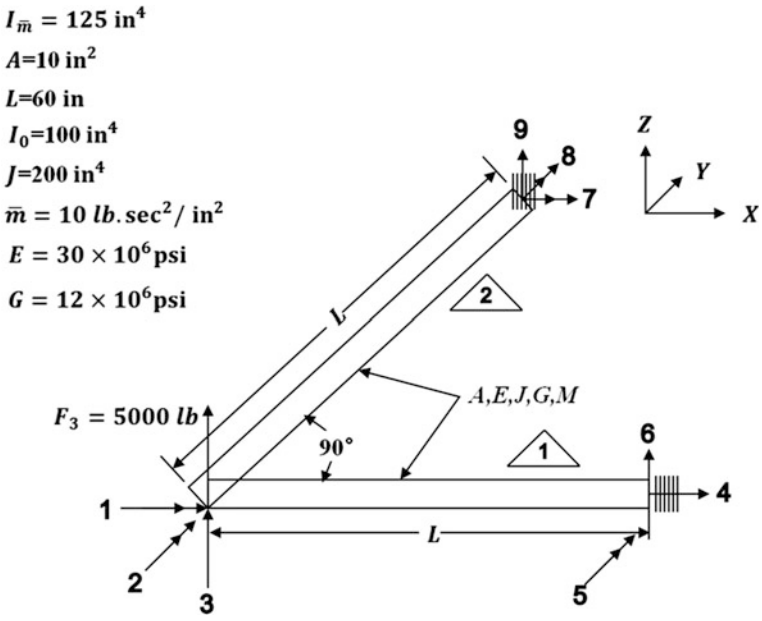

Fig. 12.4 Grid frame of Illustrative Example 12.1

$$[T_2] = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

so that

$$\left[\bar{K}_2\right] = [T_2]^T[K_2][T_2]$$

$$[\bar{K}_2] = 10^6 \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \\ \begin{bmatrix} 200 & 0 & 5 & 100 & 0 & 5 \\ 0 & 40 & 0 & 0 & -40 & 0 \\ 5 & 0 & 0.167 & -5 & 0 & 0.167 \\ 100 & 0 & 5 & 200 & 0 & -5 \\ 0 & -40 & 0 & 0 & 40 & 0 \\ -5 & 0 & -0.167 & -5 & 0 & 0.167 \end{bmatrix} \end{array} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{matrix}$$

The system matrix $[K_s]$ assembled from $\left[\bar{K}_1\right]$ and $\left[\bar{K}_2\right]$ is

$$[K_s] = 10^6 \begin{bmatrix} 240 & 0 & 5 \\ 0 & 240 & -5 \\ 5 & -5 & 0.333 \end{bmatrix}$$

Analogously, for the mass, we have from Eq. (12.11)

$$[M_1] = \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \\ \begin{bmatrix} 2500 & 0 & 0 & 1250 & 0 & 0 \\ 0 & 20,570 & 1886 & 0 & -15,430 & 1114 \\ 0 & 1886 & 223 & 0 & -1114 & 77 \\ 1250 & 0 & 0 & 2500 & 0 & 0 \\ 0 & -15,430 & -1114 & 0 & 20,570 & -1886 \\ 0 & 1114 & 77 & 0 & -1886 & 223 \end{bmatrix} \end{array} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{matrix}$$

We then calculate using Eq. (12.23)

$$\left[\bar{M}_1\right] = [M_1]$$

since

$$[T_1] = [I]$$

and analogously

$$[\bar{M}_2] = \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \\ \begin{bmatrix} 20,570 & 0 & -1886 & 15,430 & 0 & 1114 \\ 0 & 2500 & 0 & 0 & 1250 & 0 \\ -1886 & 0 & 223 & 1114 & 0 & 77 \\ 15,430 & 0 & 1114 & 20,570 & 0 & 1886 \\ 0 & 1250 & 0 & 0 & 2500 & 0 \\ 1114 & 0 & 77 & 1886 & 0 & 223 \end{bmatrix} \end{array} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{matrix}$$

From $[\bar{M}_1]$ and $[\bar{M}_2]$ we assemble the system mass matrix and obtain

$$[M_s] = \begin{bmatrix} 23,070 & 0 & 1886 \\ 0 & 23,070 & -1886 \\ 1886 & -1886 & 446 \end{bmatrix}$$

The natural frequencies and mode shapes are obtained from the solution of the eigenproblem

$$\left([K_s] - \omega^2[M_s]\right)\{a\} = \{0\}$$

which gives the eigenvalues (squares of the natural frequencies)

$$\omega_1^2 = 198 \qquad \omega_2^2 = 10,402, \quad \text{and} \qquad \omega_3^2 = 47,849$$

Then

$$\omega_1 = 14.06 \text{ rad/ sec}, \qquad \omega_2 = 101.99 \text{ rad/ sec}, \quad \text{and} \qquad \omega_3 = 218.74 \text{ rad/ sec}$$

and the eigenvectors ordered in the column of the modal matrix:

$$[\text{a}] = \begin{bmatrix} -1.000 & 1.000 & -1.000 \\ 1.000 & 1.000 & 1.000 \\ 43.82 & 0 & 9.072 \end{bmatrix}$$

The eigenvectors are conveniently normalized by dividing the columns of the modal matrix, respectively, by the factors

$$\sqrt{\{a_1\}^T [M_s]\{a_1\}} = 1,110$$
$$\sqrt{\{a_2\}^T [M_s]\{a_2\}} = 214.81$$
$$\sqrt{\{a_3\}^T [M_s]\{a_{31}\}} = 119.99$$

The normalized eigenvectors are arranged in columns of the modal matrix, so that

$$[\Phi] = \begin{bmatrix} -0.0009 & 0.0047 & 0.0083 \\ 0.0009 & 0.0047 & -0.0083 \\ 0.0395 & 0 & 0.0756 \end{bmatrix}$$

Illustrative Example 12.2
Determine the response of the grid frame shown in Fig. 12.4 when subjected to a suddenly applied force $F_3 = 5000$ lb. as shown in the figure.

Solution:
The natural frequencies and modal shapes for this structure were calculated in Example 12.1. The modal equation is given in general as

$$\ddot{z}_n + \omega_n^2 z_n = P_n \qquad (n = 1, 2, 3)$$

where

$$P_n = \sum_{i=1}^{3} \phi_{in} F_i$$

and F_i the external forces at the nodal coordinates which for this example are $F_1 = F_2 = 0$ and $F_3 = 5000$ lb. Hence, we obtain

$$\ddot{z}_1 + 198 z_1 = 197.35$$
$$\ddot{z}_2 + 10,402 z_2 = 0$$
$$\ddot{z}_3 + 47,849 z_3 = 378.05$$

The solution of these equations for zero initial conditions is

$$z_1 = \frac{197.35}{198}(1 - \cos\ 14.06t)$$
$$z_2 = 0$$
$$z_3 = \frac{378.05}{47,849}(1 - \cos\ 218.74t)$$

The displacements at the nodal coordinates are calculated from

$$\{u\} = [\Phi]\{z\}$$
$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{bmatrix} -0.0009 & 0.0047 & 0.0083 \\ 0.0009 & 0.0047 & -0.0083 \\ 0.0395 & 0 & 0.0756 \end{bmatrix} \begin{bmatrix} 0.9967 & (1-\cos 14.06t) \\ & 0 \\ 0.0079 & (1-\cos 218.74t) \end{bmatrix}$$

and finally

$$u_1 = 10^{-3}(-0.8315 + 0.897\cos\ 14.06t - 0.0656\cos\ 218.74t)\ \text{in}$$
$$u_2 = 10^{-3}(0.9626 - 0.897\cos\ 14.06t - 0.0656\cos\ 218.74t)\ \text{in}$$
$$u_3 = 10^{-3}(39.97 - 39.37\cos\ 14.06t - 0.60\cos\ 154.49t)\ \text{radian}$$

12.7 Modeling Structures as Grid Frames Using MATLAB

MATLAB calculates the stiffness and mass matrices for a grid frame and stores the coefficients of these matrices in a file. After calculating system matrices, the natural frequencies and mode shapes can be estimated. After solving modal equations, the MATLAB can plot the responses.

Illustrative Example 12.3
For the grid frame shown in Fig. 12.4 and analyzed in the previous examples, (a) model this structure, (b) calculate the natural frequencies and mode shapes, and (c) determine the response to a constant force of 5000 lb. suddenly applied for 0.1 s as indicated in the figure.

Solution:
This MATLAB file is to yield the results of sections (a) and (b). Two function files, GridFrameElement.m and GridConMass.m are needed. After assembling matrices, the system matrix can be found using System.m file. Using system matrices, the natural frequencies and mode shapes will be found.

```
clc
close all
clear all

%_____________________________________________________________________
%  Determine System Matrices/Determine Force
%_____________________________________________________________________

%%%%-GIVEN VALUES-%%%%

E=30*10^6; A=10;            %E (psi), A (in.^2)
Iz = 100;                   %Second Moment of Inertia (in^4)
J = 200;                    %Torsional constant
G = 12*10^6;                %Modulus of rigidity (psi)
I0 = 125;                   %Polar moment of inertia of cross sectional area(in^4)


%%%Create frame model (ith row of nodes is ith node)
nodes = [0, 0; 60,0; 0,60];
%%%Element number (ith row = ith element with two nodes)
conn=[1,2; 1,3];
%%%Dofs for ith element (ith row)
lmm=[1:3,4:6; 1:3,7:9];


m_bar = 10;                 %Mass per unit length (lb-sec^2/in/in)

dof = 3*length(nodes);      % Total No. dofs

K= zeros(dof);
M= zeros(dof);

%%%Generate equations for each element and assemble them.
for i=1:2
lm=lmm(i,:);
con=conn(i,:);
ke = GridFrameElement(E, Iz, G, J, nodes(con,:));
    K(lm, lm) = K(lm, lm) + ke;
end

%%%Generate mass matrix for each element and assemble them.
for i=1:2
lm=lmm(i,:);
con=conn(i,:);
    m=GridConsMass(m_bar, I0, A, nodes(con,:));
    M(lm, lm) = M(lm, lm) + m;
end

K;

M;

%%%Define the load vector size
F = zeros(dof,1); F(3) = 5000;              %Applied force at specific dofs

%%%System Matrices
[Kf, Mf, Rf] = System(K, M, F, [4:6,7:9]);

Kf

Mf

Rf
%_____________________________________________________________________
% Solve the eigenvalue problem and normalized eigenvectors
%_____________________________________________________________________

%%%Solve for eigenvalues (D) and eigenvectors (a)
[a, D] = eig(Kf, Mf);

[omegas,ii] = sort(sqrt(diag(D))); %Natural Frequencies

omegas

a = a(:,ii)                         %Mode Shapes

T = 2*pi./omegas;                   %Natural Periods

save ('temp0.mat', 'Mf', 'Kf' ,'Rf');
```

The function file of MATLAB is used to assemble the stiffness matrix of grid frame element for global stiffness matrix.

```
functionke = GridFrameElement(E, Iz, G, J, coord)
% ke = GridFrameElement(e, Iz, J, coord)
% Generates equations for a space frame element
% E = modulus of elasticity
% G = Modulus of rigidity (psi)
% Iz = moment of inertia about element z axe
% J = torsional constant
% coord = coordinates at the element ends

EI=E*Iz;  GJ=G*J;
x1=coord(1,1); y1=coord(1,2);
x2=coord(2,1); y2=coord(2,2);
L=sqrt((x2-x1)^2+(y2-y1)^2);
C=(x2-x1)/L; S=(y2-y1)/L;

T = [C S 0 0 0 0;
    -S C 0 0 0 0;
       0 0 1 0 0 0;
     0 0 0 C S 0;
     0 0 0 -S C 0;
     0 0 0 0 0 1];

ke = EI/L^3*T'*[GJ*L^2/EI, 0, 0, -GJ*L^2/EI, 0, 0;
         0, 4*L^2, -6*L, 0, 2*L^2, 6*L;
         0, -6*L, 12, 0, -6*L, -12;
         -GJ*L^2/EI, 0, 0, GJ*L^2/EI, 0, 0;
         0, 2*L^2, -6*L, 0, 4*L^2, 6*L;
         0, 6*L, -12, 0, 6*L, 12]*T;
```

The function file of MATLAB is used to assemble the mass matrix of each grid frame element for global mass matrix.

```
function m = GridConsMass(m_bar,I0, A, coord)
% FrameConsMass(m_bar, nodes(con,:))
% Generates mass matrix for a grid frame element
% m = mass (lb.sec^2/in.^2)
% L = length
% A = area of cross-section
% I0 = polar moment of inertia of cross sectional area(in^4)
% coord = coordinates at the element ends

x1=coord(1,1); y1=coord(1,2);
x2=coord(2,1); y2=coord(2,2);
L=sqrt((x2-x1)^2+(y2-y1)^2);
C=(x2-x1)/L; S=(y2-y1)/L;


T = [C S 0 0 0 0;
    -S C 0 0 0 0;
       0 0 1 0 0 0;
     0 0 0 C S 0;
     0 0 0 -S C 0;
     0 0 0 0 0 1];

m = m_bar*L/420*T.'*[140*I0/A  0         0     70*I0/A     0           0  ;
                     0       4*L^2       22*L    0       -3*L^2      13*L;
                     0       22*L        156     0       -13*L       54;
                     70*I0/A     0       0       140*I0/A     0      0;
                     0         -3*L^2 -13*L        0       4*L^2     -22*L;
                     0       13*L        54        0         -22*L   156]*T;
```

This MATLAB file is to yield the results of section (c).

After running the program above, the following MATLAB program will yield the response. The duration of response is from 0 to 5 s with the interval of 0.01 s (Fig. 12.5).

```
function  q = MDOFP(t, q)
load ('temp2.mat', 'omega', 'P', 'm' ,'zeta')

if t > 0.1
     P=0;
else
     P=P;
end

q = [q(2); -omega*omega*q(1)-2*zeta*omega*q(2)+P];
```

The maximum displacements at the nodal coordinates were estimated using MATLAB.

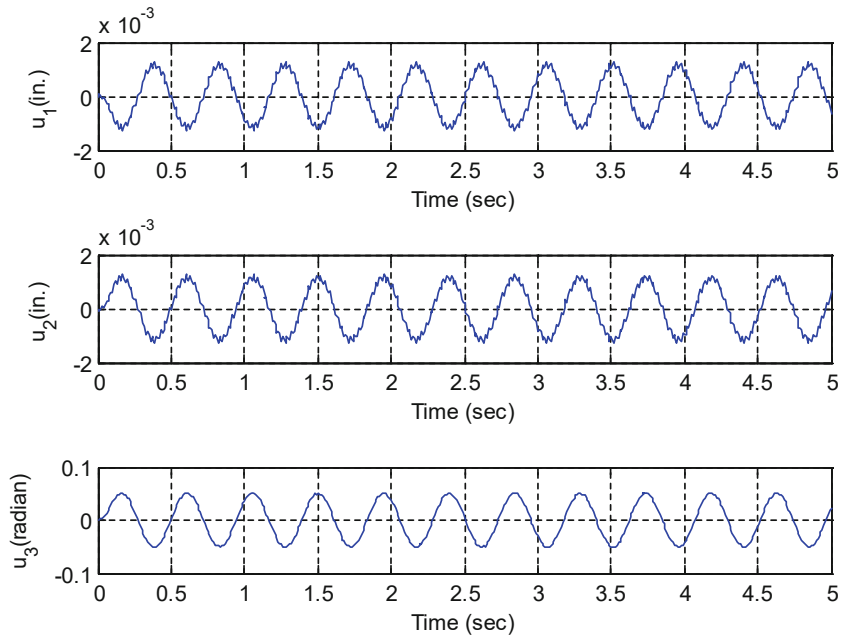

Fig. 12.5 Response of Illustrative Example 12.3

$$u_{1\max} = 0.0013 \text{ in} \quad u_{2\max} = 0.0013 \text{ in} \quad u_{3\max} = 0.0517 \text{ radian}$$

12.8 Summary

This chapter has presented the dynamic analysis of structures modeled as grid supporting loads applied normally to its plane. The dynamic analysis of grids requires the inclusion of torsional effects in the element stiffness and mass matrices. It also requires a transformation of coordinates of the element matrices previous to the assembling of the system matrix. The required matrices for torsional

effects are developed and a computer program for the dynamic analysis of grids is presented. This program is also organized along the same pattern of the programs in the two preceding chapters for the dynamic analysis of beams and plane frames.

12.9 Problems

The following problems are intended for hand calculation, though it is recommended that whenever possible solutions should also be obtained using MATLAB to model the structure and calculate natural frequencies and to solve for the response.

Problem 12.1

Use MATLAB: (1) Model the plane grid frame shown in Fig. P12.1 using a total of four beam elements, (2) Determine the first three natural frequencies and corresponding mode shapes, and (3) Calculate the response due to the force $F(t) = 5000$ lb. applied in the Z direction at joint 1 suddenly for 0.1 sec and then decreasing linearly to zero at time 0.5 s.

Problem 12.2

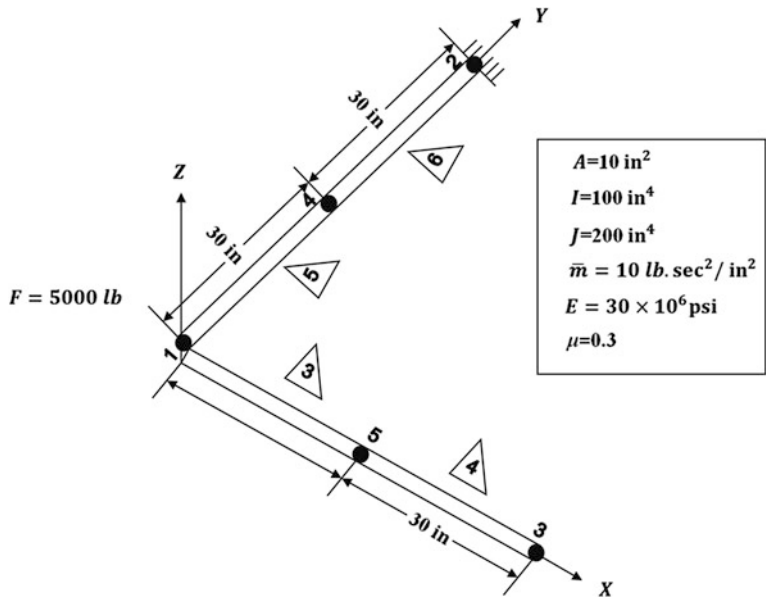

Fig. P12.1

For the grid shown in Fig. P12.2 determine the system stiffness and mass matrices. Base the analysis on the three nodal coordinates indicated in the figure. Use consistent mass method.

Problem 12.3

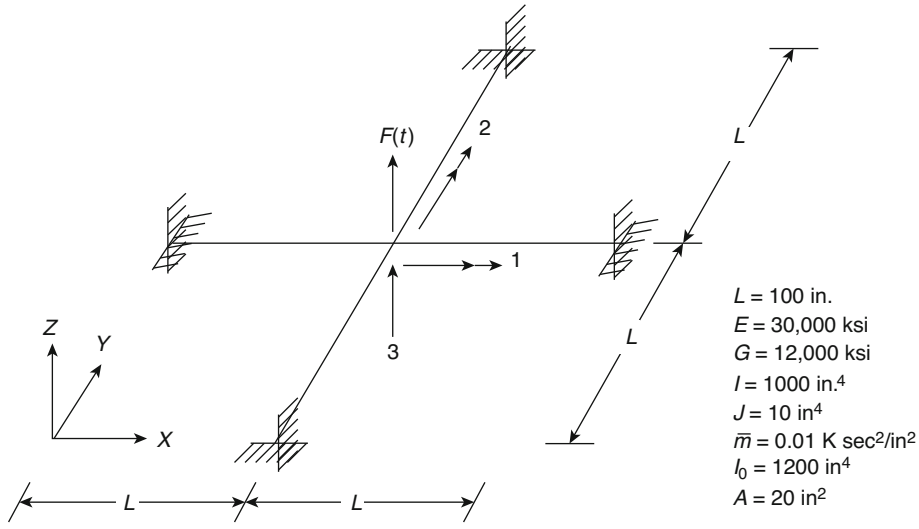

Fig. P12.2

Use static condensation to eliminate the rotational degrees of freedom and determine the transformation matrix and the reduced stiffness and mass matrices in Problem 12.2.

Problem 12.4
Determine the natural frequency for the reduced system in Problem 12.3.

Problem 12.5
Determine the natural frequencies and corresponding normal modes for the grid analyzed in Problem 12.2.

Problem 12.6
Determine the response of the grid shown in Fig. P12.1 when acted upon by a force $F(t) = 10$ Kip suddenly applied for 1 s at the nodal coordinate 3 as shown in the figure. Use results of Problem 12.2 to obtain the equation of motion for the condensed system. Assume 10% modal damping.

Problem 12.7
Use results from Problem 12.4 to solve Problem 12.5 on the basis of the three nodal coordinates as indicated in Fig. P12.2.

Problem 12.8
Determine the steady-state response of the grid shown in Fig. P12.2 when subjected to harmonic force $F(t) = 10 \sin 50\, t$ (Kip) along nodal coordinate 3. Neglect damping in the system.

Problem 12.9
Repeat Problem 12.8 assuming that the damping is proportional to the stiffness of the system, $[C] = a_0\,[K]$, where $a_0 = 0.3$.

Problem 12.10
Determine the equivalent nodal forces for a member of a grid loaded with a dynamic force, $P(t) = P_0 f(t)$, uniformly distributed along its length.

Problem 12.11
Determine the equivalent nodal forces for a member of a grid supporting a concentrated dynamic force $F(f)$ as shown in Fig. P12.11.

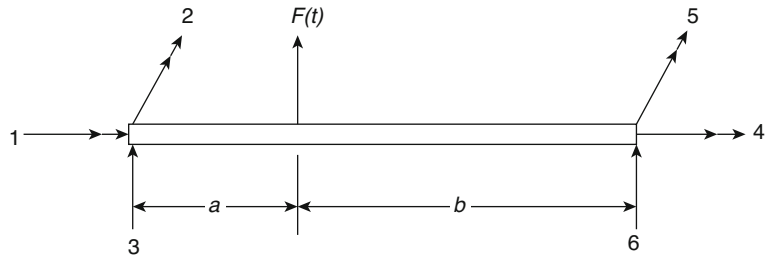

Fig. P12.11

The following problems are intended for computer solution using MATLAB, to model the structure and to determine natural frequencies and nodal shapes and to calculate the response.

Problem 12.12
Determine the natural frequencies and corresponding normal modes for the grid shown in Fig. P12.2.

Problem 12.13
Determine the response of the grid shown in Fig. P12.2 when acted upon by the force depicted in Fig. P12.2 acting along nodal coordinate 3. Neglect damping in the system.

Problem 12.14
Repeat Problem 12.13 for 15% damping in all the modes. Use modal super-position method.

Problem 12.15
Repeat Problem 12.13. Use step-by-step linear acceleration method. Neglect damping Fig. P12.15.

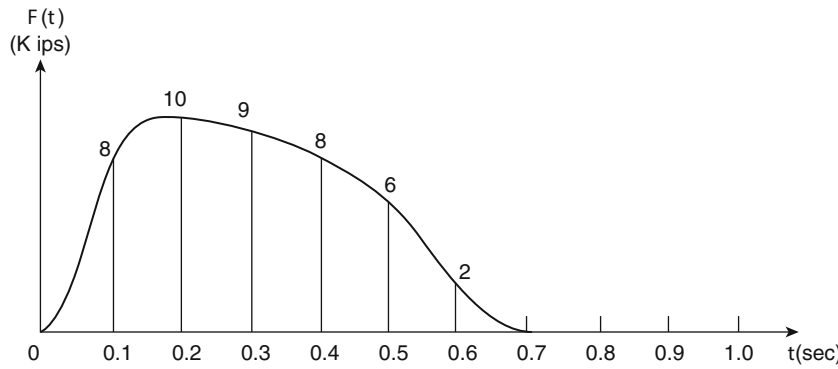

Fig. P12.15

13 Dynamic Analysis of Three-Dimensional Frames

The stiffness method for dynamic analysis of frames presented in Chap. 11 for plane frames and in Chap. 12 for grid frames can readily be expanded for the analysis of three-dimensional space frames. Although for the plane frame or for the grid there were only three nodal coordinates at each joint, the three-dimensional frame has a total of six possible nodal displacements at each unconstrained joint: three translation components along the x, y, z axes and three rotational components about these axes. Consequently, a beam element of a three-dimensional frame or a space frame has for its two joints a total of 12 nodal coordinates; hence the resulting element matrices will be of dimension 12×12.

The dynamic analysis of three-dimensional frames resulting in a comparatively longer computer program in general, requiring more input data as well as substantially more computational time. However, except for size, the analysis of three-dimensional frames by the stiffness method of dynamic analysis is basically identical to the analysis of plane frames or grid frames.

13.1 Element Stiffness Matrix

Figure 13.1 shows a beam segment of a space frame with its 12 nodal coordinates numbered consecutively. The convention adopted is to label first the three translatory displacements of the first joint followed by the three rotational displacements of the same joint, then to continue with the three translatory displacements of the second joint and finally the three rotational displacements of this second joint. The double arrows used in Fig. 13.1 serve to indicate rotational nodal coordinates; thus, these are distinguished from translational nodal coordinates for which single arrows are used.

The stiffness matrix for a three-dimensional uniform beam segment is readily written by the superposition of the axial stiffness matrix from Eq. (11.3), the torsional stiffness matrix from Eq. (12.6), and the flexural stiffness matrix from Eq. (10.20). The flexural stiffness matrix is used twice in forming the stiffness matrix of a three-dimensional beam segment to account for the flexural effects in the two principal planes of the cross section. Proceeding to combine in an appropriate manner these matrices, we obtain in Eq. (13.1) the stiffness equation for a uniform beam segment of a three-dimensional frame, namely

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M. Paz, Y. H. Kim, *Structural Dynamics*, https://doi.org/10.1007/978-3-319-94743-3_13

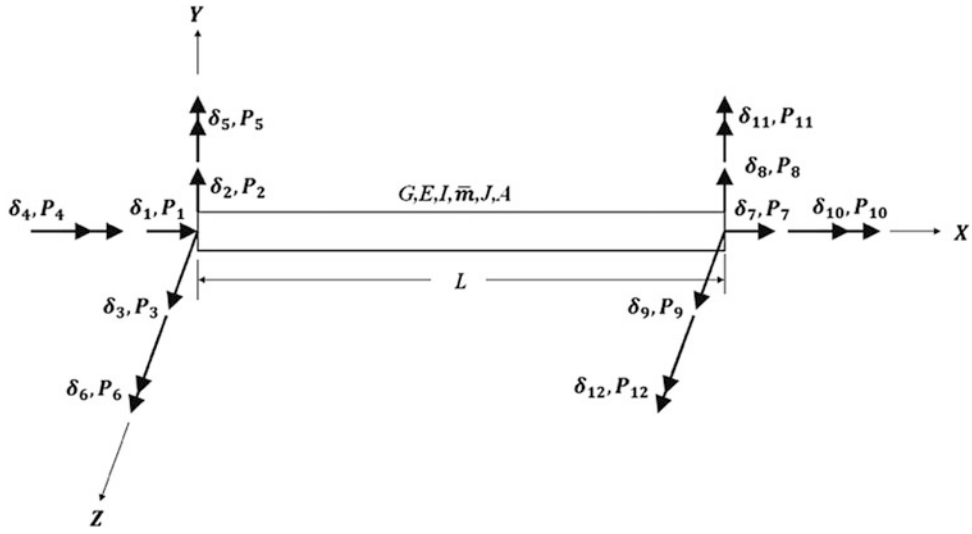

Fig. 13.1 Beam segment of a space frame showing forces and displacements at the nodal coordinates

$$
\begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_6 \\ P_7 \\ P_8 \\ P_9 \\ P_{10} \\ P_{11} \\ P_{12} \end{bmatrix} =
\begin{bmatrix}
\frac{EA}{L} & & & & & & & & & & & \\
0 & \frac{12EI_z}{L^3} & & & & & & & & & & \\
0 & 0 & \frac{12EI_y}{L^3} & & & & & \text{Symmertic} & & & & \\
0 & 0 & 0 & \frac{GI}{L} & & & & & & & & \\
0 & 0 & \frac{-6EI_y}{L^2} & 0 & \frac{4EI_y}{L} & & & & & & & \\
0 & \frac{6EI_z}{L^2} & 0 & 0 & 0 & \frac{4EI_y}{L} & & & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{EA}{L} & & & & & \\
\frac{-EA}{L} & \frac{-12EI_z}{L^3} & 0 & 0 & 0 & \frac{-6EI_z}{L^2} & 0 & \frac{12EI_z}{L^3} & & & & \\
0 & 0 & \frac{-12EI_y}{L^3} & 0 & \frac{6EI_y}{L^2} & 0 & 0 & 0 & \frac{12EI_y}{L^3} & & & \\
0 & 0 & 0 & \frac{-GI}{L} & 0 & 0 & 0 & 0 & 0 & \frac{GI}{L} & & \\
0 & 0 & \frac{-6EI_y}{L^2} & 0 & \frac{2EI_y}{L} & 0 & 0 & 0 & \frac{6EI_y}{L^2} & 0 & \frac{4EI_y}{L} & \\
0 & \frac{6EI_z}{L^2} & 0 & 0 & 0 & \frac{2EI_z}{L} & 0 & \frac{-6EI_z}{L^2} & 0 & 0 & 0 & \frac{4EI_z}{L}
\end{bmatrix}
\begin{bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \\ \delta_5 \\ \delta_6 \\ \delta_7 \\ \delta_8 \\ \delta_9 \\ \delta_{10} \\ \delta_{11} \\ \delta_{12} \end{bmatrix} \tag{13.1}
$$

or in condensed notation

$$\{P\} = [K] = \{\delta\} \tag{13.2}$$

in which I_y and I_z are, respectively, the cross-sectional moments of inertia with respect to the principal axes labeled as y and z in Fig. 13.1, and L, A, and J are respectively the length, cross-sectional area, and torsional constant of the beam element.

13.2 Element Mass Matrix

The lumped mass matrix for the uniform beam segment of a three-dimensional frame is simply a diagonal matrix in which the coefficients corresponding to translatory and torsional displacements are equal to one-half of the total inertia of the beam segment while the coefficients corresponding to flexural rotations are assumed to be zero. The diagonal lumped mass matrix for the uniform beam of distributed mass $\bar{m}$ and polar mass moment $I_{\bar{m}} = \bar{m}I_o/A$ of inertia per unit of length may be written conveniently as

$$[M_L] = \frac{\bar{m}L}{2}\lceil 1 \;\; 1 \;\; 1 \;\; I_o/A \;\; 0 \;\; 0 \;\; 1 \;\; 1 \;\; 1 \;\; I_o/A \;\; 0 \;\; 0\rceil \tag{13.3}$$

in which I_o is the polar moment of inertia of the cross-sectional area A.

The consistent mass matrix for a uniform beam segment of a three-dimensional frame is readily obtained combining the consistent mass matrices, Eq. (11.26) for axial effects, Eq. (12.8) for torsional effects, and Eq. (10.34) for flexural effects. The appropriate combination of these matrices results in the consistent mass matrix for the uniform beam segment of a three-dimensional frame, namely,

$$\begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_6 \\ P_7 \\ P_8 \\ P_9 \\ P_{10} \\ P_{11} \\ P_{12} \end{bmatrix} = \frac{\bar{m}L}{420} \begin{bmatrix} 140 & & & & & & & & & & & \\ 0 & 156 & & & & & & & & & & \\ 0 & 0 & 156 & & & & & & & & & \\ 0 & 0 & 0 & \frac{140I_o}{A} & & & & & & & & \\ 0 & 0 & -22L & 0 & 4L^2 & & & & & & & \\ 0 & 22L & 0 & 0 & 0 & 4L^2 & & & & & & \\ 70 & 0 & 0 & 0 & 0 & 0 & 140 & & & & & \\ 0 & 54 & 0 & 0 & 0 & 13L & 0 & 156 & & & & \\ 0 & 0 & 54 & 0 & -13L & 0 & 0 & 0 & 156 & & & \\ 0 & 0 & 0 & \frac{70I_o}{A} & 0 & 0 & 0 & 0 & 0 & \frac{140I_o}{A} & & \\ 0 & 0 & 13L & 0 & -3L^2 & 0 & 0 & 0 & 22L & 0 & 4L^2 & \\ 0 & -13L & 0 & 0 & 0 & -3L^2 & 0 & -22L & 0 & 0 & 0 & 4L^2 \end{bmatrix} \begin{bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \\ \delta_5 \\ \delta_6 \\ \delta_7 \\ \delta_8 \\ \delta_9 \\ \delta_{10} \\ \delta_{11} \\ \delta_{12} \end{bmatrix} \tag{13.4}$$

or in condensed notation

$$\{P\} = [M]\{\ddot{\delta}\} \tag{13.5}$$

13.3 Element Damping Matrix

The damping matrix for a uniform beam segment of a three-dimensional frame may be obtained in a manner entirely similar to those of the stiffness, Eq. (13.1), and mass, Eq. (13.4), matrices. Nevertheless, as was discussed in Sect. 10.5, in practice, damping is generally expressed in terms of damping ratios for each mode of vibration. Therefore, if the response is sought using the modal superposition

method, these damping ratios are directly introduced in the modal equations. When the damping matrix is required explicitly, it may be determined from given values of damping ratios by the methods presented in Chap. 22.

13.4 Transformation of Coordinates

The stiffness and the mass matrices, respectively, given by Eqs. (13.1) and (13.4), are referred to local coordinates axes fixed on the beam segment. Inasmuch as the elements of these matrices corresponding to the same nodal coordinates of the structure should be added to obtain the system stiffness and mass matrices, it is necessary first to transform these matrices to the same reference system, the global system of coordinates. Figure 13.2 shows these two reference systems, the x, y, z axes representing the local system of coordinates and the X, Y, Z axes representing the global system of coordinates. Also shown in this figure is a general vector A with its components X, Y Z along the global coordinates. This vector $\boldsymbol{A}$ may represent any force or displacement at the nodal coordinates of one of the joints of the structure. To obtain the components of vector $\boldsymbol{A}$ along one of the local axes x, y, z, it is necessary to add the projections along that axis of the components X, Y, Z. For example, the component x of vector $\boldsymbol{A}$ along the x coordinate is given by

$$x = X_{\cos}\, xX + Y \cos xY + Z \cos xZ \tag{13.6a}$$

in which $\cos xY$ is the cosine of the angle between axes x and Y and corresponding definitions for other cosines. Similarly, the y and z components of A are

$$y = X_{\cos}\, yX + Y \cos yY + Z \cos yZ \tag{13.6b}$$

$$z = X_{\cos}\, zX + Y \cos zY + Z \cos zZ \tag{13.6c}$$

Equations (13.6a, 13.6b, and 13.6c) are conveniently written in matrix notation as

$$\begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \begin{Bmatrix} \cos xX & \cos xY & \cos xZ \\ \cos yX & \cos yY & \cos yZ \\ \cos zX & \cos zY & \cos zZ \end{Bmatrix} \begin{Bmatrix} X \\ Y \\ Z \end{Bmatrix} \tag{13.7}$$

or in short notation

$$\{A\} = \{T_1\}\{\bar{A}\} \tag{13.8}$$

in which $\{A\}$and $\{\bar{A}\}$ are, respectively, the components in the local and global systems of the general vector **A** and $[T_1]$ the transformation matrix given by

$$[T_1] = \begin{bmatrix} \cos xX & \cos xY & \cos xZ \\ \cos yX & \cos yY & \cos yZ \\ \cos zX & \cos zY & \cos zZ \end{bmatrix} \tag{13.9}$$

The cosines required in the transformation matrix $[T_1]$ are usually calculated in computer codes from the global coordinates of three points. The two points defining the two ends of the beam element along the local x axis and any third point located in x-y local plane in which y is one of the principal axes of the cross-sectional area of the member. The input data containing the global coordinates of these three points are sufficient for the evaluation of all the cosine terms in Eq. (13.9). To demonstrate

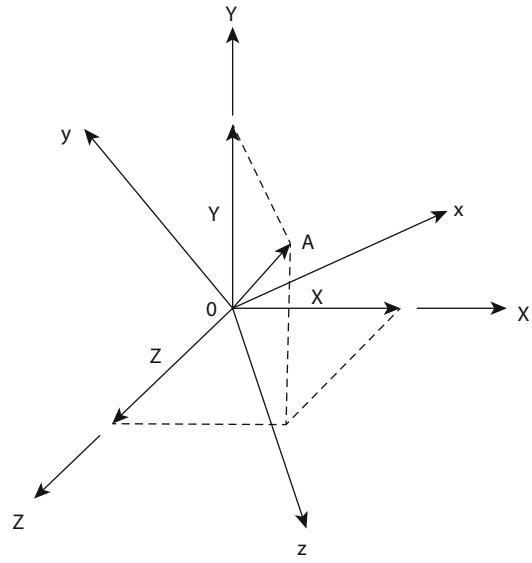

Fig. 13.2 Components of a general vector ***A*** in local and global coordinates

this fact let us designate by x_i, y_i, z_i, and x_j, y_j, z_j the coordinates of point ① and ② at the two ends of a beam element and by x_p, y_p, z_p, the coordinates of a point P placed on the local x-y plane. Then the direction cosines of local axis x along the beam element are given by

$$\cos xX = \frac{x_j - x_i}{L}, \quad \cos xY = \frac{y_j - y_i}{L}, \quad \cos xZ = \frac{z_j - z_i}{L} \tag{13.10}$$

where L is the length of the beam element given by

$$L = \sqrt{\left(x_j - x_i\right)^2 + \left(y_j - y_i\right)^2 + \left(z_j - z_i\right)^2} \tag{13.11}$$

The direction cosines of the z axis can be calculated from the condition that any vector $\mathbf{Z}$ along the z axis must be perpendicular to the plane formed by any two vectors in the local x-y plane. These two vectors could simply be the vector $\boldsymbol{X}$ from point ① to point ② along the x axis and the vector $\boldsymbol{P}$ from point ① to point $\boldsymbol{P}$. The orthogonality condition is then expressed by the cross product between vectors $\boldsymbol{X}$ and $\boldsymbol{P}$ as

$$\mathbf{Z} = \mathbf{X} \times \mathbf{P} \tag{13.12}$$

or substituting the components of these vectors as

$$z_x\widehat{i} + z_y\widehat{j} + z_z\widehat{k} = \begin{vmatrix} \widehat{i} & \widehat{j} & \widehat{k} \\ x_j - x_i & y_j - y_i & z_j - z_i \\ x_p - x_i & y_p - y_i & z_p - z_i \end{vmatrix} \tag{13.13}$$

where $\widehat{i}$, $\widehat{j}$, and $\widehat{k}$ are unit vectors along the global coordinate axes X, Y, and Z, respectively. Consequently the direction cosines of axis z are given by

$$\cos zX = \frac{z_x}{|Z|}, \quad \cos zY = \frac{z_y}{|Z|}, \quad \cos zZ = \frac{z_x}{|Z|} \tag{13.14}$$

where

$$\begin{aligned} z_x &= \left(y_j - y_i\right)\ \left(z_p - z_i\right) - \left(z_j - z_i\right)\ \left(y_p - y_i\right) \\ z_y &= \left(z_j - z_i\right)\ \left(x_p - x_i\right) - \left(x_j - x_i\right)\ \left(z_p - z_i\right) \\ z_z &= \left(x_j - x_i\right)\ \left(y_p - y_i\right) - \left(y_j - y_i\right)\ \left(x_p - x_i\right) \end{aligned} \tag{13.15}$$

and

$$|Z| = \sqrt{z_x^2 + z_y^2 + z_z^2} \tag{13.16}$$

Analogously, the direction cosines of the local axis y are calculated from the condition of orthogonality between a vector $\boldsymbol{Y}$ along the y axis and the unit vectors $\boldsymbol{X_1}$ and $\boldsymbol{Z_1}$ along the x and z aces, respectively. Hence,

$$\boldsymbol{Y} = \boldsymbol{X}_1 \times \boldsymbol{Z}_1$$

or in expanded notation

$$y_x\widehat{i} + y_y\widehat{j} + y_z\widehat{k} = \begin{vmatrix} \widehat{i} & \widehat{j} & \widehat{k} \\ \cos xX & \cos xY & \cos xZ \\ \cos zX & \cos zY & \cos zZ \end{vmatrix} \tag{13.17}$$

Therefore,

$$\cos yX = \frac{y_x}{|Y|}, \quad \cos yY = \frac{y_y}{|Y|}, \quad \cos yZ = \frac{y_z}{|Y|}$$

where

$$\begin{aligned} y_x &= \cos xY \cos zZ - \cos xZ \cos zY \\ y_y &= \cos zX \cos zX - \cos xX \cos zZ \\ y_z &= \cos xX \cos zY - \cos xY \cos zX \end{aligned} \tag{13.18}$$

and

$$|Y| = \sqrt{y_x^2 + y_y^2 + y_z^2}$$

We have, therefore, shown that knowledge of the coordinates of points at the two ends of an element of a point P on the local plane x-y suffices to calculate the direction cosines of the transformation matrix $[T_1]$ in Eq. (13.9). The choice of point P is generally governed by the geometry of the structure and the orientation of the principal directions of the cross section of the member. Quite often point P is selected as a known point in the structure, which is placed on the local axis y, although, as it has been shown, the point P could be any point in the plane formed by the local x-y axes.

Alternatively, the direction cosines in the transformation matrix, Eq. (13.9) may be calculated from the nodal coordinates (x_{i}, y_{i}, z_{i},) and (x_{j}, y_{j}, z_{j}) at the two ends of the beam element and the knowledge of an angle known as the angle of rolling. This angle provides the rational information of the principal axes of the cross-sectional area with respect to an orientation defined as the standard orientation of these axes. The analytical development to implement the calculation of the direction cosines using the angle of rolling is presented at the end of this chapter as Problems 13.1 and 13.2.

We have, therefore, shown that the knowledge of the coordinates at the two ends of an element, together with the knowledge of either a third point located in the x-y local plane in which y is one of

the principal axes of the cross-sectional area of the member, or alternatively, the knowledge of the angle of rolling, suffices to calculate the direction cosines of the transformation matrix $[T_1]$ in Eq. (13.9).

For the beam segment of a three-dimensional frame, the transformation of the nodal displacement vectors involve the transformation of linear and angular displacement vectors at each joint of the segment. Therefore, a beam element of a space frame requires, for the two joints, the transformation of a total of four displacement vectors. This transformation of the 12 nodal displacements $\{\bar{\delta}\}$ global coordinates to the displacement $\{\delta\}$ in local coordinates may be written in abbreviated form as

$$\{\delta\} = [T]\{\bar{\delta}\} \tag{13.19}$$

in which

$$[T] = \begin{bmatrix} [T_1] & & & \\ & [T_1] & & \\ & & [T_1] & \\ & & & [T_1] \end{bmatrix}$$

Analogously, the transformation from nodal forces $\{\bar{P}\}$ in global coordinates to nodal forces $\{P\}$ in local coordinates is given by

$$\{P\} = [T]\{\bar{P}\} \tag{13.20}$$

Finally, to obtain the stiffness matrix $[\bar{K}]$ and the mass matrix $\lfloor \bar{M} \rfloor$ in reference to the global system of coordinates, we simply substitute, into Eq. (13.2), {δ} from Eq. (13.19) and [P] from Eq. (13.21) to obtain

$$[T]\{\bar{P}\} = [K][T]\{\bar{\delta}\}$$

or

$$\{\bar{P}\} = [T]^T[K][T]\{\bar{\delta}\} \tag{13.21}$$

since $[T]$ is an orthogonal matrix. From Eq. (13.21), we may write

$$\{\bar{P}\} = [\bar{K}]\{\bar{\delta}\} \tag{13.22}$$

in which $\lfloor \bar{K} \rfloor$ is defined as

$$[\bar{K}] = [T]^T[K][T] \tag{13.23}$$

Analogously, the mass matrix in Eq. (13.5) is transformed from local to global coordinates by

$$[\bar{M}] = [T]^T[M][T] \tag{13.24}$$

and the damping matrix [C] by

$$[\bar{C}] = [T]^T[C][T] \tag{13.25}$$

13.5 Differential Equation of Motion

The direct method which was explained in detail in Chap. 10 may also be used to assemble the stiffness, mass, and damping matrices from the corresponding matrices for a three-dimensional beam segment, Eqs. (13.23), (13.24) and (13.25), which are referred to the global system of coordinates. The differential equations of motion referred to the global system of coordinates. The differential equations of motion which are obtained by establishing the dynamic equilibrium among the inertial, damping, and elastic forces with the external forces may be expressed in matrix notation as

$$[M]\{\ddot{u}\} + [C]\{\dot{u}\} + [K]\{u\} = \{F(t)\} \tag{13.26}$$

in which $[M]$, $[C]$, and $[K]$ are, respectively, the system mass, damping, and stiffness matrices, $\{\ddot{u}\}$, $\{\dot{u}\}$, and $\{u\}$ are the system acceleration, velocity, and displacement vectors, and $\{F(t)\}$ is the force vector which includes the forces applied directly to the joints of the structure and the equivalent nodal forces for the forces not applied at the joints.

13.6 Dynamic Response

The integration of the differential equations of motion, Eq. (13.26), may be accomplished by any of the methods presented in previous chapters to obtain the response of structures modeled as beams, plane frames, or grids. The selection of the particular method of solution depends, as discussed previously, on the linearity of the differential equation, that is, whether the stiffness matrix $[K]$ or any other coefficient matrix is constant, and also depends on the complexity of the excitation as a function of time. When the differential equations of motion, Eq. (13.26), are linear, the modal superposition method is applicable. This method, as we have seen in the preceding chapters, requires the solution of an eigenproblem to uncouple the differential equations resulting in the modal equations of motion.

If the structure is assumed to follow an elastoplastic behavior or any other form of nonlinearity, it is necessary to resort to some kind of numerical integration in order to solve the differential equations of motion, Eq. (13.26). In Chap. 16, the linear acceleration method with a modification known as the Wilson-θ method is presented for analysis of linear structures with an elastic behavior.

13.7 Modeling Structures as Space Frames Using MATLAB

MATLAB calculates the stiffness and mass matrices for a three-dimensional frame and stores the coefficients of these matrices.

Illustrative Example 13.1
For the three-dimensional frame shown in Fig. 13.3, determine (a) the stiffness and mass matrices, (b) the natural frequencies and corresponding modal shapes. Model the structure with four beam elements and use consistent mass matrix formulation, and (c) determine the response to a constant force of 5000 lb. suddenly applied for 0.1 s at node 1 in z axis.

Solution:

As the first step in the analysis, the frame is divided in four elements, as indicated in Fig. 13.3. This division of the structure results in five nodes with a total of 30 nodal coordinates, of which 24 are fixed. The numerical values needed for analysis of this structure are given in Table 13.1.

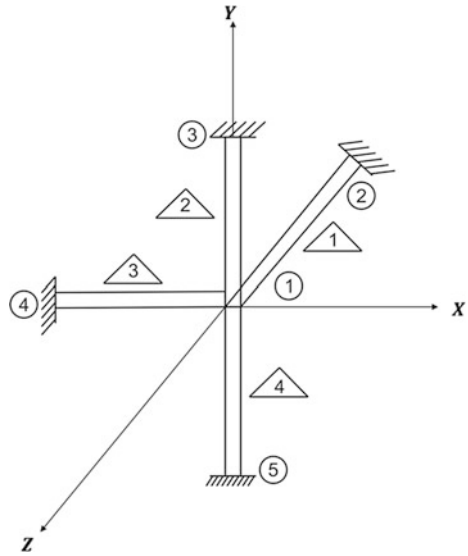

Fig. 13.3 Space Frame of Example 13.1

Table 13.1 Input Data for Illustrative Example 13.1

Quantity	Members 1,3	Members2,4
Modulus of elasticity (psi)	30×10^6	30×10^6
Modulus of rigidity (psi)	12×10^6	12×10^6
Distributed mass (lb · sec^2 /in^2)	0.2	0.1
Cross-sectional y moment of inertia (in^4)	200	64
Cross-sectional z moment of inertia (in ^{4})	200	64
Torsional constant (in^4)	40.0	12.8
Cross-sectional area (in^2)	50	28
Cross-sectional moment of inertia (in^4)	205	68

This MATLAB file is to yield the results of sections (a) and (b). Two function files, SpaceFrameElement.m and SpaceConMass.m are needed. After assembling matrices, the system matrix can be found using System.m file. Using system matrices, the natural frequencies and mode shapes will be found.

```
clc
clear all
close all

%_____________________________________________________________________
%  Determine System Matrices/Determine Force
%_____________________________________________________________________

%%%%-GIVEN VALUES-%%%%

E=30*10^6;                  %E (psi)
G = 12*10^6;                %G, Modulus of rigidity (psi)


% Members 1 & 3
A1 = 50;                    %Cross-sectional area A (in^2)
Iz1 = 200;                  %Second Moment of Inertia in z axes (in^4)
Iy1 = 200;                  %Second Moment of Inertia in y axes (in^4)
J1 = 40;                    %Torsional constant
m_bar1 = 0.2;               %Distributed mass (lb-sec^2/in/in)
I0_1 = 205;                 %Polar moment of inertia of cross sectional area (in^4)


% Members 2 & 4
A2 = 28;                    %Cross-sectional area A (in^2)
Iz2 = 64;                   %Second Moment of Inertia in z axes (in^4)
Iy2 = 64;                   %Second Moment of Inertia in y axes (in^4)
J2 = 12.8;                  %Torsional constant
m_bar2 = 0.1;               %Distribution mass (lb-sec^2/in^2)
I0_2 =  68;                 %Polar moment of inertia of cross sectional area (in^4)

%%%Create frame model (ith row of nodes is ith node)
nodes = [0, 0, 0; 0, 0, -200; 0, 200, 0; -200, 0, 0; 0, -200, 0];
%%%Element number (ith row = ith element with two nodes)
conn=[1,2,3; 1,3,2; 1,4,2; 1,5,2];
%%%Dofs for ith element (ith row)
lmm=[1:12; [1:6 13:18]; [1:6 19:24]; [1:6 25:30]];
dof=6*length(nodes);        % Total No. dofs

K= zeros(dof);
M= zeros(dof);

%%%Generate equations for each element and assemble them.
for i=1
    lm=lmm(i,:);
    con=conn(i,:);
    ke = SpaceFrameElement(E, G, Iz1, Iy1, J1, A1, nodes(con,:));
    K(lm, lm) = K(lm, lm) + ke;
end

for i=2
    lm=lmm(i,:);
    con=conn(i,:);
    ke = SpaceFrameElement(E, G, Iz2, Iy2, J2, A2, nodes(con,:));
    K(lm, lm) = K(lm, lm) + ke;
end

for i=3
    lm=lmm(i,:);
    con=conn(i,:);
    ke = SpaceFrameElement(E, G, Iz1, Iy1, J1, A1, nodes(con,:));
    K(lm, lm) = K(lm, lm) + ke;
end

for i=4
    lm=lmm(i,:);
    con=conn(i,:);
    ke = SpaceFrameElement(E, G, Iz2, Iy2, J2, A2, nodes(con,:));
    K(lm, lm) = K(lm, lm) + ke
end
```

```
K;
M;
%%%Define the load vector
F = zeros(dof,1); F(3) = 5000;      %Applied force at specific dofs
%%%System Matrices
[Kf, Mf, Rf] = System(K, M, F,[7:30]);
Kf
Mf
Rf
%_____________________________________________________________
% Solve the eigenvalue problem and normalized eigenvectors
%_____________________________________________________________
%%%Solve for eigenvalues (D) and eigenvectors (a)
[a, D] = eig(Kf, Mf);
[omegas,ii] = sort(sqrt(diag(D))); %Natural Frequencies
omegas
a = a(:,ii)                        %Mode Shapes
T = 2*pi./omegas;                  %Natural Periods
save ('temp0.mat', 'Mf', 'Kf' ,'Rf');
```

The function file of MATLAB is used to assemble the stiffness matrix of space frame element for global stiffness matrix.

```
function ke = SpaceFrameElement(E, G, Iz, Iy, J, A, coord)
% ke = SpaceFrameElement(E, G, Iz, Iy, J, A, wz, wy, coord)
% Generates equations for a space frame element
% E = modulus of elasticity (psi)
% G = Modulus of rigidity (psi)
% Iz, Iy = moment of inertias about element z and y axes (in^4)
% J = torsional rigity (in^4)
% A = area of cross-section (in^2)
% coord = coordinates at the element ends

EIz=E*Iz; EIy=E*Iy; GJ=G*J; EA = E*A;
n1=coord(1,1:3); n2=coord(2,1:3); n3=coord(3,1:3);
L=sqrt(dot((n2-n1),(n2-n1)));
ex = (n2 - n1)/L;
eyy = cross(n3 - n1, n2 - n1);
ey = eyy/sqrt(dot(eyy,eyy));
ez = cross(ex, ey);
H = [ex; ey; ez];
T = zeros(12);
T([1, 2, 3], [1, 2, 3]) = H;
T([4,5,6], [4,5,6]) = H;
T([7,8,9], [7,8,9]) = H;
T([10,11,12], [10,11,12]) = H;
TT = T.';
ke = [EA/L, 0, 0, 0, 0, 0, -(EA/L), 0, 0, 0, 0, 0;
    0, (12*EIz)/L^3, 0, 0, 0, (6*EIz)/L^2, 0, -((12*EIz)/L^3), ...
        0,0, 0, (6*EIz)/L^2;
    0, 0, (12*EIy)/L^3, 0, -((6*EIy)/L^2), 0, ...
        0, 0, -((12*EIy)/L^3), 0, -((6*EIy)/L^2), 0;
    0, 0, 0, GJ/L, 0, 0, 0, 0, 0, -(GJ/L), 0, 0;
    0, 0, -((6*EIy)/L^2), 0, (4*EIy)/L, 0, 0, 0, (6*EIy)/L^2, ...
        0,(2*EIy)/L, 0;
    0, (6*EIz)/L^2, 0, 0, 0, (4*EIz)/L, 0, ...
        -((6*EIz)/L^2), 0, 0, 0, (2*EIz)/L;
    -(EA/L), 0, 0, 0, 0, 0, EA/L, 0, 0, 0, 0, 0;
    0, -((12*EIz)/L^3), 0, 0, 0, -((6*EIz)/L^2), ...
        0, (12*EIz)/L^3, 0, 0, 0, -((6*EIz)/L^2);
    0, 0, -((12*EIy)/L^3), 0, (6*EIy)/L^2, 0, 0, 0, ...
        (12*EIy)/L^3, 0, (6*EIy)/L^2, 0;
    0, 0, 0, -(GJ/L), 0, 0, 0, 0, 0, GJ/L, 0, 0;
    0, 0, -((6*EIy)/L^2), 0, (2*EIy)/L, 0, 0,0, (6*EIy)/L^2, ...
        0, (4*EIy)/L, 0;
    0, (6*EIz)/L^2, 0, 0, 0, (2*EIz)/L, 0, -((6*EIz)/L^2), 0, ...
        0, 0, (4*EIz)/L];
```

The function file of MATLAB is used to assemble the mass matrix of each space frame element for global mass matrix.

```
function m = SpaceFrameConsMass(m_bar,I0, A, coord)
% FrameConsMass(m_bar, nodes(con,:))
% Generates stiffness matrix for a space frame element
% m = distributed mass (lb.sec^2/in/in)
% L = length (in.)
% A = area of cross-section (in^2)
% I0 = Polar moment of inertia of cross sectional area (in^4)
% coord = coordinates at the element ends

n1=coord(1,1:3); n2=coord(2,1:3); n3=coord(3,1:3);
L=sqrt(dot((n2-n1),(n2-n1)));
ex = (n2 - n1)/L;
eyy = cross(n3 - n1, n2 - n1);
ey = eyy/sqrt(dot(eyy,eyy));
ez = cross(ex, ey);
H = [ex; ey; ez];
T = zeros(12);
T([4,5,6], [4,5,6]) = H;
T([7,8,9], [7,8,9]) = H;
T([10,11,12], [10,11,12]) = H;


m = m_bar*L/420*T.'*[140  0   0   0   0   0   70*I0/A  0   0   0   0   0 ;
                     0  156   0   0   0   22*L   0     54  0   0   0 -13*L;
                     0   0  156   0  22*L  0   0  0  54    0  -13*L  0 ;
                     0   0   0  140*I0/A  0    0   0  0  0  70*I0/A 0 0;
                     0   0  22*L  0   4*L^2   0   0  0  13*L 0 -3*L^2  0;
                     0  22*L  0    0   0  4*L^2  0  13*L 0   0   0 -3*L^2;
                     70  0    0    0    0  0     140  0   0  0  0  0;
                     0  54    0    0    0  13*L  0    156  0  0 0 -22*L;
                     0  0    54   0    13*L    0   0   0 156  0 -22*L  0;
                     0  0    0  70*I0/A  0   0  0  0  0  140*I0/A  0  0;
                     0  0  -13*L  0 -3*L^2   0  0  0 -22*L   0  4*L^2  0;
                     0 -13*L 0  0   0  -3*L^2  0 -22*L  0 0 0 4*L^2]*T;
```

This MATLAB file is to yield the results of section (c).

The response can be obtained from the similar manner in the previous chapters. The force is the same as the case of Example 12.3. The force is applied at node 1 in z axis with 5000 lb. for 0.1 s.

The maximum displacements at the nodal coordinates were estimated using MATLAB.

$$u_{1\max} = 8.814 \times 10^{-5} \text{ in.} \quad u_{2\max} = 2.6552 \times 10^{-7} \text{ in.} \quad u_{3\max} = 0.9827 \text{ radian}$$

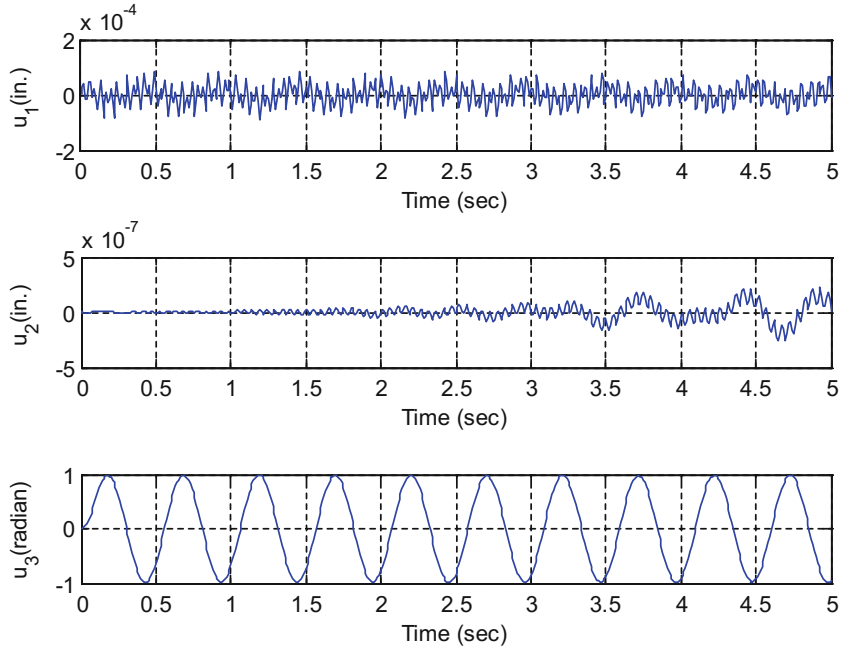

Fig. 13.4 Response of Illustrative Example 13.1

13.8 Summary

This chapter presented the formulation of the stiffness and mass matrices for an element of a space frame, as well as the transformation of coordinates required to refer these matrices to the global system of coordinates. Except for the larger dimensions of the matrices resulting from modeling a space frame, the procedure is identical to the case of a beam, plane frame, or grid frame described in the preceding chapters.

Problem 13.1

For the three-dimensional frame shown in Fig. 13.4a, determine: (a) The first 6 natural frequencies an corresponding modal shapes and (b) The dynamic response when the frame is acted upon by the impulsive force depicted in Fig. 13.4b applied at joint 8 of the frame along the Y- direction as shown in the figure (Fig. 13.5).

Problem Data (for all members):

Modulus of elasticity:	$E = 30 \times 106$ psi
Modulus of rigidity:	$G = 12 \times 106$ psi
Distributed mass:	$m = 0.2$ lb. $\cdot$ sec^2/in.
Concentrated masses:	$m = 10$ lb. $\cdot$ sec^2/in.
Cross-sectional y moment of inertia:	$I_y = 300$ in.^{4}
Cross-sectional z moment of inertia:	$I_z = 400$ in.^{4}
Torsional constant:	$J = 500$ in.^{4}
Cross-sectional area:	$A = 20$ in.^{2}

Problem 13.2

Determine the dynamic response of the three dimensional frame of Problem 13.1 subjected to the harmonic excitation of acceleration a(t) = 0.3 sin 5.3 t applied at support 3 in the Y-direction.

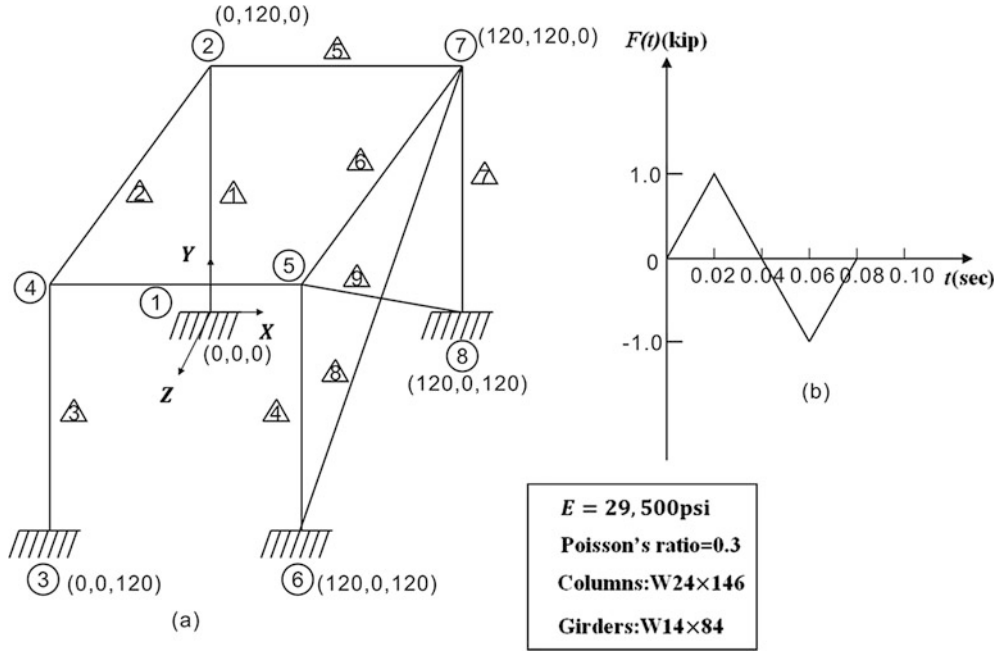

Fig. 13.5 Space frame for Problem 13.1

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14 Dynamic Analysis of Trusses

The static analysis of trusses whose members are pin-connected reduces to the problem of determining the bar forces due to a set of loads applied at the joints. When the same trusses are subjected to the action of dynamics forces, the simple situation of only axial stresses in the members is no longer present. The inertial forces developed along the members of the truss will, in general, produce flexural bending in addition to axial forces. The bending moments at the ends of the truss members will still remain zero in the absence of external joint moments. The dynamic stiffness method for the analysis of trusses is developed as in the case of framed structures by establishing the basic relations between external forces, elastic forces, damping forces, inertial forces and the resulting displacements, velocities, and accelerations at the nodal coordinates, that is, by determining the stiffness, damping, and mass matrices for a member of the truss. The assemblage of system stiffness, damping, and mass matrices of the truss as well as the solution for the displacements at the nodal coordinates follows along the standard method presented in the preceding chapters for framed structures.

14.1 Stiffness and Mass Matrices for the Plane Truss

A member of a plane truss has two nodal coordinates at each joint, that is, a total of four nodal coordinates (Fig. 14.1). For small deflections, it may be assumed that the force-displacement relationship for the nodal coordinates along the axis of the member (coordinates 1 and 3 in Fig. 14.1) are independent of the transverse displacements along nodal coordinates 2 and 4. This assumption is equivalent to stating that a displacement along nodal coordinates 1 or 3 does not produce forces along nodal coordinates 2 or 4 and vice versa.

The stiffness and mass coefficients corresponding to the axial nodal coordinates were derived in Chap. 11 and are given, in general, by Eq. (11.17) for the stiffness coefficients and by Eq. (11.23) for consistent mass coefficients. Applying these equations to a uniform beam element, we obtain, using the notation of Fig. 14.1, the following coefficients:

$$k_{11} = k_{33} = \frac{AE}{L}, \quad k_{13} = k_{31} = -\frac{AE}{L} \tag{14.1}$$

$$m_{11} = m_{33} = \frac{\bar{m}L}{3}, \quad m_{13} = m_{31} = \frac{\bar{m}L}{6} \tag{14.2}$$

in which $\bar{m}$ is the mass per unit length, A is the cross-sectional area, and L is the length of the element.

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M. Paz, Y. H. Kim, *Structural Dynamics*, https://doi.org/10.1007/978-3-319-94743-3_14

The stiffness coefficients, for pin-ended elements, corresponding to the nodal coordinates 2 and 4 are all equal to zero, since a force is not required to produce displacements at these coordinates. Therefore, arranging the coefficients given by Eq. (14.1), we obtain the stiffness equation for a uniform member of a truss as

$$\begin{Bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{Bmatrix} = \frac{AE}{L} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \end{Bmatrix} \tag{14.3}$$

or in condensed notation

$$\{P\} = [K]\{\delta\} \tag{14.4}$$

in which $[K]$ is the element stiffness matrix.

The consistent mass matrix is obtained, as previously demonstrated, using expressions for static displacement functions in the application of the principle of virtual work. The displacement functions corresponding to a unit deflection at nodal coordinates 2 and 4 indicated in Fig. 14.2 are given by

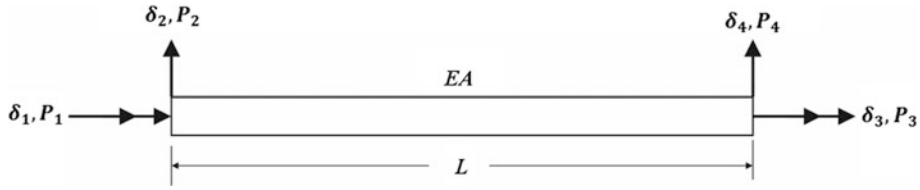

Fig. 14.1 Member of a plane truss showing nodal displacements and forces

$$u_2 = 1 - \frac{x}{L} \tag{14.5}$$

and

$$u_4 = \frac{x}{L} \tag{14.6}$$

The consistent mass coefficients are given by the general expression. Eq. (11.23), which is repeated here for convenience, namely,

$$m_{ij} = \int_0^L \bar{m}(x) u_i(x) u_j(x) dx \tag{14.7}$$

For a uniform member of mass $\bar{m}$ per unit length, the substitution of Eqs. (14.5) and (14.6) into Eq. (14.7) yields

$$\begin{aligned} m_{22} = m_{44} &= \frac{\bar{m}L}{3} \\ m_{24} = m_{42} &= \frac{\bar{m}L}{6} \end{aligned} \tag{14.8}$$

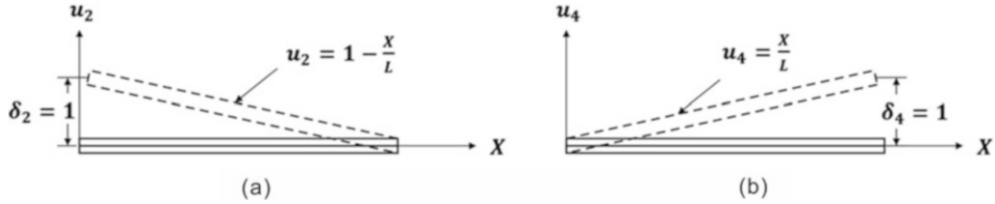

Fig. 14.2 Displacement functions. (**a**) For a unit displacement $\delta_2 = 1$. (**b**) For a unit displacement $\delta_4 = 1$

Finally, the combination of the mass coefficients from Eqs. (14.2) and (14.8) forms the consistent mass matrix relating forces to accelerations at the nodal coordinates for a uniform member of a plane truss, namely,

$$\begin{Bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{Bmatrix} = \frac{\bar{m}E}{6} \begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix} \begin{Bmatrix} \ddot{\delta}_1 \\ \ddot{\delta}_2 \\ \ddot{\delta}_3 \\ \ddot{\delta}_4 \end{Bmatrix} \tag{14.9}$$

or in concise notation

$$\{P\} = [M]\{\ddot{\delta}\} \tag{14.10}$$

14.2 Transformation of Coordinates

The stiffness matrix, Eq. (14.3), and the mass matrix, Eq. (14.9), were derived in reference to nodal coordinates associated with the local or element system of coordinates. As discussed before in the chapters on framed structures, it is necessary to transform these matrices to a common system of reference, the global coordinate system. The transformation of displacements and forces at the nodal coordinates is accomplished, as was demonstrated in Chap. 11, performing a rotation of coordinates. Deleting the angular coordinates in Eq. (11.31) and relabeling the remaining coordinates result in the following transformation for the nodal forces:

$$\begin{Bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{Bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \\ 0 & 0 & -\sin\theta & \cos\theta \end{bmatrix} \begin{Bmatrix} \bar{P}_1 \\ \bar{P}_2 \\ \bar{P}_3 \\ \bar{P}_4 \end{Bmatrix} \tag{14.11}$$

or in condensed notation,

$$\{P\} = [T]\{\bar{\mathrm{P}}\} \tag{14.12}$$

in which $\{P\}$ and $\{\bar{P}\}$ are the nodal forces in reference to local and global coordinates, respectively, and $[T]$ the transformation matrix defined in Eq. (14.11). The same transformation matrix $[T]$ also serves to transform the nodal displacement vector $\{\bar{\delta}\}$ in the global coordinate system to the nodal displacement vector $\{\delta\}$ in local coordinates:

$$\{\delta\} = [T]\{\bar{\delta}\} \tag{14.13}$$

The substitution of Eqs. (14.12) and (14.13) into the stiffness Eq. (14.4) gives

$$[T]\left[\bar{P}\right] = [K][T]\{\bar{\delta}\}$$

Since $[T]$ is an orthogonal matrix ($[T]^{-1} = [T]^T$), it follows that

$$\{\bar{P}\}[T]^T[K][T]\{\bar{\delta}\}$$

or

$$\{\bar{P}\} = \left[\bar{K}\right]\{\bar{\delta}\} \tag{14.14}$$

in which

$$\left[\bar{K}\right] = [T]^T[K][T] \tag{14.15}$$

is the element stiffness matrix in the global coordinate system. Analogously, substituting Eqs. (14.12) and (14.13) into Eq. (14.10) results in

$$\{\bar{P}\} = [T]^T[M][T]\{\bar{\delta}\} \tag{14.16}$$

or

$$\{\bar{P}\} = \left[\bar{M}\right]\{\bar{\delta}\} \tag{14.17}$$

$$\left[\bar{M}\right] = [T]^T[M][T] \tag{14.18}$$

in which $\left[\bar{M}\right]$ is the element mass matrix referred to the global system of coordinates. However, there is no need to use Eq. (14.18) to calculate matrix $\left[\bar{M}\right]$. This matrix is equal to the mass matrix $[M]$ in reference to local axes of coordinates. To verify this fact, we substitute into Eq. (14.18) matrices $[M]$ and $[T]$, respectively, from Eqs. (14.9) and (14.11) to obtain

$$\begin{aligned}
\left[\bar{M}\right] &= \frac{\bar{m}L}{6}\begin{bmatrix} c & -s & 0 & 0 \\ s & c & 0 & 0 \\ 0 & 0 & c & -s \\ 0 & 0 & s & c \end{bmatrix}\begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix}\begin{bmatrix} c & s & 0 & 0 \\ -s & c & 0 & 0 \\ 0 & 0 & c & s \\ 0 & c & s & c \end{bmatrix} \\
\left[\bar{M}\right] &= \frac{\bar{m}L}{6}\begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix} = [M]
\end{aligned} \tag{14.19}$$

in which we use the notation $c = \cos\theta$, $s = \sin\theta$ and the fact that $\cos^2\theta + \sin^2\theta = 1$.

A similar relationship is also obtained for the element damping matrix, namely,

$$\left[\bar{C}\right] = [T]^T[C][T] \tag{14.20}$$

in which $\left[\bar{C}\right]$ and $[C]$ are the damping matrices referred, respectively, to the global and the local systems of coordinates.

Illustrative Example 14.1

The plane truss shown in Fig. 14.3 which has only three members is used to illustrate the application of the stiffness method for trusses. For this truss determine the system stiffness and the system consistent mass matrices.

Solution:

The stiffness matrix, Eq. (14.3), the mass matrix, Eq. (14.9), and the transformation matrix, Eq. (14.11), are applied to the three members of this truss. For member 1, $\theta = 90°$,

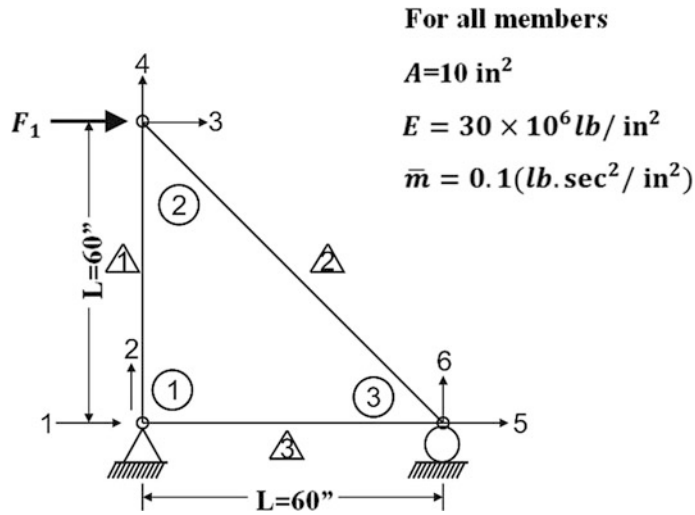

Fig. 14.3 Plane truss for Illustrative Example 14.1

$$[K_1] = \frac{AE}{L}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad [\overline{M_1}] = [M_1] = \frac{\bar{m}E}{6}\begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix}$$

and

$$[T_1] = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

Then from Eqs. (14.15)

$$[\bar{K}_1] = [T_1]^T[K_1][T_1] = \frac{AE}{L}\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

For member 2, $\theta = 135^\circ$,

$$[K_2] = \frac{AE}{\sqrt{2}L}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad [\overline{M_2}] = [M_2] = \frac{\bar{m}\sqrt{2}L}{6}\begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix}$$

and

$$[T_2] = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & -1 \end{bmatrix}$$

Then from Eqs. (14.15)

$$[\bar{K}_2] = [T_2]^T[K_2][T_2] = \frac{AE}{2\sqrt{2}L}\begin{bmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

For member 3, $\theta = 0^\circ$,

$$[\bar{K}_3] = [K_3] = \frac{AE}{L}\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[\bar{M}_3] = [M_3] = \frac{\bar{m}L}{6}\begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix}$$

substituting the proper numerical values for this example; $L = 60$ in, $A = 10$ in^2, $\bar{m} = 0.1$ lb. • sec^2/in^2, $E = 30 \times 10^6$ lb./in^2, and following the rules of the direct method of assembling the system stiffness and mass matrix from the above element matrices, we obtain

$$[K_s] = 10^6\begin{bmatrix} 1.768 & -1.768 & -1.768 \\ -1.768 & 6.768 & 1.768 \\ -1.768 & 1.768 & 6.768 \end{bmatrix}$$

$$[M_s] = \begin{bmatrix} 4.828 & 0 & 1.414 \\ 0 & 4.828 & 0 \\ 1.414 & 0 & 4.828 \end{bmatrix}$$

where $[K_s]$ and $[M_s]$ are, respectively, the system stiffness and mass matrices for the truss shown in Fig. 14.3.

Illustrative Example 14.2
Determine the natural frequencies and normal modes for the truss of Example 14.1.

Solution:
The differential equations of motion for this system are

$$[M_S]\{\ddot{u}\} + [K_S]\{u\} = 0 \tag{a}$$

Substituting $\{u\} = \{a\} \sin \omega t$, we obtain

$$\left([K_S] - \omega^2[M_S]\right)\{a\} = \{0\} \tag{b}$$

For the nontrivial solution, we require

$$\left|[K_S] - \omega^2[M_S]\right| = 0 \tag{c}$$

Substituting from Example 14.1 $[K_S]$ and $[M_S]$ and expanding the above determinant give a cubic equation in $\lambda = \omega^2 \bar{m} L^2/6AE$ which has the following roots

$$\lambda_1 = 0.00344 \text{ or } \omega_1 = 415 \text{ rad/ sec}$$
$$\lambda_2 = 0.0214 \text{ or } \omega_2 = 1034 \text{ rad/ sec}$$
$$\lambda_3 = 0.0466 \text{ or } \omega_3 = 1526 \text{ rad/ sec}$$

Substituting in turn ω_1, ω_2 and ω_3 into Eq. (b), setting $a_1 = 1$, and solving for a_2 and a_3 give the modal vectors.

$$\{a_1\} = \begin{Bmatrix} 1.000 \\ 0.216 \\ 0.274 \end{Bmatrix}, \{a_2\} = \begin{Bmatrix} 1.000 \\ 5.488 \\ -4.000 \end{Bmatrix}, \{a_3\} = \begin{Bmatrix} 1.000 \\ -1.000 \\ -1.524 \end{Bmatrix}$$

which may be normalized using the factors

$$\sqrt{\{a_1\}^T[M_S]\{a_1\}} = 2.489, \sqrt{\{a_2\}^T[M_S]\{a_2\}} = 14.695$$
$$\sqrt{\{a_3\}^T[M_S]\{a_3\}} = 4.066$$

This normalization results in

$$\{\phi_1\} = \begin{Bmatrix} 0.402 \\ 0.087 \\ 0.110 \end{Bmatrix}, \{\phi_2\} = \begin{Bmatrix} 0.068 \\ 0.373 \\ -0.272 \end{Bmatrix}, \{\phi_3\} = \begin{Bmatrix} 0.246 \\ -0.246 \\ -0.375 \end{Bmatrix}$$

These normalized eigenvectors form the modal matrix:

$$[\Phi] = \begin{bmatrix} 0.402 & 0.068 & 0.246 \\ 0.087 & 0.373 & -0.246 \\ 0.110 & -0.272 & -0.375 \end{bmatrix}$$

The MATLAB program is presented to demonstrate Illustrative Examples 14.1 and 14.2.

```
clc
close all
clear all

%_____________________________________________________________________
%  Determine System Matrices/Determine Force
%_____________________________________________________________________

%%%%-GIVEN VALUES-%%%%

E=30*10^6; A=10;                                %E (psi), A (in.^2)
P = 5000;                                       %P (lbs)

%%%Create frame model (ith row of nodes is ith node)
nodes = 60*[0, 0; 0, 1; 1, 0];                  % Unit: in.
%%%Element number (ith row = ith element with two nodes)
conn = [1,2; 2,3; 1,3];
%%%Dofs for ith element (ith row)
lmm = [1, 2, 3, 4; 3, 4, 5, 6; 1, 2, 5, 6];
m_bar = 0.1;                                    %Distributed mass (lb-sec^2/in/in)

dof = 2*length(nodes);                          %Total No.dofs

K= zeros(dof);
M= zeros(dof);

%%%Generate equations for each element and assemble them.
for i=1:3
    lm=lmm(i,:);
    con=conn(i,:);
    k=PlaneTrussElement(E, A, nodes(con,:));
    K(lm, lm) = K(lm, lm) + k;
end

%%%Generate mass matrix for each element and assemble them.
for i=1:3
    lm=lmm(i,:);
    con=conn(i,:);
    m=PlaneTrussMass(m_bar, nodes(con,:));
    M(lm, lm) = M(lm, lm) + m;
end

K

M

%%%Define the load vector
F = zeros(dof,1); F(3) = 5000;              %Applied force at specific dofs

%%%System Matrices
[Kf, Mf, Rf] = System(K, M, F, [1,2,6]);
Kf

Mf

Rf
%_____________________________________________________________________
% Solve the eigenvalue problem and normalized eigenvectors
%_____________________________________________________________________

%%%Solve for eigenvalues (D) and eigenvectors (a)
[a, D] = eig(Kf, Mf);

[omegas,ii] = sort(sqrt(diag(D)));        %Natural Frequencies

omegas

a = a(:,ii)                               %Mode Shapes

T = 2*pi./omegas;                         %Natural Periods

save ('temp0.mat', 'Mf', 'Kf' ,'Rf');
```

The function file of MATLAB is used to assemble the stiffness matrix of plane truss element for global stiffness matrix.

```
function k = PlaneTrussElement(E, A, coord)
% PlaneTrussElement(e, A, coord)
% Generates stiffness matrix for a plane truss element
% E = modulus of elasticity (psi)
% A = area of cross-section (in^2)
% coord = coordinates at the element ends

x1=coord(1,1); y1=coord(1,2);
x2=coord(2,1); y2=coord(2,2);
L=sqrt((x2-x1)^2+(y2-y1)^2);
ls=(x2-x1)/L; ms=(y2-y1)/L;
T = [ls ms 0 0; -ms ls 0 0; 0 0 ls ms; 0 0 -ms ls];
k = E*A/L*T.'*[1 0 -1 0;
               0 0  0 0;
               -1 0 1 0;
                0 0 0 0]*T;
```

The function file of MATLAB is used to assemble the mass matrix of each plane truss element for global mass matrix.

```
function m = PlaneTrussMass(m_bar, coord)
% PlaneTrussMass(m_bar, coord)
% Generates mass matrix for a plane truss element
% m = distributed mass (lb.sec^2/in/in)
% L = length (in.)
% coord = coordinates at the element ends

x1=coord(1,1); y1=coord(1,2);
x2=coord(2,1); y2=coord(2,2);
L=sqrt((x2-x1)^2+(y2-y1)^2);
ls=(x2-x1)/L; ms=(y2-y1)/L;
T = [ls ms 0 0; -ms ls 0 0; 0 0 ls ms; 0 0 -ms ls];
m = m_bar*L/6*T.'*[2 0 1 0;
                   0 2 0 1;
                   1 0 2 0;
                   0 1 0 2]*T;
```

Illustrative Example 14.3

Determine the response of the truss in Examples 14.1 and 14.2 when a constant force $F_1 = 5000$ lb. is suddenly applied at node 2 in x axis as shown in Fig. 14.3.

Solution:

The modal equations are given in general [Eq. (8.6)] by

$$\ddot{z}_n + \omega_n^2 z_n = P_n$$

in which the modal force.

$$P_n = \sum_l \phi_{in} F_i$$

Hence using the results that were calculated in Example 18.2 we obtain.

$$\ddot{z}_1 + (415)^2 z_1 = 2010$$
$$\ddot{z}_2 + (1034)^2 z_2 = 340$$
$$\ddot{z}_3 + (1526)^2 z_3 = 1230$$

The solution of the above equations for zero initial conditions $(z_n = 0, \dot{z}_n = 0)$ is given by Eqs. (4.5) as.

$$z_1 = \frac{2010}{(415)^2}(1 - \cos\ 415t)$$
$$z_2 = \frac{340}{(1034)^2}(1 - \cos\ 1034t)$$
$$z_3 = \frac{1230}{(1526)^2}(1 - \cos\ 1526t)$$

The response at the nodal coordinates is then calculated from

$$\{u\} = [\Phi]\{z\}$$

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{bmatrix} 0.402 & 0.068 & 0.246 \\ 0.087 & 0.373 & -0.246 \\ 0.110 & -0.272 & -0.375 \end{bmatrix} \begin{Bmatrix} z_1 \\ z_2 \\ z_3 \end{Bmatrix}$$

or

$$u_1 = 10^{-3}[4.843 - 4.692\cos\ 415t - 0.022\cos\ 1034t - 0.130\cos\ 1526t]$$
$$u_2 = 10^{-3}[1.004 - 1.015\cos\ 415t - 0.119\cos\ 1034t + 0.130\cos\ 1526t]$$
$$u_3 = 10^{-3}[0.999 - 1.284\cos\ 415t + 0.087\cos\ 1034t + 0.198\cos\ 1526t]$$

The system matrices in Example 14.2 are used to plot the responses over 5 s at 0.01 s interval. The initial conditions are zeros for both displacement and velocity. The program prints out the maximum responses and figure below.

```
clc
clear all
close all


%_____________________________________________________________________
% Inputs:
%   M, K
%   F = forcing function
%   t = Time period
%   u0 = initial displacement
%   v0 = initial velocity
%_____________________________________________________________________
 t = 0:0.01:5;

load ('temp0.mat', 'Mf', 'Kf', 'Rf')

%%%Deifne Mass Matrix
```

```
M = Mf

%%%Deifne Stiffness Matrix
K = Kf

[n,n]= size(M);


F = Rf;

nstep = size(t');

u0 = zeros(n,1); u0(1) =0;
v0 = zeros(n,1); v0(1) =0;
[n,n]= size(M);

%_____________________________________________________________________________
% Solve the eignevalue problem and normalized the eigenvectors
%_____________________________________________________________________________

[a, D] = eig(K, M)                  % Solve for eigenvalues (D) and eigenvectors (a)

[omegas,k] = sort(sqrt(diag(D)));  % Natural Frequencies
a = a(:,k)

T = 2*pi./omegas;                   % Natural Periods

aMa = diag(a'*M*a);                  % aMa = {a}'*[M]*{a}

nom_phi = (a)*inv(sqrt(diag(aMa))); % Normalized modal matrix

orth_M = nom_phi'*M*nom_phi         % Check the orthogonality condition for Mass Matrix

orth_K = nom_phi'*K*nom_phi         % Check the orthogonality condition for Stiffness
Matrix
%_____________________________________________________________________________
% Initial conditions
%_____________________________________________________________________________

P = nom_phi'*F                     % Normalized force, P = nom_F
q0 = nom_phi'*M*u0;
dq0 = nom_phi'*M*v0;

%_____________________________________
% Damping matrix using the proportional damping matrix
% [C] = a[M]+b[K]
% zetas = damping ratios
%_____________________________________
a = 0;
b = 0;
nom_C = nom_phi'*(a*M+b*K)*nom_phi;
zetas = diag((1/2)*nom_C*inv(diag(omegas)));

save ('temp1.mat', 'omegas', 'P' ,'zetas');
q = [];
r = [];
for i=1:n
q0_i = q0(i,:);
dq0_i = dq0(i,:);

load temp1.mat
omega = omegas(i,:);
P = P(i,:);
m = M(i,i);
zeta = zetas(i,:);
save ('temp2.mat', 'omega', 'P', 'm', 'zeta');

[t,q] = ode45(@MDOFP, t, [q0_i dq0_i]',[]);

r(:,i) = q(:,1);
save ('temp3.mat', 'r')

end
```

```
load ('temp3.mat', 'r');

yim = nom_phi*[r'];

save ('response.mat','yim');

figure
subplot(3,1,1);          % Node 2: x displ (in.).
plot(t, yim(1,:))
xlabel ('Time (sec)'); ylabel ('u_1(in.)'); grid on

subplot(3,1,2);          % Node 2: y displ (in.).
plot(t, yim(2,:))
xlabel ('Time (sec)'); ylabel ('u_2(in.)'); grid on

subplot(3,1,3);          % Node 3: x displ (in.).
plot(t, yim(3,:))
xlabel ('Time (sec)'); ylabel ('u_3(in.)'); grid on

umax_1=max(abs(yim(1,:)))
umax_2=max(abs(yim(2,:)))
umax_3=max(abs(yim(3,:)))
```

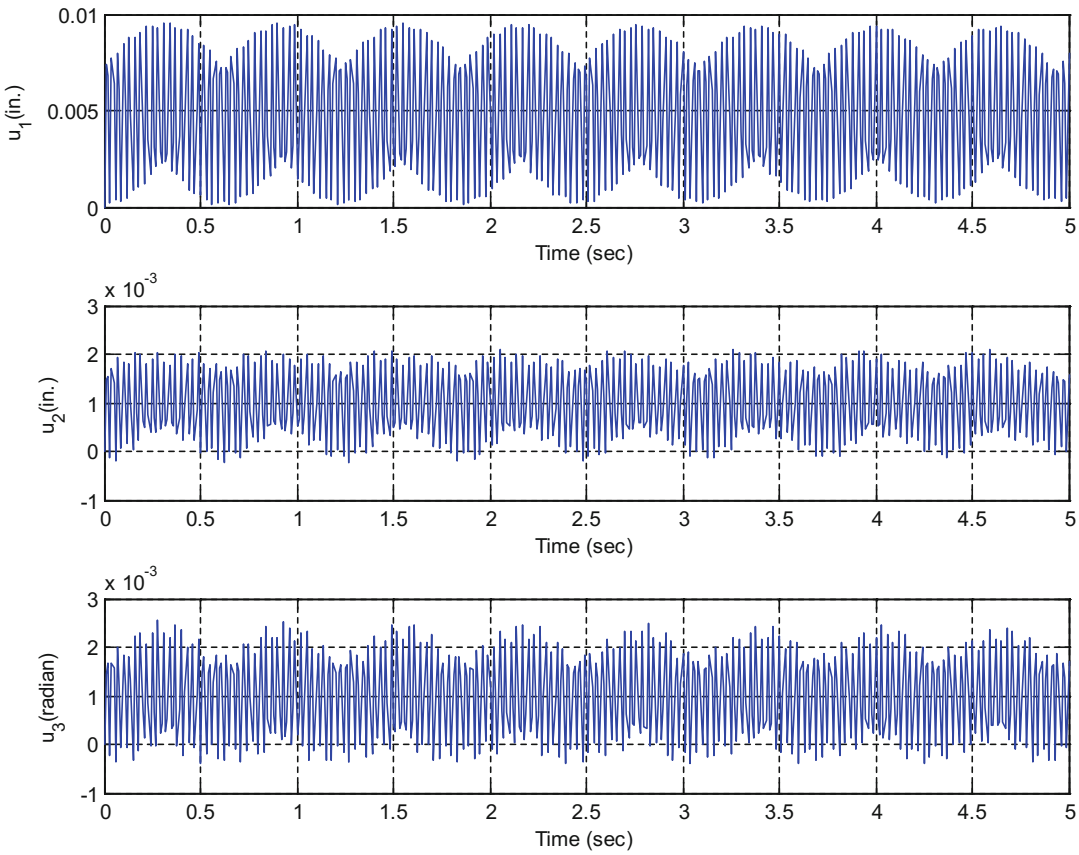

Fig. 14.4 Response of Illustrative Example 14.3

The maximum displacements at the nodal coordinates were estimated using MATLAB.

$$u_{1\max} = 0.0096 \text{ in} \quad u_{2\max} = 0.0021 \text{ in} \quad u_{3\max} = 0.0025 \text{ in}$$

14.3 Stiffness and Mass Matrices for Space Trusses

The stiffness matrix and the mass matrix for a space truss can be obtained as an extension of the corresponding matrices for the plane truss. Figure 14.4 shows the nodal coordinates in the local system (unbarred) and in the global system (barred) for a member of a space truss. The local x axis is directed along the longitudinal axes of the member while the y and z axes are set to agree with the principal directions of the cross section of the member. The following matrices may then be written for a uniform member of a space truss as an extension of the stiffness, Eq. (14.3), and the mass, Eq. (14.9), matrices for a member of a plane truss.

Stiffness matrix:

$$[K] = \frac{AE}{L}\begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{14.21}$$

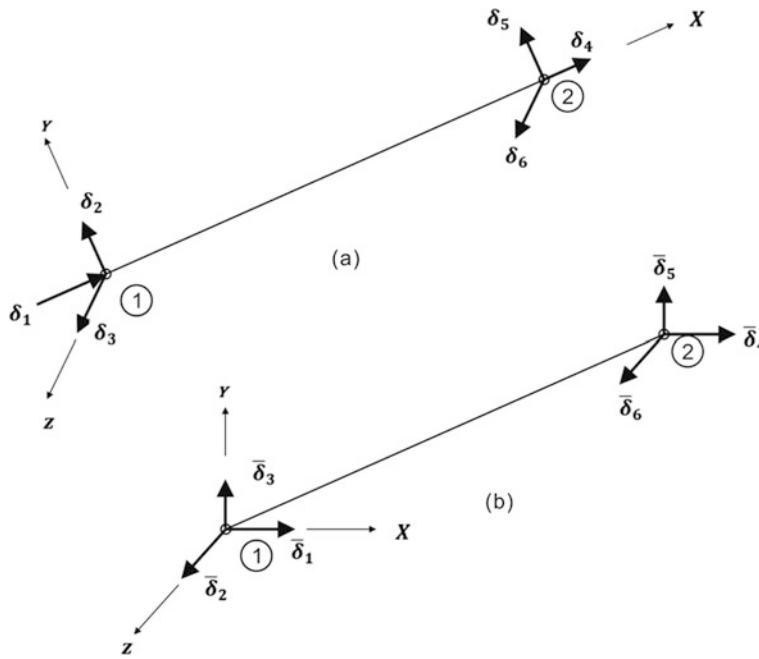

Fig. 14.5 Member of a space truss showing nodal coordinates. (**a**) In the local system (unbarred). (**b**) In the global system (barred)

Consistent mass matrix:

$$[M_c] = \frac{\bar{m}L}{6}\begin{bmatrix} 2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 \\ 1 & 0 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 2 \end{bmatrix} \tag{14.22}$$

Lumped mass matrix:

$$[M_L] = \frac{\bar{m}L}{2}[1 \;\; 1 \;\; 1 \;\; 1 \;\; 1 \;\; 1] \tag{14.23}$$

The transformation matrix $[T_1]$ corresponding to three nodal coordinates at a joint is given by Eq. (13.9). It is repeated here for convenience.

$$[T_1] = \begin{bmatrix} \cos xX & \cos xY & \cos xZ \\ \cos yX & \cos yY & \cos yZ \\ \cos zX & \cos zY & \cos zZ \end{bmatrix} \tag{14.24}$$

The direction cosines in the first row of Eq. (14.24), $c_1 = \cos xX$, $c_2 = \cos xY$, and $c_3 = \cos xZ$, may be calculated from the coordinates of the two points P_1 (X_1, Y_1, Z_1) and P_2 (X_2, Y_2, Z_2) at the ends of the truss element, that is

$$c_1 = \frac{X_1 - X_2}{L}, c_2 = \frac{Y_2 - Y_1}{L} \text{ and } c_3 = \frac{Z_2 - Z_1}{L} \tag{14.25}$$

with

$$L = \sqrt{(X_2 - X_1)^2 + (Y_2 - Y_1)^2 + (Z_2 - Z_1)^2} \tag{14.26}$$

The transformation matrix for the nodal coordinates at the two ends of a truss member is then given by

$$[T] = \begin{bmatrix} [T_1] & [0] \\ [0] & [T_1] \end{bmatrix} \tag{14.27}$$

in which $[T_1]$ is given by Eq. (14.24). The following transformations are then required to obtain the member stiffness matrix $\left[\bar{K}\right]$ and the member mass matrix $\left[\bar{M}\right]$ in reference to the global system of coordinates:

$$\left[\bar{K}\right] = [T]^T[K][T] \tag{14.28}$$

and

$$\left[\bar{M}\right] = [T]^T[M][T] \tag{14.29}$$

In the case of an element of a space truss, it is only necessary to calculate the direction cosines of the centroidal axis of the element which are given by Eq. (14.25). The other direction cosines in

Eq. (14.24) do not appear in the final expression for the element stiffness matrix $[\bar{K}]$ as may be verified by substituting Eqs. (14.24) and (14.27) into Eq. (14.28) and proceeding to multiply the matrices indicated in this last equation. The final result of this operation may be written as follows.

$$[K] = \frac{AE}{L}\begin{bmatrix} c_1^2 & c_1c_2 & c_1c_3 & -c_1^2 & -c_1c_2 & -c_1c_3 \\ c_2c_1 & c_2^2 & c_2c_3 & -c_2c_1 & -c_2^2 & -c_2c_3 \\ c_3c_1 & c_3c_2 & c_3^2 & -c_3c_1 & -c_3c_2 & -c_3^2 \\ -c_1^2 & -c_1c_2 & -c_1c_3 & c_1^2 & c_1c_2 & c_1c_3 \\ -c_2c_1 & -c_2^2 & -c_2c_3 & c_2c_1 & c_2^2 & c_2c_3 \\ -c_3c_1 & -c_3c_2 & -c_3^2 & c_3c_1 & c_3c_2 & c_3^2 \end{bmatrix} \tag{14.30}$$

Consequently, the determination of the stiffness matrix for an element of a space truss, in reference to the global system of coordinates by Eq. (14.30), requires the evaluation by Eq. (14.25) of only the direction cosines of the local axis x along the element.

Also, analogously, to Eq. (14.19) for an element of a plane truss, the mass matrix $[\bar{M}]$ for an element of a space truss in reference to global coordinates is equal to the mass matrix of the element $[M]$ in local coordinates. Thus, there is no need to perform the operations indicated in Eq. (14.29). The substitution into this equation of $[M]$ from Eq. (14.22) and $[T]$ from Eqs. (14.24) and (14.27) results, after performing the multiplications established by Eq. (14.29), in the same matrix $[M]$, that is,

$$[\bar{M}] = [M] \tag{14.31}$$

14.4 Equation of Motion for Space Trusses

The dynamic equilibrium conditions at the nodes of the space truss result in the differential equations of motion which in matrix notation may be written as follows:

$$[\bar{M}]\{\ddot{u}\} + [\bar{C}]\{\dot{u}\} + [\bar{K}]\{u\} = \{F(t)\} \tag{14.32}$$

in which $\{u\}$, $\{\dot{u}\}$, and $\{\ddot{u}\}$ are, respectively, the displacement, velocity, and acceleration vectors at the nodal coordinates, $\{F(t)\}$ is the vector of external nodal forces, and $[\bar{M}]$, $[\bar{C}]$, and $[\bar{K}]$ are the system mass, damping, and stiffness matrices.

In the stiffness method of analysis, the system matrices in Eq. (14.32) are obtained by appropriate superposition of the corresponding member matrices using the direct method as we have shown previously for the framed structures. As was discussed in the preceding chapters, the practical way of evaluating damping is to prescribe damping ratios relative to the critical damping for each mode:. Consequently, when Eq. (14.32) is solved using the modal superposition method, the specified modal damping ratios are introduced directly into the modal equations. In this case, there is no need for explicitly obtaining the system damping matrix $[C]$. However, this matrix is required when the solution of Eq. (14.32) is sought by other methods of solution, such as the step-by-step integration method. In this case, the system damping matrix $[C]$ can be obtained from the specified modal damping ratios by any of the methods discussed in Chap. 22.

14.5 Modeling Structures as Space Trusses Using MATLAB

The MATLAB calculates the stiffness and the mass matrices for the space truss. MATLAB can determine natural frequencies or calculate the response of the structure subjected to external excitation.

Illustrative Example 14.4
Determine the stiffness and mass matrices for the space truss shown in Fig. 14.5. The mass per unit length of any member is $\bar{M} = 0.1$ (lb · sec^2/in^2). Also determine the first three natural frequencies of the truss. Plot the response of the strcture when $F_1 = 5000$ lb is suddenly applied at node 3 in z axis (Fig. 14.6).

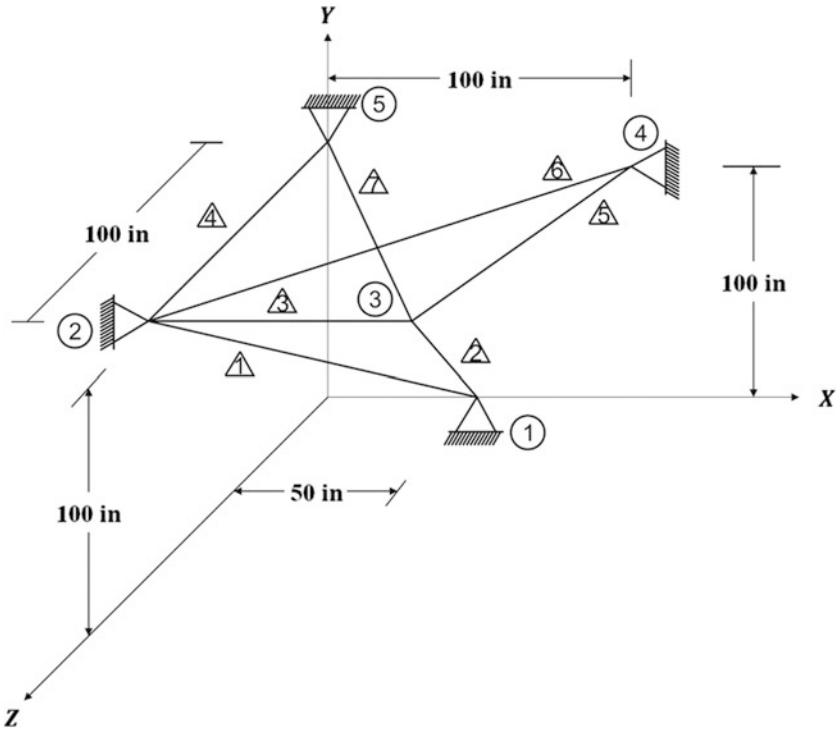

Fig. 14.6 Space truss for Example 14.4

Solution:
The force is applied at node 3 in z axis (which is corresponding to 9th degree-of-freedom).

```
clc
close all
clear all

%_____________________________________________________________________
%  Determine System Matrices/Determine Force
%_____________________________________________________________________

%%%%-GIVEN VALUES-%%%%

E=30*10^6; A=10;                                %E (psi), A (in^2)
P = 5000;                                       %P (lbs)


%%%Create frame model (ith row of nodes is ith node)
nodes =     [50, 0, 0;
             0,  100, 100;
             100,100, 100;
             100,100, 0;
             0, 100,  0];                       % Unit: in.

%%%Element number (ith row = ith element with two nodes)
conn = [2,1;
        3,1;
        2,3;
        2,5;
        3,4;
        2,4;
        5,3];

%%%Dofs for ith element (ith row)
lmm = [4, 5, 6, 1, 2, 3;
       7, 8, 9, 1, 2, 3;
       4, 5, 6, 7, 8, 9;
       4, 5, 6, 13,14,15;
       7, 8, 9, 10,11,12;
       4, 5, 6, 10,11,12;
       13,14,15, 7, 8, 9];


m_bar = 0.1;                            %Distributed mass (lb-sec^2/in/in)

dof = 3*length(nodes);                  %Total No.dofs

K= zeros(dof);
M= zeros(dof);

%%%Generate equations for each element and assemble them.
for i=1:7
    lm=lmm(i,:);
    con=conn(i,:);
    k=SpaceTrussElement(E, A, nodes(con,:));
    K(lm, lm) = K(lm, lm) + k;
end

%%%Generate mass matrix for each element and assemble them.
for i=1:7
    lm=lmm(i,:);
    con=conn(i,:);
    m=SpaceTrussMass(m_bar, nodes(con,:));
    M(lm, lm) = M(lm, lm) + m;
end
```

```
K

M

%%%Define the load vector
F = zeros(dof,1); F(9) = 5000;              %Applied force at specific dofs

%%%System Matrices
[Kf, Mf, Rf] = System(K, M, F, [1:6,10:dof])

Kf

Mf

Rf
%_____________________________________________________________________
% Solve the eigenvalue problem and normalized eigenvectors
%_____________________________________________________________________

%%%Solve for eigenvalues (D) and eigenvectors (a)
[a, D] = eig(Kf, Mf);

[omegas,ii] = sort(sqrt(diag(D)));   %Natural Frequencies

omegas

a = a(:,ii)                          %Mode Shapes

T = 2*pi./omegas;                    %Natural Periods

save ('temp0.mat', 'Mf', 'Kf' ,'Rf');
```

The function file of MATLAB is used to assemble the stiffness matrix of space truss element for global stiffness matrix.

```
function k = SpaceTrussElement(e, A, coord)
% k = SpaceTrussElement(e, A, coord)
% Generates stiffness matrix of a space truss element
% E = modulus of elasticity (psi)
% A = Area of cross-section (in^2)
% coord = coordinates at the element ends

x1=coord(1,1); y1=coord(1,2); z1=coord(1,3);
x2=coord(2,1); y2=coord(2,2); z2=coord(2,3);
L=sqrt((x2-x1)^2+(y2-y1)^2+(z2-z1)^2);
C1=(x2-x1)/L; C2=(y2-y1)/L; C3=(z2-z1)/L;

  k = e*A/L*[C1^2, C1*C2, C1*C3, -(C1^2), -(C1*C2), -(C1*C3);
             C2*C1, C2^2, C2*C3, -(C2*C1), -(C2^2), -(C2*C3);
             C3*C1, C3*C2, C3^2, -(C3*C1), -(C3*C2), -(C3^2);
             -(C1^2), -(C1*C2), -(C1*C3), C1^2, C1*C2, C1*C3;
             -(C2*C1), -(C2^2), -(C2*C3), C2*C1, C2^2, C2*C3;
             -(C3*C1), -(C3*C2), -(C3^2), C3*C1, C3*C2, C3^2];
```

The function file of MATLAB is used to assemble the mass matrix of each space truss element for global mass matrix.

```
function m = SpaceTrussMass(m_bar, coord)
% SpaceTrussElement(e, A, coord)
% Generates mass matrix for a space truss element
% m = distributed mass (lb.sec^2/in/in)
% L = length (in.)
% coord = coordinates at the element ends

x1=coord(1,1); y1=coord(1,2); z1=coord(1,3);
x2=coord(2,1); y2=coord(2,2); z2=coord(2,3);
L=sqrt((x2-x1)^2+(y2-y1)^2+(z2-z1)^2);

m = m_bar*L/6*[2 0 0 1 0 0;
               0 2 0 0 1 0;
               0 0 2 0 0 1;
               1 0 0 2 0 0;
               0 1 0 0 2 0;
               0 0 1 0 0 2];
```

The same MATLAB file can be used in Example 14.4 to plot the response of the space truss structures (Fig. 14.7).

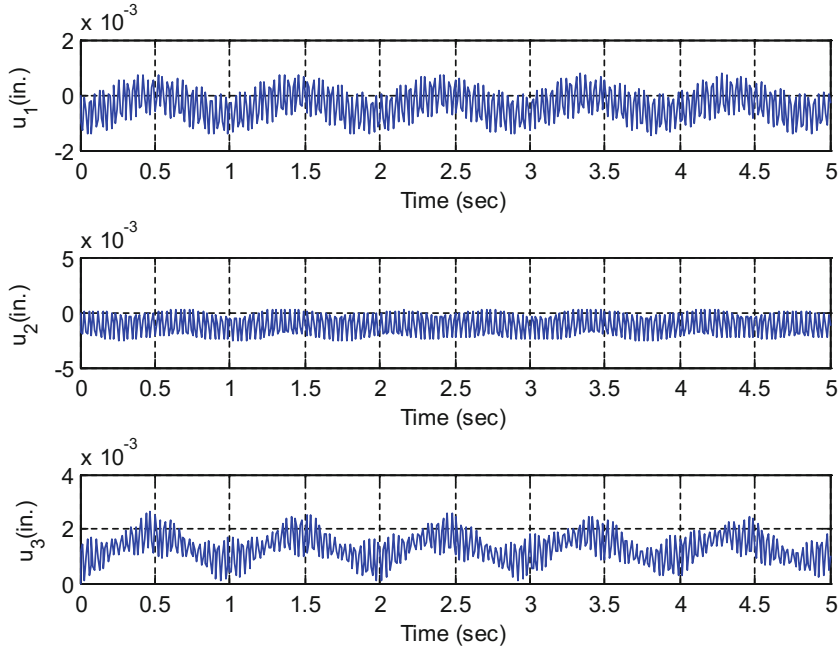

Fig. 14.7 Response of Illustrative Example 14.4

The maximum displacements at the nodal coordinates were estimated using MATLAB.

$$u_{1\max} = 0.0014 \text{ in.} \quad u_{2\max} = 0.0026 \text{ in.} \quad u_{3\max} = 0.0026 \text{ in.}$$

14.6 Summary

The dynamic analysis of trusses by the stiffness matrix method was presented in this chapter. As in the case of framed structures, discussed in the preceding chapters, the stiffness and mass matrices for a member of a truss were developed. The system matrices for a truss are assembled as explained for framed structures by the appropriate superposition of the coefficients in the matrices of the elements.

14.7 Problems

Problem 14.1
For the plane truss shown in Fig. P14.1 determine the system stiffness and mass matrices corresponding to the two nodal coordinates indicated in the figure.

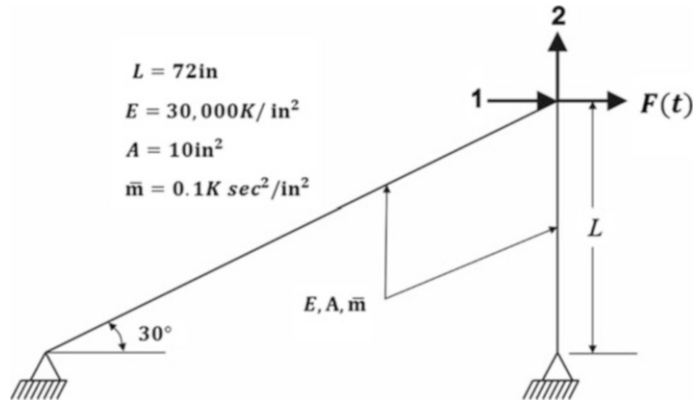

Fig. P14.1

Problem 14.2
Determine the natural frequencies and corresponding normal modes for the truss shown in Fig. P14.1.

Problem 14.3
Determine the response of the truss shown in Fig. P14.1 when subjected to a force $F(t) = 10$ Kip suddenly applied for 2 s at nodal coordinate 1. Use the results of Problem 14.2 to obtain the modal equations. Neglect damping in the system.

Problem 14.4
Solve Problem 14.3 assuming 10% damping in all the modes.
Determine the maximum response of the truss shown in Fig. P14.1 when subjected to a rectangular pulse of magnitude $F_0 = 10$ Kip and duration $t_d = 0.1$ sec. Use the appropriate response spectrum to determine the maximum modal response (Fig. 4.4). Neglect damping in the system.

Problem 14.5
Determine the dynamic response of the frame shown in Fig. P14.1 when subjected to a harmonic force $F(t) = 10 \sin 10\,t$ (Kips) along nodal coordinate 1. Neglect damping in the system.

Problem 14.6
Repeat Problem 14.6 assuming that the damping in the system is proportional to the stiffness, $[C] = a_0\,[K]$ where $a_0 = 0.1$.

Problem 14.7
Determine the response of the truss shown in Fig. P14.8 when acted upon by the forces $F_1\,(t) = 10\,t$ and $F_2\,(t) = 5\,t^2$ during 1 sec. Neglect damping.

Problem 14.8
Determine the response of the truss shown in Fig. P14.8 when acted upon by the forces $F_1(t) = 10$ t and $F_2(t) = 5\ t^2$ during 1 s. Neglect damping.

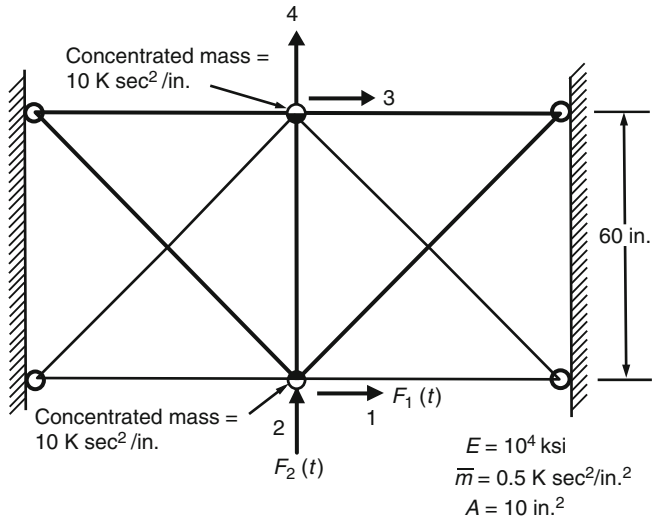

Fig. P14.8

Problem 14.9
Solve problem 14.8 assuming 10% modal damping in all the modes.

15 Dynamic Analysis of Structures Using the Finite Element Method

In the preceding chapters, we considered the dynamic analysis of structures modeled as beams, frames, or trusses. The elements of all these types of structures are described by a single coordinate along their longitudinal axis; that is, these are structures with unidirectional elements, called, "skeletal structures." They, in general, consist of individual members or elements connected at points designated as "nodal points" or "joints." For these types of structures, the behavior of each element is first considered independently through the calculation of the element stiffness matrix and the element mass matrix. These matrices are then assembled into the system stiffness matrix and the system mass matrix in such a way that the equilibrium of forces and the compatibility of displacements are satisfied at each nodal point. The analysis of such structures is commonly known as the Matrix Structural Method and could be applied equally to static and dynamic problems.

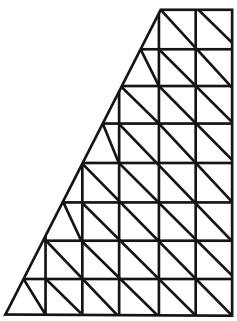

Fig. 15.1 Finite element modeling of thin plate with triangular elements

The structures presented in this chapter are continuous structures which are conveniently idealized as consisting of two-dimensional elements connected only at the selected nodal points. For example, Fig. 15.1 shows a thin plate idealized with plane triangular elements. The static or dynamic analysis of such idealized structures is known as the Finite Element Method (FEM). This is a powerful method for the analysis of structures with complex geometrical configurations, material properties or loading conditions. This method is entirely analogous to Matrix Structural Analysis for skeletal structures (beams, frames, and trusses) presented in the preceding chapters. The Finite Element Method differs from the Matrix Structural Method only in two respects: (1) the selection of elements and nodal points

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M. Paz, Y. H. Kim, *Structural Dynamics*, https://doi.org/10.1007/978-3-319-94743-3_15

are not naturally or clearly established by the geometry as it is for skeletal structures and (2) the displacements at internal points of an element are expressed by approximate interpolating functions and not by an exact analytical relationship as it is in the Matrix Structural Method. Furthermore, for skeletal structures, the displacement of an interior point of an element is governed by an ordinary differential equation, while for a continuous two-dimensional element it is governed by a partial differential equation of much greater complexity.

15.1 Plane Elasticity Problems

Plane elasticity problems refer to plates that are loaded in their own planes. Out-of-plane displacements are induced when plates are loaded by normal forces that are perpendicular to the plane of the plate, such problems are generally referred to as plate bending. (Plate bending is considered in Sect. 15.2.) There are two different types of plane elasticity problems: (1) plane stress and (2) plane strain. In the plane stress problems, the plate is thin relative to the other dimensions and the stresses normal to the plane of the plate are not considered. Figure 15.2 shows a perforated strip-plate in tension as an example of a plane stress problem. For plane strain problems, the strain normal to the plane of loading is suppressed and assumed to be zero. Figure 15.3 shows a transverse slice of a retaining wall as an example of a plane strain problem.

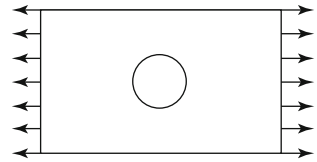

Fig. 15.2 Perforated plate tension element as an example of a structural member load in plane stress

In the analysis of plane elasticity problems, the continuous plate is idealized as finite elements interconnected at their nodal points. The displacements at these nodal points are the basic unknowns as are the displacements at the joints in the analysis of beams, frames or trusses. Consequently, the first step in the application of the FEM is to model the continuous system into discrete elements. The most common geometric elements used for plane elasticity problems are triangular, rectangular or quadrilateral, although other geometrical shapes could be used as well.

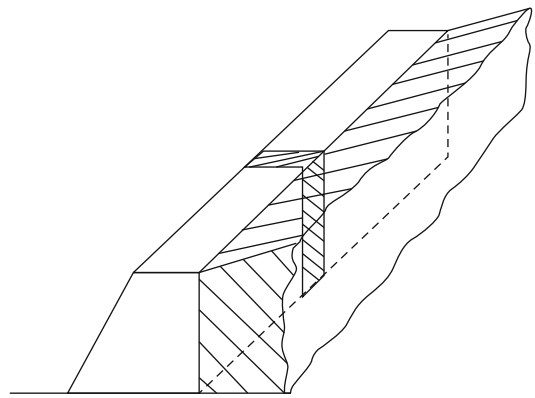

Fig. 15.3 Retaining wall showing a plate slice as an example of plane strain conditions

15.1.1 Triangular Plate Element for Plane Elasticity Problems

The following steps are used in the application of the FEM for the analysis of structural problems:

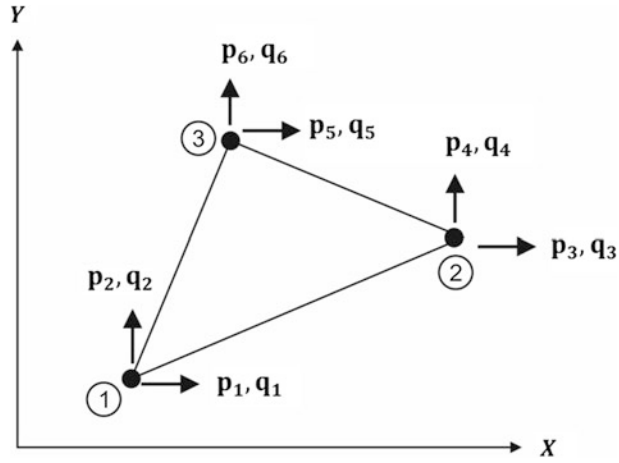

Fig. 15.4 Triangular plate element showing nodal forces P_i and corresponding nodal displacements q_i at the three nodes.

Step I: Modeling the Structure

Figure 15.4 Shows a triangular plate element with nodal forces $\{P\}_e$ and corresponding nodal displacements $\{q\}_e$ with components in the x and y directions at the three nodes of this element. In dynamic analysis, the plate element is assumed to be loaded by external forces distributed in the plane of the plate (Fig. 15.5). These external forces are: (1) body forces with components of forces per unit of volume in the x and y directions, b_x and b_y, conveniently arranged in the vector $\{\boldsymbol{b}\}_e$ and 2) inertial forces, $\rho\{\ddot{q}\}$, per unit of volume in which ρ is the mass density of the plate and $\{\ddot{q}\}$ is the acceleration of a point in the plate element with components $\ddot{q}_x$ and $\ddot{q}_y$ along the coordinates x and y. In addition to these two types of forces, other external forces may also be considered in the analysis.

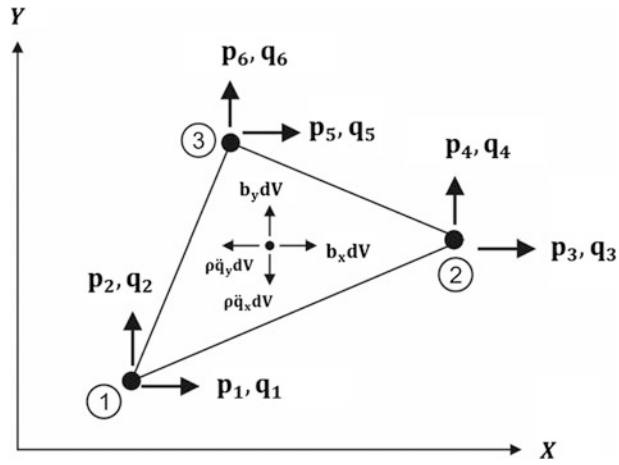

Fig. 15.5 Triangular plate element showing external forces

In a plane elasticity problem, the triangular element with three nodes has two degrees of freedom at each node, as shown in Fig. 15.5, resulting in a total of six degrees of freedom. Thus, the nodal displacement vector $\{\boldsymbol{q}\}_e$ and corresponding nodal force vector $\{\boldsymbol{P}\}_e$ for the plane triangular element have six terms. These vectors may be written as:

$$\{q\}_e = \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{Bmatrix} \quad \{\mathrm{P}\}_e = \begin{Bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_6 \end{Bmatrix} \tag{15.1}$$

Therefore, for this plane triangular element, the element stiffness matrix $[K]_e$ relating nodal forces and nodal displacements and the element mass matrix $[M]_e$ relating nodal forces and nodal accelerations, are of dimension 6×6.

Step II: Selection of a Suitable Displacement Function

The displacements, $u = u\ (x, y)$ and $v = v\ (x, y)$, respectively in the x and y directions at any interior point $P\ (x, y)$ of the triangular element, are expressed approximately by polynomial functions with a total of six coefficients equal to the number of possible nodal displacements. In this case, the simplest expressions for the displacement functions, $u\ (x, y)$ and $v\ (x, y)$, at an interior point of the triangular element are:

$$\begin{aligned} u(x, y) &= c_1 + c_2 x + c_3 y \\ v(x, y) &= c_4 + c_5 x + c_6 y \end{aligned} \tag{15.2}$$

or

$$\{q(x, y)\} = [g(x, y)]\{c\} \tag{15.3}$$

where $\{c\}$ is a vector containing the six coefficients c_i, $[g\ (x, y)]$ a matrix function of the position coordinates (x, y) and $\{q\ (x, y)\}$ a vector with the displacement components $u(x, y)$ and $v(x, y)$ at an interior point along x and y directions, respectively.

Step III: Displacements {q (x, y)} at a Point in the Element Are Expressed in Terms of the Nodal Displacements, $\{q\}_e$

The evaluation of Eq. (15.3) for the displacements of the three nodal joints of the triangular element followed by the solution of the coefficients c_i $(i = 1, 2. . .6)$ results in

$$\{q\}_e = [A]\{c\} \tag{15.4}$$

and

$$\{c\} = [A]^{-1}\{q\}_e \tag{15.5}$$

The subsequent introduction of $\{c\}$ from Eq. (15.5) into Eq. (15.3) gives the displacements at any interior point of the element in terms of the nodal displacements $\{q\}_e$ as

$$\{q(x,y)\} = \begin{Bmatrix} u(x,y) \\ v(x,y) \end{Bmatrix} = [g(x,y][A]^{-1}\{q\}_e \tag{15.6}$$

or

$$\{q(x,y)\} = \begin{Bmatrix} u(x,y) \\ v(x,y) \end{Bmatrix} = [f(x,y)]\{q\}_e \tag{15.7}$$

where the matrix $[f(x, y)] = [A]^{-1}[g(x, y)]$ is only a function of the coordinates x, y of a point in the triangular element. Because the displacement functions $u(x, y)$ and $v(x, y)$ in Eq. (15.7) are both linear in x and y, displacement continuity is ensured along the interface adjoining elements for any value of nodal displacements.

Step IV: Relation between Strain, ε{(x, y)} at any Point within the Element to the Displacements {q (x, y)} and Hence to the Nodal Displacements $\{q\}_e$.

In Theory of Elasticity (Timoshenko and Goodier, 1970), it is shown that the linear strain vector, $\{\varepsilon(x, y)\}$, with axial strain components along the x, y directions, ε_x, ε_y and shearing strain γ_{xy} is given by differentiation of the displacement functions $u(x, y)$ and $v(x, y)$ as follows:

$$\{\varepsilon(x,y)\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{Bmatrix} \tag{15.8}$$

The strain components may then be expressed in terms of the nodal displacements $\{q\}_e$ by substituting from Eq. (15.7) the derivatives of $u(x, y)$ and $v(x, y)$ into Eq. (15.8):

$$\{\varepsilon(x,y)\} = [B]\{q\}_e \tag{15.9}$$

in which the matrix $[\boldsymbol{B}]$ is solely a function of the coordinates (x, y) at an interior point of the element.

Step V: Relationship between Internal Stresses {σ (x, y)} to Strains {ε (x, y)} and Hence to the Nodal Displacements $\{q\}_e$

For plane elasticity problems, the relationship between the normal stresses σ_x, σ_y and shearing stress τ_{xy} to the corresponding strains ε_x, ε_y, and γ_{xy} may be expressed, in general as:

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} d_{11} & d_{12} & 0 \\ d_{21} & d_{22} & 0 \\ 0 & 0 & d_{33} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} \tag{15.10}$$

or in a short matrix notation as:

$$\{\sigma(x,y)\} = [D]\{\varepsilon(x,y)\} \tag{15.11}$$

The substitution of $\{\varepsilon\ (x, y)$ from Eq. (15.9) into Eq.(15.11) gives the desired relationship between stresses $\{\sigma\ (x, y)\}$ at an interior point in the element and the displacements $\{q\}_e$ at the nodes as:

$$\{\sigma(x,y)\} = [D][B]\{q\}_{\mathrm{e}} \tag{15.12}$$

The terms d_{ij} of the matrix $\boldsymbol{D}$ in Eq. (15.10) have different expressions for plane stress problem than for plane strain problem. These expressions as given by the theory of elasticity are:

For plane stress problems:

$$\begin{aligned} d_{11} &= d_{22} = \frac{E}{1\text{-}v^2} \\ d_{12} &= d_{21} = \frac{vE}{1\text{-}v^2} \quad 1 \\ d_{33} &= \frac{E}{2(1+v)} \end{aligned} \tag{15.13}$$

For plane strain problems:

$$\begin{aligned} d_{11} &= d_{22} = \frac{E}{1-v^2} \\ d_{12} &= d_{21} = \frac{vE}{1-v^2} \\ d_{33} &= \frac{E}{2(1+v)} \end{aligned} \tag{15.14}$$

in which E is the modulus of elasticity and v is the Poisson's ratio.

Step VI: Element Stiffness and Mass Matrices

Use is made of the Principle of Virtual Work to establish the expressions for the element stiffness matrix $[K]_e$ and the element mass matrix $[M]_e$ This principle states that for structures in dynamic equilibrium subjected to small, compatible virtual displacements, $\{\delta q\}$, the virtual work, δW_E, of the external forces is equal to the virtual work of internal stresses, δW_I, that is.

$$\delta W_I = \delta W_E \tag{15.15}$$

In applying this principle to a finite element, we assume a vector of virtual displacements $\delta q\{(x, y)\}$. Hence, by Eq. (15.7).

$$\delta\{q(x,y)\} = [f(x,y)]\delta\{q\}_e \tag{15.16}$$

Using the strain-displacement relationships, Eq. (15.9), we obtain.

$$\delta\{\varepsilon\} = [B]\delta\{q\}_e \tag{15.17}$$

The internal work over the volume of the element during this virtual displacement is then given by the product of the virtual strain and the stress integrated over the volume of the element:

$$\delta W_I = \int_v \delta\{\varepsilon\}^T\{\sigma(x,y)\}dV \tag{15.18}$$

The external virtual work W_E includes the work of the applied body forces $\{b\}_e\, dV$; the work of inertial forces $\rho\{\ddot{q}\}dV$; and the work of the nodal forces $P_i\,(i = 1, 2 \ldots 6)$, shown in Fig. 15.5. The total external virtual work is equal to the product of these forces times the corresponding virtual displacements integrated over the volume of the element, namely.

$$\delta W_e = \int_v \delta\{q\}^T\{b\}_e dV - \int_v \delta\{q\}^T\rho\{\ddot{q}\}dV + \delta\{q\}_e^T\{P\}_e \tag{15.19}$$

The second term of Eq. (15.19) is negative because the inertial forces act in directions that are opposite to the positive sense of the accelerations. The substitution of Eqs. (15.18) and (15.19) into Eq. (15.15) results in

$$\int_v \delta\{\varepsilon\}^T\{\sigma(x,y)\}dV = \int_v \delta\{q\}^T\{b\}_e dV - \int_v \delta\{q\}^T\rho\{\ddot{q}\}dV + \delta\{q\}_e^T\{P\}_e \tag{15.20}$$

By substituting into Eq. (15.20) $\sigma(x, y) = [\boldsymbol{D}][B]\{q\}_e$ from Eq. (15.12), and the transposes of $\delta\{q\}^T$ and $\delta\{\varepsilon\}^T$ from Eqs. (15.16) and (15.17) respectively, we obtain, after cancellation of the common factor $\delta\{q\}^T$, the equation of motion for the element:

$$[\boldsymbol{M}]_e\{q\}_e + [\boldsymbol{K}]_e\{q\}_e = \{\boldsymbol{P}\}_e + \{\boldsymbol{P}_b\}_e \tag{15.21}$$

where

$$[K]_e = \int_v [B]^T[D][B]dV \tag{15.22}$$

$$[\boldsymbol{M}]_e = \int_v \rho[f(x,y)]^T[f(x,y)]dV \tag{15.23}$$

and

$$\{\boldsymbol{P}_b(t)\}_e = \int_v [f(x,y)]\{b\}_e dV^2 \tag{15.24}$$

Matrix, $[\boldsymbol{K}]_e$, in Eq. (15.22) is the element stiffness matrix with the terms expressing the force for unit displacement at a nodal coordinate of the element. Eq. (15.23) gives the consistent mass matrix for the element with components expressing forces due to a unit value of the acceleration at a nodal coordinate. Finally, the vector $\{\boldsymbol{P}\}_e$ contains the external forces applied to the nodes of the element and the vector $\{\boldsymbol{P}_b(t)\}_e$ is the consistent nodal forces vector due to the body forces $\{b\}_e$ on the element.

The element stiffness matrix, $[\boldsymbol{K}]_e$, and the element mass matrix, $[\boldsymbol{M}]_e$, as well as the equivalent vector of the applied body forces, $\{\boldsymbol{P}_t\}_e$, may be readily obtained explicitly from Eqs.(15.22), (15.23), and (15.24) respectively, for the simple plane elasticity element. However, computer codes are generally written to calculate these matrices by numerical methods, particularly when the structure is modeled using more advanced elements developed from higher order polynomials.

Step VII: Assemblage of the System Stiffness Matrix [K], the System Mass Matrix [M], and the System Equivalent Nodal Force Vector $\{\mathbf{P_b}\}_\mathbf{s}$ Due to the Body Forces.

The system stiffness matrix $[\boldsymbol{K}]$ and the system mass matrix $[\boldsymbol{M}]$ are assembled from the appropriate summations of the corresponding element matrices, by exactly the same process used in the previous

chapters to assemble these matrices for skeletal-type structures; the system force vector is assembled from the element equivalent nodal forces. Hence, we may symbolically write

$$[\boldsymbol{K}] = \boldsymbol{\Sigma}[\boldsymbol{K}]_e \tag{15.25}$$

$$[\boldsymbol{M}] = \boldsymbol{\Sigma}[\boldsymbol{M}]_e \tag{15.26}$$

and

$$\{\boldsymbol{P}_b\} = \boldsymbol{\Sigma}\{\boldsymbol{P}_b\}_e \tag{15.27}$$

The system of differential equations of motion is then given by

$$[\boldsymbol{M}]\{\ddot{u}\} + [\boldsymbol{K}]\{\mathrm{u}\} = \{\boldsymbol{F}\} \tag{15.28}$$

in which $\{\boldsymbol{F}\}$ is the vector of the external forces at the system nodal coordinates $\{\mathrm{u}\}$ which includes the equivalent nodal forces for the body forces and for any other forces distributed over the structural element.

Step VIII: Solution of the Differential Equations of Motion
The solution of the system of differential equations of motion in terms of the nodal displacements $\{\mathrm{u}\}$ is usually obtained by the modal superposition method presented in Chap. 8. Damping in the system can readily be included in the analysis by the simple addition of the modal damping term to the modal equation. For systems with nonlinear behavior, the modal superposition is not valid and the solution must be obtained using a numerical method such as the step-by-step integration method presented for a single-degree-of-freedom system in Chap. 6 and for a multiple-degree-of freedom system in Chap. 16.

Step IX: Determination of Nodal Stresses
The final step is the calculation of stresses at the nodal points. These stresses can be calculated from the element nodal displacements $\{q\}_e$ selected from the system nodal displacements $\{\mathrm{u}\}$ already determined. The element nodal stresses, $\{\sigma\{x_j, y_j)\}_e$, for node j of an element, are given from Eq. (15.12) by.

$$\left\{\sigma\left(x_j, y_j\right)\right\}_e = [\boldsymbol{D}]_j[\boldsymbol{B}]_j\{q\}_e \tag{15.29}$$

in which the matrices $[\boldsymbol{D}\}_j$ and $[\boldsymbol{B}]_j$ are evaluated for the coordinates of the node j of the element.

15.2 Plate Bending

The application of the finite element method is now considered for the analysis of plate bending, that is, plates loaded by forces that are perpendicular to the plane of the plate. The presentation that follows is based on two assumptions: (1) the thickness of the plate is assumed to be small compared to other dimensions of the plate and (2) the deflections of the loaded plate under the load are assumed to be small relative to its thickness. These assumptions are not particular to the application of finite element method; they are also made in the classical theory of elasticity for bending of thin plates. These two assumptions are necessary because if the thickness of the plate is large, then the plate has to be analyzed as a three-dimensional problem and if the deflections are also large, then in-plane membrane forces are developed and should be accounted for in the analysis. The analysis of plates can be undertaken by the finite element method without these two assumptions. The computer

program such as SAP2000 used in this book is general and may be applied for the analysis of either thin plates undergoing small deflections or to thick plates in which these two assumptions are not required. However, the presentation in this section is limited to thin plates that undergo small deflections.

15.2.1 Rectangular Element for Plate Bending

The derivation of the stiffness and mass matrices as well as the vector of equivalent nodal forces for body forces, inertial forces or any other forces distributed on the plate elements is obtained following the same steps used for derivation presented for a triangular element subjected to in-plane loads in the preceding section.

Step I: Modeling the Structure

A suitable system of coordinates and node numbering is defined in Fig. 15.6a with x, y axes along the continuous sides of the rectangular plate element, and the z axis normal to the plane of the plate, completing a right-hand system of Cartesian coordinates. This rectangular element has three nodal coordinates at each of its four nodes, a rotation about the x-axis (θ_x), a rotation about the y-axis (θ_y), and a normal displacement (w) along the z-axis transverse to the plane of the plate. These nodal displacements are shown in their positive sense and labeled q_i (i = 1, 2... 12) in Fig. 15.6b.

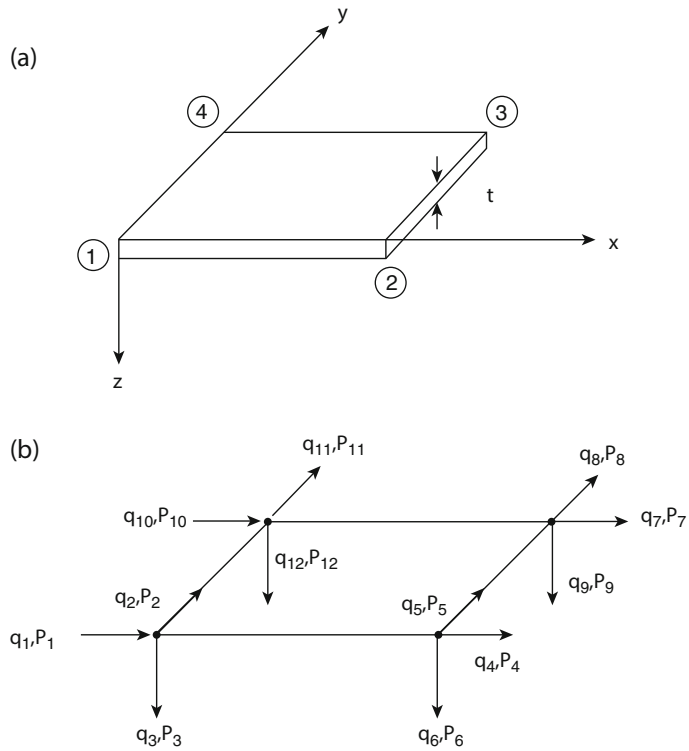

Fig. 15.6 Rectangular plate bending element. (**a**) Coordinate system and node numbering. (**b**) Nodal coordinates q_i and corresponding nodal forces p_i

Corresponding to the three nodal displacements at each node, two moments and a force which are labeled P_i, are also shown in this figure. These 12 element nodal displacements and 12 nodal forces are conveniently arranged in two vectors of 12 components, $\{\boldsymbol{q}\}_e$ and $\{\boldsymbol{P}\}_e$. Therefore, the stiffness matrix or the mass matrix, respectively, relating the nodal forces and the nodal displacements, or nodal forces and nodal accelerations for this rectangular plate bending element with four nodes are of dimension 12 × 12. The angular displacements θ_x and θ_y at any point (x, y) of the plate element are related to the normal displacement w by the following expressions:

$$\theta_x = -\frac{\partial w}{\partial y} \quad \theta_y = \frac{\partial w}{\partial x} \tag{15.30}$$

The positive directions of θ_x and θ_y, are chosen to agree with the angular nodal displacement q_1, q_2, q_4 q_5, etc. is selected in Fig. 15.6b. Therefore, after a function, $w = w\,(x, y)$, is chosen for the lateral displacement w, the angular displacements are determined through the relations in Eq. (15.30).

Step II: Selection of a Suitable Displacement Function

Since the rectangular element in plate bending has twelve degrees of freedom, the polynomial expression chosen for the normal displacements w, must contain 12 constants. A suitable polynomial function is given by.

$$w = c_1 + c_2x + c_3y + c_4x^2 + c_5xy + c_6y^2 + c_7x^3 + c_8x^2y + c_9xy^2 + c_{10}y^3 + c_{11}x^3\mathrm{y} + c_{12}xy^3 \tag{15.31}$$

The displacement function for the rotations θ_x and θ_y are then obtained from Eqs. (15.30) and (15.31) as

$$\theta_x = -\frac{\partial w}{\partial y} = -\left(c_3 + c_5x + 2c_6y + c_8x^2 + 2c_9xy + 3c_{10}y^2 + c_{11}x^3 + 3c_{12}xy^2\right) \tag{15.32}$$

and

$$\theta_y = \frac{\partial w}{\partial x} = c_2 + 2c_4x + c_5y + 3c_7x^2 + 2c_8xy + c_9y^2 + 3c_{11}x^2y + c_{12}y^3 \tag{15.33}$$

By considering the displacements at the edge of one element, that is, in the boundary between adjacent elements, it may be demonstrated that there is continuity of normal lateral displacements and of the rotational displacement in the direction of the boundary line, but not in the direction transverse to this line as it is shown graphically in Fig. 15.7. The displacement function in Eq. (15.31) is called "a non-conforming function" because it does not satisfy the condition of continuity at the boundaries between elements for all three displacements w, θ_x, and θ_y.

Step III: Displacements {q (x, y)} at a Point within the Element Are Expressed in Terms of the Nodal Displacements $\{q\}_e$

Writing Eq. (15.31), (15.32), and (15.33) in matrix notation, evaluating the displacements at the nodal coordinates and solving for the unknown constants results in

$$\{q(x,y)\} = \begin{Bmatrix} \theta_x \\ \theta_y \\ w \end{Bmatrix} = [g(x,y)]\{c\} \tag{15.34}$$

$$\{q\}_e = [\boldsymbol{A}]\{c\} \tag{15.35}$$

and

$$\{c\} = [\boldsymbol{A}]^{-1}\{q\}_e \tag{15.36}$$

where $[\boldsymbol{A}]^{-1}$ is the inverse of the matrix $[\boldsymbol{A}]$ in Eq. (15.35), $[g\ (x, y)]$ is a function of the coordinates x, y at a point in the element, and $\{q\}_e$ is the vector of the 12 displacements at the nodal coordinates of the element (Fig. 15.7).

The substitution of the vector of constants $\{c\}$ from Eq. (15.36) into Eq. (15.34) provides the required relationship for displacements $\{q(x, y)\}$ at an interior point in the rectangular element and the displacements $\{q\}_e$ at the nodes.

$$\{q(x,y)\} = [g(x,y)][\boldsymbol{A}]^{-1}\{q\}_e \tag{15.37}$$

or using Eq. (15.30):

$$\{q(x,y)\} = \begin{Bmatrix} -\dfrac{\partial w}{\partial y} \\ \dfrac{\partial w}{\partial x} \\ w \end{Bmatrix} = [f(x,y)]\{q\}_e \tag{15.38}$$

in which $[f(x, y)] = [g\ (x, y)][\boldsymbol{A}]^{-1}$ is solely a function of the coordinates x, y at a point within the element.

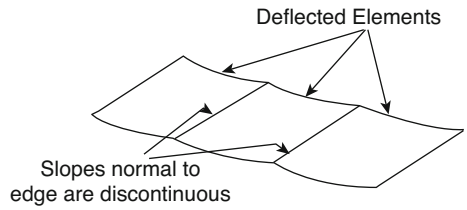

Fig. 15.7 Deflected continuous rectangular elements

Step IV: Relationship between Strains (ε (x, y) at any Point within the Element to Displacements {q (x, y)} and Hence to the Nodal Displacements $\{q\}_e$.

For plate bending, the state of strain at any point of the element may be represented by three components: the curvature in the x direction, the curvature in the y direction, and a component representing torsion in the plate. The curvature in the x direction is equal to the rate of change of the slope in that direction, that is, to the derivative of the slope,

$$-\frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x}\right) = -\frac{\partial^2 w}{\partial x^2} \tag{15.39}$$

Similarly, the curvature in the y direction is

$$-\frac{\partial}{\partial y}\left(\frac{\partial w}{\partial y}\right) = -\frac{\partial^2 w}{\partial y^2} \tag{15.40}$$

Finally, the torsional strain component is equal to the rate of change, with respect to y, of the slope in the x direction, that is

$$\frac{\partial}{\partial y}\left(\frac{\partial w}{\partial x}\right) = \frac{\partial^2 w}{\partial x \partial y} \tag{15.41}$$

The bending moments M_x and M_y and the torsional moments M_{xy} and M_{yx} each act on two opposites sides of the element, but since M_{xy} is numerically equal to M_{yx}, one of these torsional moments, M_{xy}, can be considered to act in all four sides of the element, thus allowing for simply doubling the torsional strain component. Hence, the "strain" vector, $\{\varepsilon\ (x,\ y)\}$ for a plate bending element can be expressed by

$$\{\varepsilon(x,y)\} = \begin{Bmatrix} -\dfrac{\partial^2 w}{\partial x^2} \\ -\dfrac{\partial^2 w}{\partial y^2} \\ 2\dfrac{\partial^2 w}{\partial x \partial y} \end{Bmatrix} \tag{15.42}$$

The substitution in Eq. (15.42) of the second derivatives obtained by differentiation of Eq. (15.38) yields

$$\{\varepsilon(x,y)\} = [\boldsymbol{B}]\{q\}_e \tag{15.43}$$

in which $[\boldsymbol{B}]$ is a function of the coordinates $(x,\ y)$ only.

Step V: Relationship between Internal {σ (x, y)} to Internal Strains {ε (x, y)} and Hence to Nodal Displacements $\{\mathbf{q}\}_\mathbf{e}$

In a plate bending, the internal "stresses" are expressed as bending and twisting moments, and the "strains" are the curvatures and the twist. Thus, for plate bending, the state of internal "stresses" can be represented by

$$\{\sigma(x,y)\} = \begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} \tag{15.44}$$

where M_x and M_y are internal bending moments per unit of length and M_{xy} is the internal twisting moment per unit of length. For a small rectangular element of the plate bending, these internal moments are shown in Fig. 15.8 The moment-curvature relationships obtained from plate bending theory (Timoshenko and Goodier, 1970) are:

$$
\begin{aligned}
M_x &= -\left(D_x \frac{\partial^2 w}{\partial x^2} + D_1 \frac{\partial^2 w}{\partial y^2}\right) \\
M_y &= -\left(D_y \frac{\partial^2 w}{\partial y^2} + D_1 \frac{\partial^2 w}{\partial x^2}\right) \\
M_{xy} &= 2D_{xy} \frac{\partial^2 w}{\partial x \partial y}
\end{aligned}
\tag{15.45}
$$

These relations are written in general for an orthotopic plate, i.e. a plate which has different elastic properties in two perpendicular directions, in which D_x and D_y are flexural rigidities in the x and y directions, respectively, D_1 is a "coupling" rigidity coefficient representing the Poisson's ratio type of effect and D_{xy} is the torsional rigidity.

For an isotropic plate which has the same properties in all directions, the flexural and twisting rigidities are given by

$$
\begin{aligned}
D_x = D_y = D &= \frac{Et^3}{12(1-v^2)} D \\
D_1 = vD \quad D_{xy} &= \frac{(1-v)}{2}
\end{aligned}
\tag{15.46}
$$

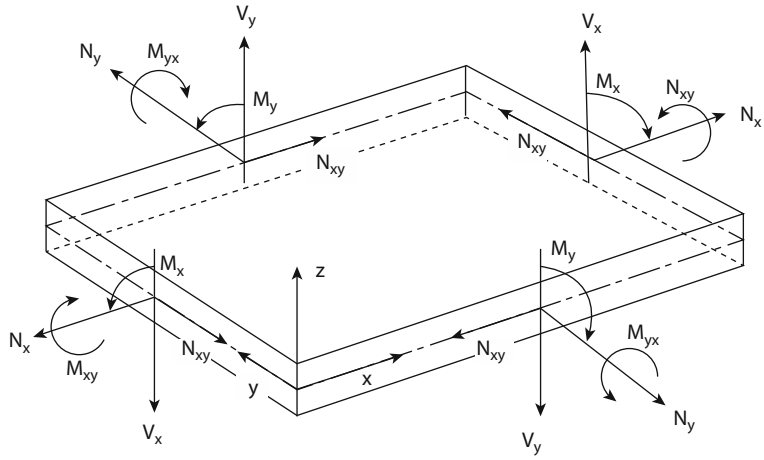

Fig. 15.8 Direction of force and moment per unit length as defined for thin shells

Equations (15.45) may be written in matrix form as

$$
\{\sigma(x,y)\} = \begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} D_x & D_1 & 0 \\ D_1 & D_y & 0 \\ 0 & 0 & D_{xy} \end{bmatrix} \begin{Bmatrix} -\dfrac{\partial^2 w}{\partial x^2} \\ -\dfrac{\partial^2 w}{\partial y^2} \\ 2\dfrac{\partial^2 w}{\partial x \partial y} \end{Bmatrix}
\tag{15.47}
$$

or symbolically as

$$\{\sigma(x,y)\} = [\boldsymbol{D}]\{\varepsilon(x,y)\} \tag{15.48}$$

where the matrix $[\boldsymbol{D}]$ is defined in Eq. (15.47). The substitution into Eq. (15.48) of $\{\varepsilon\ (x, y)\}$ from Eq. (15.43) results in the required relationship between element stresses and nodal displacements as

$$\{\sigma(x,y)\} = [\boldsymbol{D}][\boldsymbol{B}]\{q\}_e \tag{15.49}$$

Step VI: Element Stiffness and Mass Matrices
The stiffness matrix and the mass matrix for an element of plate bending obtained by applying the Principle of Virtual Work results in Eqs. (15.22) and (15.23), and for the equivalent forces due to the applied body forces on the element in Eq.(15.24). The matrices $[f\,(x, y)]$, $[\boldsymbol{B}]$, and $[\boldsymbol{D}]$ required in these equations are defined, respectively, in Eqs. (15.38), (15.43), and (15.47). The calculation of these matrices and also of the integral indicated in Eqs.(15.22), (15.23), and (15.24), is usually undertaken by numerical methods implemented in the coding of computer programs.

Step VII: Assemblage of the System Stiffness Matrix [K], the System Mass Matrix [M], and the Vector of the External Forces {F} at the System Nodal Coordinates {y}, which Includes the Equivalent Forces for the Body forces $\{P_b\}$ and for any Other Forces Distributed over the Structural Element.
The system stiffness and mass matrices, as well as the nodal force vector, due to applied body forces, are assembled from the corresponding element matrices and vector as given in Eqs. (15.25), (15.26), and (15.27).

Step VIII: Solution of the Differential Equations of Motion
The system differential equation of motion is given by Eq. (15.28) in which $[\boldsymbol{M}]$ and $[\boldsymbol{K}]$ are the assembled mass and stiffness matrices, respectively and $\{F\}$ is the equivalent vector of the external forces. The solution of Eq. (15.28) thus provides the system vector of nodal displacements $\{y\}$.

Step IX: Determination of Nodal Stresses
The element nodal stresses, $\{\sigma\ (x_j, y_j)\}_e$, for node j of an element are given from Eq. (15.49) as

$$\left\{\sigma\left(x_j, y_j\right)\right\}_e = [\boldsymbol{D}]_j[\boldsymbol{B}]_j\{\boldsymbol{q}\}_e \tag{15.50}$$

in which the matrices $[\boldsymbol{D}]_j$ and $[\boldsymbol{B}]_j$ are evaluated for the coordinates of node j of the element.

15.3 Summary

In this chapter we have presented an introduction to the Finite Element Method (FEM) for the analysis of problems in Structural Dynamics. The theory of FEM in structural dynamics was formulated through the following steps:

1. Modeling of the entire structure into one-dimensional, two-dimensional, or three-dimensional beam, rod, triangular, quadrilateral, rectangular, or other types of structural elements.
2. Identifying nodes and nodal coordinates at joints between structural elements.

3. Selecting an interpolating function, which usually is a polynomial to express the displacements at an interior point in the element in terms of the
   (a) displacements at its nodal coordinates.
4. Establishing the relationships at the nodal coordinates of a structural element between forces and displacements (the element stiffness matrix) and between forces and accelerations (the element mass matrix.)
5. Obtaining the vector of the equivalent nodal forces for the body or other external forces acting on the element.
6. Assembling the system stiffness matrix, the system mass matrix, and the system vector of the equivalent nodal forces, respectively, from the element stiffness matrices, element mass matrices, and the element vectors of the equivalent nodal forces.
7. Establishing the dynamic equilibrium at the system nodal coordinates, among which $[\boldsymbol{M}]$ and $[\boldsymbol{K}]$ are, respectively, the system mass matrix and the system stiffness matrix, $\{F(t)\}$ is the system vector of the equivalent nodal forces, and $\{\mathrm{u}\}$ and $\{\ddot{u}\}$ are, respectively, the displacement and the acceleration vectors at the system nodal coordinates, the elastic forces, inertial forces and the external forces to obtain the system differential equation of motion:

$$[\boldsymbol{M}]\{\ddot{u}\} + [\boldsymbol{K}]\{\mathrm{u}\} = \{\boldsymbol{F}(t)\} \quad (15.28 \text{ repeated})$$

8. Introducing in the system differential equation the boundary conditions restricting displacements at specified nodal coordinates.
9. Solving of the system differential equation of motion, Eq. (15.28), to obtain the system nodal displacements $\{u\}$.
10. Determining the element nodal stresses $\{\sigma\ (x_j,\ y_j)\}$ from the calculated element nodal displacements $\{q\}_e$.

In engineering practice, the solution of problems by the Finite Element Method is obtained with the computer and appropriate software, such as the computer program such as SAP2000 used in this text. In the implementation of the computer program such as SAP2000, items are selected from menus presented by the program and data is supplied in response at the prompts in the program.

15.4 Problems

Problem 15.1
The steel plate shown in Fig. P15.1 of dimensions 20 in $\times$ 20 in and thickness 0.10 in with a circular hole of radius $r = 50$ in is subjected to a suddenly applied in-plane lateral compressive pressure along the edges AD and BC of magnitude $p = 100$ psi. Model the plate with elements PLANE2D on a 6×6 mesh in each quarter section of the plate. Determine the first five natural frequencies and plots of the response for displacements and stresses on the elements.

Problem 15.2
Solve Problem 15.1 taking advantage of the double symmetry of geometry and load introducing appropriate boundary conditions and analyzing only a quarter of the structure.

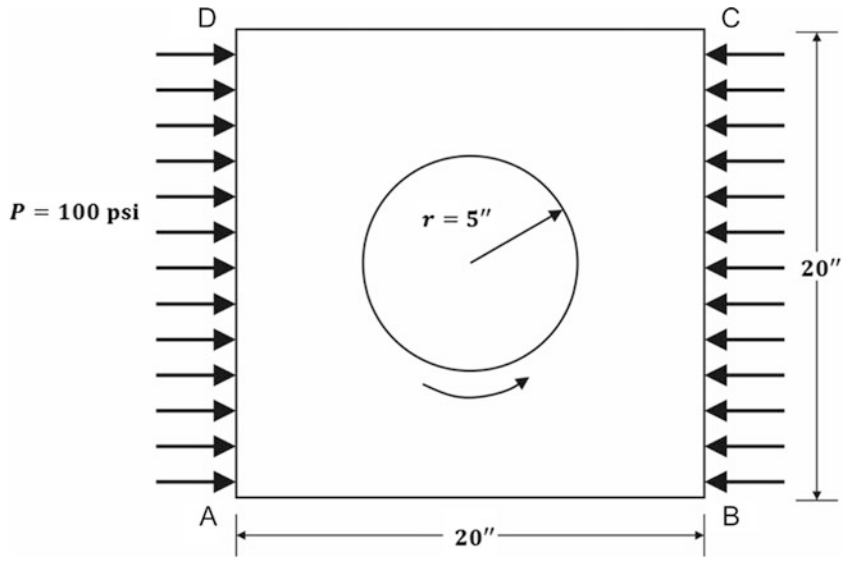

Fig. P15.1

Problem 15.3

A simple supported deep steel beam (Modulus of Elasticity, $E = 29E06$) shown in Fig. 15.6, with a distributed mass of 0.03 (lb.sec^2/in^2) is loaded at its center by the impulsive force F(t) depicted in Fig. 15.6b. Determine the first five natural periods and the time-displacement response at the center of the beam. Use computer program with rectangular shell elements to model this structure.

Problem 15.4

A square steel plate 40 in. by 40 in. and thickness 0.10 in., assumed to be fixed at the supports on its four sides (Fig. P15.4) is acted upon by a harmonic force $F(t) = 0.1 \sin 5.3\,t$ (kip) applied normally at its center. Determine: (a) the first five natural frequencies, (b) the time-displacement function at the center of the plate, and (c) the time-stress function at the center of plate. Use time step $\Delta t = 0.01$ s. ($E = 29{,}500$ ksi, $\upsilon = 0.3$).

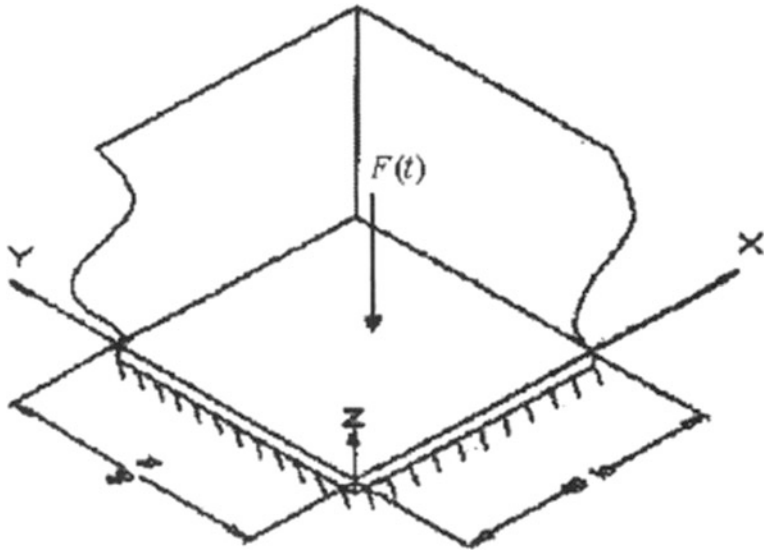

Fig P15.4 Square plate of Illustrative Example 15.2 supporting a normal harmonic force at the center.

16 Time History Response of Multi-Degree-of-Freedom Systems

The nonlinear analysis of a single-degree-of-freedom system using the step-by-step linear acceleration method was presented in Chap. 6. The extension of this method with a modification known as the Wilson-θ method, for the solution of structures modeled as multidegree-of-freedom systems is developed in this chapter. The modification introduced in the method by Wilson et al. 1973 serves to assure the numerical stability of the solution process regardless of the magnitude selected for the time step; for this reason, such a method is said to be *unconditionally stable*. On the other hand, without Wilson's modification, the step-by-step linear acceleration method is only conditionally stable and for numerical stability of the solution it may require such an extremely small time step as to make the method impractical if not impossible. The development of the necessary algorithm for the linear and nonlinear multidegree-of-freedom systems by the step-by-step linear acceleration method parallels the presentation for the single-degree-of-freedom system in Chap. 6.

Another well-known method for step-by-step numerical integration of the equations of motion of a discrete system is the Newmark beta method. This method which also may be considered a generalization of the linear acceleration method is presented later in this chapter after discussing in detail the Wilson-θ method.

16.1 Incremental Equations of Motion

The basic assumption of the Wilson-θ method is that the acceleration varies linearly over the time interval from t to $t + \theta\Delta\, t$. Where $\theta \geq 1.0$. The value of the factor θ is determined to obtain optimum stability of the numerical process and accuracy of the solution. It has been shown by Wilson that, for $\theta \geq 1.38$, the method becomes unconditionally stable.

The equation of motion evaluated at time t_i for a mutidegree-of-freedom system in matrix notation is given by

$$\mathbf{M}\ddot{\mathbf{u}}_\mathbf{i} + \mathbf{C}\left(\dot{\mathbf{u}}\right)\dot{\mathbf{u}}_\mathbf{i} + \mathbf{K}(\mathbf{u})\mathbf{u}_\mathbf{i} = \mathbf{F}_\mathbf{i}(t) \tag{16.1}$$

where **M, C** and **K** are, respectivly the mass, damping and stiffness matrices of the system; $\mathbf{u_i}, \dot{\mathbf{u}}_\mathbf{i}, \ddot{\mathbf{u}}_\mathbf{i}$ the displacement, velocity and acceleration vectors; and $\mathbf{F_i}$(t) the external force vector.

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M. Paz, Y. H. Kim, *Structural Dynamics*, https://doi.org/10.1007/978-3-319-94743-3_16

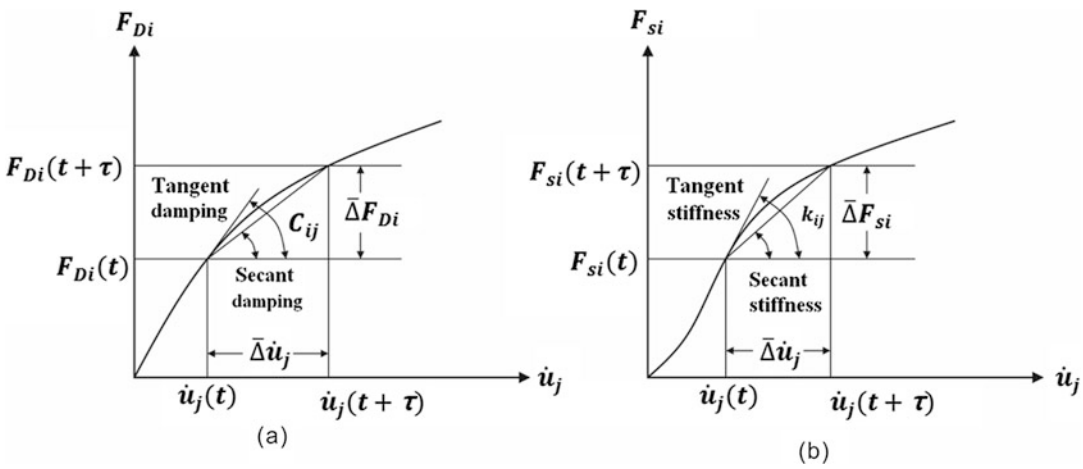

Fig. 16.1 Definition of influence coefficients, (**a**) Nonlinear viscous damping, c_{ij}. (**b**) Nonlinear stiffness, k_{ij}

Then the equations expressing the incremental equilibrium conditions for a multidegree-of-freedom system can be derived as the matrix[1] equivalent of the incremental equation of motion for the single degree-of-freedom system, Eq. (6.6). Thus taking the difference between dynamic equilibrium conditions defined at time t_i and at $t_i + \tau$, where $\tau = \theta\Delta\ \ t$, we obtain the incremental equations of motion:

$$\mathbf{M}\widehat{\Delta}\ddot{\mathbf{u}}_\mathbf{i}+\mathbf{C}(\dot{\mathbf{u}})\widehat{\Delta}\dot{\mathbf{u}}_\mathbf{i}+\mathbf{K}(\mathbf{u})\widehat{\Delta}\mathbf{u}_\mathbf{i}=\widehat{\Delta}\mathbf{F}_\mathbf{i} \tag{16.2}$$

in which the circumflex over Δ indicates that the increments are associated with the extended time step $\tau = \theta\Delta\ \ t$. Thus

$$\widehat{\Delta}u_i = u(t_i+\tau) - u(t_i) \tag{16.3}$$

$$\widehat{\Delta}\dot{u}_i = \dot{u}(t_i+\tau) - \dot{u}(t_i) \quad \ldots \tag{16.4}$$

$$\widehat{\Delta}\ddot{u}_i = \ddot{u}(t_i+\tau) - \ddot{u}(t_i) \tag{16.5}$$

and

$$\widehat{\Delta}F_i = F(t_i+\tau) - F(t_i) \tag{16.6}$$

In writing Eq. (16.2), we assumed, as explained in Chap. 6 for single-degree-of-freedom systems, that the stiffness and damping are obtained for each time step as the initial values of the tangent to the corresponding curves as shown in Fig. 16.1 rather than the slope of the secant line which requires iteration. Hence the stiffness coefficient is defined as

$$k_{ij} = \frac{dF_{si}}{du_j} \tag{16.7}$$

and the damping coefficient as

[1] Matrices and vectors are denoted with boldface lettering throughout this chapter.

$$c_{ij} = \frac{dF_{Di}}{d\dot{u}_j} \tag{16.8}$$

in which F_{si}, and F_{Di} are, respectively, the elastic and damping forces at nodal coordinate i and u_j and u_j are, respectively, the displacement and velocity at nodal coordinate j,.

16.2 The Wilson-θ Method

The integration of the nonlinear equations of motion by the step-by-step linear acceleration method with the extended time step introduced by Wilson is based, as has already been mentioned, on the assumption that the acceleration may be represented by a linear function during the time step $\tau = \theta \Delta t$ as shown in Fig. 16.2. From this figure we can write the linear expression for the acceleration during the extended time step as

$$\ddot{u}(t) = \ddot{u}_i + \frac{\widehat{\Delta}\ddot{u}_i}{\tau}(t - t_i) \tag{16.9}$$

in which $\widehat{\Delta}\ddot{u}_i$ is given by Eq. (16.5). Integrating Eq. (16.9) twice between limits t_i and t yields

$$\dot{u}(t) = \dot{u}_i + \ddot{u}_i(t - t_i) + \frac{1}{2}\frac{\widehat{\Delta}\ddot{u}_i}{\tau}(t - t_i)^2 \tag{16.10}$$

and

$$u(t) = u_i + \dot{u}_i(t - t_i) + \frac{1}{2}\ddot{u}_i(t - t_i)^2 + \frac{1}{6}\frac{\widehat{\Delta}\ddot{u}_i}{\tau}(t - t_i)^3 \tag{16.11}$$

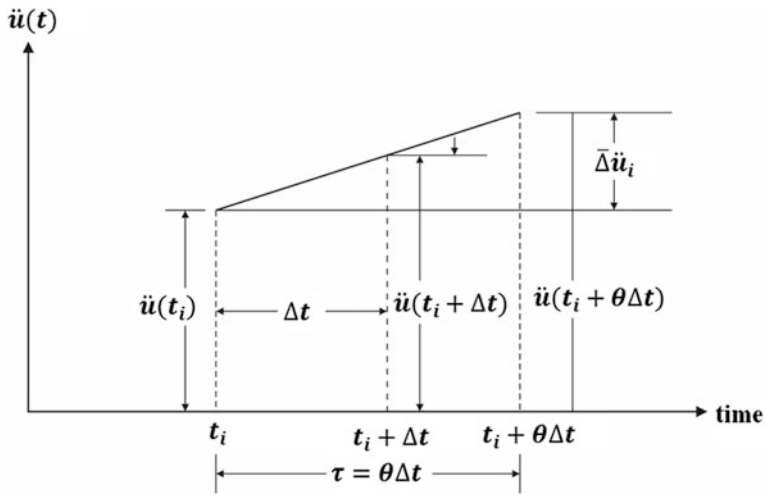

Fig. 16.2 Linear acceleration assumption in the extended time interval

Evaluation of Eqs. (16.10) and (16.11) at the end of the extended interval $t = t_i + \tau$ gives

$$\widehat{\Delta}\dot{u}_i = \ddot{u}_i\tau + \frac{1}{2}\widehat{\Delta}\ddot{u}_i\tau \tag{16.12}$$

and

$$\widehat{\Delta}u_i = \dot{u}_i\tau + \frac{1}{2}\ddot{u}_i\tau^2 + \frac{1}{6}\widehat{\Delta}\ddot{u}_i\tau^2 \tag{16.13}$$

in which $\widehat{\Delta}u_i$ and $\widehat{\Delta}\dot{u}_i$ are defined by Eqs. (16.3) and (16.4), respectively.

Now Eq. (16.13) is solved for the incremental acceleration $\widehat{\Delta}\ddot{u}_i$ and substituted in Eq. (11.12) to obtain:

$$\widehat{\Delta}\ddot{u}_i = \frac{6}{\tau^2}\widehat{\Delta}u_i - \frac{6}{\tau}\dot{u}_i - 3\ddot{u}_i \tag{16.14}$$

and

$$\widehat{\Delta}\dot{u}_i = \frac{3}{\tau}\widehat{\Delta}u_i - 3\dot{u}_i - \frac{\tau}{2}\ddot{u}_l \tag{16.15}$$

Finally, substituting Eqs. (16.14) and (16.15) into the incremental equation of motion, Eq. (16.2), results in an equation for the incremental displacement $\widehat{\Delta}u_i$, which may be conveniently written as

$$\bar{K}_i\widehat{\Delta}u_i = \overline{\widehat{\Delta}F_i} \tag{16.16}$$

where

$$\bar{K}_i = K_i + \frac{6}{\tau^2}M + \frac{3}{\tau}C_i \tag{16.17}$$

and

$$\overline{\widehat{\Delta}F_i} = \widehat{\Delta}F_i + M\left(\frac{6}{\tau}\dot{u}_i + 3\ddot{u}_i\right) + C_i\left(3\dot{u}_i + \frac{\tau}{2}\ddot{u}_i\right) \tag{16.18}$$

the matrix equation (16.16) has the same form as the static incremental equilibrium equation and may be solved for the incremental displacements $\widehat{\Delta}u_i$ by simply solving a system of linear equations.

To obtain the incremental accelerations $\widehat{\Delta}\ddot{u}_i$ for the extended time interval, the value of $\widehat{\Delta}u_i$ obtained from the solution of Eq. (16.16) is substituted into Eq. (16.14). The incremental acceleration $\widehat{\Delta}\ddot{u}_i$ for the normal time interval Δt is then obtained by a simple linear interpolation. Hence

$$\Delta\ddot{u} = \frac{\widehat{\Delta}\ddot{u}}{\theta} \tag{16.19}$$

To calculate the incremental velocity $\Delta\dot{u}_i$ and incremental displacement Δu_i, corresponding to the normal interval Δt, use is made of Eqs. (16.12) and (16.13) with the extended time interval parameter τ substituted for Δt, that is,

$$\Delta\dot{u}_i = \ddot{u}_i\Delta t + \frac{1}{2}\Delta\ddot{u}_i\Delta t \tag{16.20}$$

and

$$\Delta u_i = \dot{u}_i\Delta t + \frac{1}{2}\ddot{u}_i\Delta t^2 + \frac{1}{6}\Delta\ddot{u}_i\Delta t^2 \tag{16.21}$$

Finally, the displacement u_{i+1} and velocity $\dot{u}_{i+1}$ at the end of the normal time interval are calculated by

$$u_{i+1} = u_i + \Delta u_i \tag{16.22}$$

and

$$\dot{u}_{i+1} = \dot{u}_i + \Delta\dot{u}_i \tag{16.23}$$

As mentioned in Chap. 6 for the single degree-of-freedom system, the initial acceleration for the next step is calculated from the condition of dynamic equilibrium at the time $t + \Delta t$; thus

$$\ddot{u}_{i+1} = M^{-1}\left[F_{i+1} - F_D\left(\dot{u}_{i+1}\right) - F_S(u_{i+1})\right] \tag{16.24}$$

in which $F_D\left(\dot{u}_{i+1}\right)$ and $F_S(u_{i+1})$ represent, respectively, the damping force and stiffness force vectors evaluated at the end of the time step $t_{i+1} = t_i + \Delta t$. Once the displacement, velocity, and acceleration vectors have been determined at time $t_{i+1} = t_i + \Delta t$, the outlined procedure is repeated to calculate these quantities at the next time step $t_{i+2} = t_{i+1} + \Delta t$ and the process is continued to any desired final time.

The step-by-step linear acceleration, as indicated in the discussion for the single degree-of-freedom system, involves two basic approximations: (1) the acceleration is assumed to vary linearly during the time step, and (2) the damping and stiffness characteristics of the structure are evaluated at the initiation of the time step and are assumed to remain constant during this time interval. The algorithm for the integration process of a linear system by the Wilson-8 method is outlined in the next section. The application of this method to linear structures is then developed in the following section.

16.3 Algorithm for Step-by-Step Solution of a Linear System Using the Wilson-θ Method

16.3.1 Initialization

1. Assemble the system stiffness matrix $\boldsymbol{K}$, mass matrix $\boldsymbol{M}$, and damping matrix $\mathbf{C}$.
2. Set initial values for displacement u_0, velocity $\dot{u}_0$, and forces F_0.
3. Calculate initial acceleration $\ddot{u}_0$ from Eq. (16.1) as

$$M\ddot{u}_0 = F_0 - C\dot{u}_0 - Ku_0$$

4. Select a time step Δt, the factor θ (usually Taken as 1.4), and calculate the constants τ, a1, a2, a3 and a4 from the relationships:

$$\tau = \theta\Delta t, \quad a_1 = \frac{3}{\tau}, \quad a_2 = \frac{6}{\tau}, \quad a_3 = \frac{\tau}{2}, \quad a_4 = \frac{6}{\tau^2},$$

5. Form the effective stiffness matrix $\boldsymbol{K}$ [Eq. (16.17)], namely,

$$\bar{K} = K + a_4 M + a_1 C$$

16.3.2 For Each Time Step

1. Calculate by linear interpolation the incremental load $\widehat{\Delta}F_i$ for the time interval t_i to $t_i + \tau$, from the relationship

$$\widehat{\Delta}F_i = F_{i+1} + (F_{i+2} - F_{i+1})(\theta - 1) - F_i$$

2. Calculate the effective incremental load $\overline{\widehat{\Delta}F_i}$; for the time interval ti to ti,+ τ, from Eq. (16.18) as

$$\overline{\widehat{\Delta}F_i} = \widehat{\Delta}F_i + (a_2 M + 3C)\dot{u}_i + (3M + a_3 C)\ddot{u}_i$$

3. Solve for the incremental displacement $\widehat{\Delta}u_i$, from Eq. (16.16) as

$$\bar{K}\widehat{\Delta}u_i = \overline{\widehat{\Delta}F_i}$$

4. Calculate the incremental acceleration for the extended time interval τ, from the relation Eq. (16.14) as

$$\widehat{\Delta}\ddot{u}_i = a_4\widehat{\Delta}u_i - a_2\dot{u}_i - 3\ddot{u}_i$$

5. Calculate the incremental acceleration for the normal interval from Eq. (16.19) as

$$\Delta\ddot{u} = \frac{\widehat{\Delta}\ddot{u}}{\theta}$$

6. Calculate the incremental velocity $\Delta\dot{u}_i$, and the incremental displacement Δu_i, from time ti to ti + Δt from Eqs. (16.20) and (16.21) as

$$\Delta\dot{u}_i = \ddot{u}_i\Delta t + \frac{1}{2}\Delta\ddot{u}_i\Delta t$$

$$\Delta u_i = \dot{u}_i\Delta t + \frac{1}{2}\ddot{u}_i\Delta t^2 + \frac{1}{6}\Delta\ddot{u}_i\Delta t^2$$

7. Calculate the displacement and velocity at time $ti + 1 = ti + \Delta t$ using

$$u_{i+1} = u_i + \Delta u_i$$

$$\dot{u}_{i+1} = \dot{u}_i + \Delta\dot{u}_i$$

8. Calculate the acceleration $\ddot{u}_{i+1}$ at time $t_{i+1} = t_i + \Delta t$ directly from the equilibrium equation of motion, namely,

$$M\ddot{u}_{i+1} = F_{i+1} - C\dot{u}_{i+1} - Ku_{i+1}$$

Illustrative Example 16.1

Calculate the displacement response for a two-story shear building of Fig. 16.3 subjected to a suddenly applied force of 10 Kip at the level of the second floor. Neglect damping and assume elastic behavior.

Solution

The equations of motion, in matrix notation, for this structure are:

$$\begin{bmatrix} 0.136 & 0 \\ 0 & 0.066 \end{bmatrix}\begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{Bmatrix} + \begin{bmatrix} 75.0 & -44.3 \\ -44.3 & 44.3 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 10 \end{Bmatrix}$$

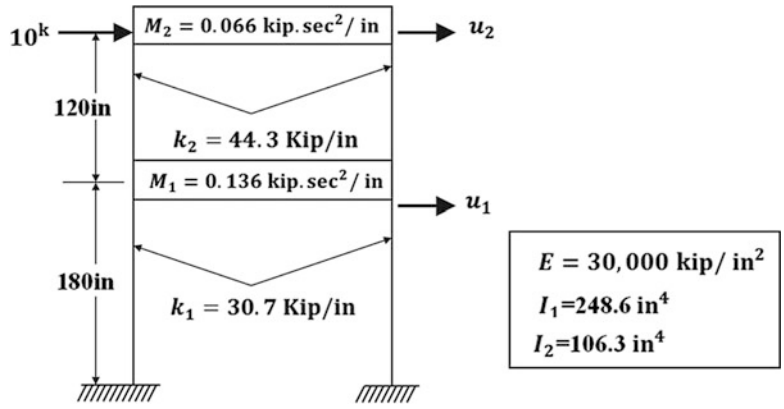

Fig. 16.3 Two-story shear building for Illustrative Examples 16.1 and 16.2

which, for free vibration, become

$$\begin{bmatrix} 0.136 & 0 \\ 0 & 0.066 \end{bmatrix} \begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{Bmatrix} + \begin{bmatrix} 75.0 & -44.3 \\ -44.3 & 44.3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 10 \end{Bmatrix}$$

Substitution of $y_i = a_i \, sin \, \omega t$ results in the eigenproblem:

$$\begin{bmatrix} 75.0 - 0.136\omega^2 & -44.3 \\ -44.3 & 44.3 - 0.066\omega^2 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

which requires for a nontrivial solution that

$$\begin{vmatrix} 75.0 - 0.136\omega^2 & -44.3 \\ -44.3 & 44.3 - 0.0666\omega^2 \end{vmatrix} = 0$$

Expansion of this determinant yields

$$\omega^4 - 1222.68\omega^2 + 151516 = 0$$

which has the roots

$$\omega_1^2 = 139.94 \qquad \text{and} \qquad \omega_2^2 = 1082.0$$

Hence, the natural frequencies are

$$\omega_1 = 11.83 \text{ rad/ sec}, \qquad \text{and} \qquad \omega_2 = 32.90 \text{ rad/ sec},$$

or

$$f_1 = 1.883 \text{ cps}, \qquad \text{and} \qquad f_2 = 5.237 \text{ cps}$$

and the natural periods

$$T_1 = 0.531 \text{ sec}, \qquad \text{and} \qquad T_2 = 0.191 \text{ sec}$$

The initial acceleration at the nodal coordinates is calculated from Eq. (16.1) after setting the initial displacement and velocity equal to zero. Thus we obtain:

$$\begin{bmatrix} 0.136 & 0 \\ 0 & 0.066 \end{bmatrix} \begin{Bmatrix} \ddot{u}_{10} \\ \ddot{u}_{20} \end{Bmatrix} \begin{bmatrix} 75.0 & -44.3 \\ -44.3 & 44.3 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 10 \end{Bmatrix}$$

giving

$$\ddot{u}_{10} = 0$$

$$\ddot{u}_{20} = 151.51 \text{ in/ sec}^2$$

Conveniently, we select $\Delta t = 0.02$ sec and $\theta = 1.4$, so $\tau = \theta \Delta t = 0.028$, and calculate the constants to obtain:

$$a_1 = \frac{3}{\tau} = 107.14, \qquad a_2 = \frac{6}{\tau} = 0.014$$
$$a_3 = \frac{\tau}{2} = 214.28, \qquad a_4 = \frac{6}{\tau^2} = 7653$$

The effective stiffness is then

$$\bar{K} = K + a_4 M + a_1 C \qquad (C = 0 \quad \text{foranundampedsysteem})$$

$$\bar{K} = \begin{bmatrix} 75.0 & -44.3 \\ -44.3 & 44.3 \end{bmatrix} + 7653 \begin{bmatrix} 0.136 & 0 \\ 0 & 0.066 \end{bmatrix}$$

$$\bar{K} = \begin{bmatrix} 1115.8 & -44.3 \\ -44.3 & 549.4 \end{bmatrix}$$

and the effective force

$$\overline{\widehat{\Delta} F_i} = \widehat{\Delta} F_i + (a_2 M + 3C)\dot{u}_i + (3M + a_3 C)\ddot{u}_i$$

$$\overline{\widehat{\Delta} F_i} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} + 21428 \begin{bmatrix} 0.136 & 0 \\ 0 & 0.066 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} + 3 \begin{bmatrix} 0.136 & 0 \\ 0 & 0.066 \end{bmatrix} \begin{Bmatrix} 0 \\ 151.51 \end{Bmatrix}$$

$$\overline{\widehat{\Delta} F_i} = \begin{Bmatrix} 0 \\ 30 \end{Bmatrix}$$

Solving for $\widehat{\Delta} u$ from $\bar{K} \widehat{\Delta} u = \overline{\widehat{\Delta} F}$ yields

$$\begin{bmatrix} 1115.8 & -44.3 \\ -44.3 & 549.4 \end{bmatrix} \begin{Bmatrix} \widehat{\Delta} u_1 \\ \widehat{\Delta} u_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 30 \end{Bmatrix}, \qquad \widehat{\Delta} u = \begin{Bmatrix} 0.002175 \\ 0.054780 \end{Bmatrix}$$

Solving for $\widehat{\Delta} \ddot{u}$ from Eq. (16.14) we obtain

$$\widehat{\Delta} \ddot{u} = 7656 \begin{Bmatrix} 0.002175 \\ 0.054780 \end{Bmatrix} - 214.28 \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} - 3 \begin{Bmatrix} 0 \\ 151.51 \end{Bmatrix}, \quad \widehat{\Delta} \ddot{u} = \begin{Bmatrix} 16.645 \\ -35.299 \end{Bmatrix}$$

Then

$$\Delta \ddot{u} = \frac{\widehat{\Delta} \ddot{u}}{\theta} = \frac{1}{1.4} \begin{Bmatrix} 16.647 \\ -35.299 \end{Bmatrix} = \begin{Bmatrix} 18.891 \\ -25.21 \end{Bmatrix}$$

From Eq. (16.20), it follows that

$$\Delta \dot{u} = \begin{Bmatrix} 0 \\ 151.51 \end{Bmatrix} (0.02) + \frac{0.02}{2} \begin{Bmatrix} 11.891 \\ -25.21 \end{Bmatrix} = \begin{Bmatrix} 0.1189 \\ 2.7781 \end{Bmatrix}$$

From Eq. (16.21),

$$\Delta u = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}(0.02) + \frac{(0.02)^2}{2}\begin{Bmatrix} 0 \\ 151.61 \end{Bmatrix} + \frac{(0.02)^2}{6}\begin{Bmatrix} 11.891 \\ -25.21 \end{Bmatrix} = \begin{Bmatrix} 0.0008 \\ 0.0286 \end{Bmatrix}$$

From Eqs. (16.21) and (16.23)

$$\{u\}_1 = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} + \begin{Bmatrix} 0.0008 \\ 0.0286 \end{Bmatrix} = \begin{Bmatrix} 0.0008 \\ 0.0286 \end{Bmatrix} \tag{a}$$

and

$$\{\dot{u}\}_1 = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} + \begin{Bmatrix} 0.1189 \\ 2.7781 \end{Bmatrix} = \begin{Bmatrix} 0.1189 \\ 2.7781 \end{Bmatrix} \tag{b}$$

From Eq. (16.23),

$$\begin{bmatrix} 0.136 & 0 \\ 0 & 0.066 \end{bmatrix}\{\ddot{u}\}_1 = \begin{Bmatrix} 0 \\ 10 \end{Bmatrix} - \begin{bmatrix} 75.0 & -44.3 \\ -44.3 & 44.3 \end{bmatrix}\begin{Bmatrix} 0.0008 \\ 0.0286 \end{Bmatrix}$$

Which gives

$$\{\ddot{u}\}_1 = \begin{Bmatrix} 8.875 \\ 132.85 \end{Bmatrix} \tag{c}$$

The results given in Eqs. (a), (b), and (c) for the displacement, velocity, and acceleration, respectively, at time $t_1 = t_0 + \Delta t$ complete a first cycle of the integration process. The continuation in determining the response for this structure is given in Illustrative Example 16.2 with the use of the computer program described in the next section.

16.4 Response by Step Integration Using MATLAB

MATLAB performs the step-by-step integration of the equations of motion for a linear system using the linear acceleration method with the Wilson-θ modification. The program requires previous modeling of the structure to determine the stiffness matrix and the mass matrix of the system.

The program performs a linear interpolation between the load data points, which result in the magnitude of the applied forces at each nodal coordinate calculated at increments of time equal to the time step Δt. The output consists of the response for each nodal coordinate in terms of displacement, velocity, and acceleration at increments of time Δt up to the maximum time specified by the duration of the forces including, if desired, an extension with forces set to zero.

Illustrative Example 16.2

Use MATLAB to determine the response of the two-story shear building shown in Fig. 16.3. The first cycle of the integration process for this structure has been hand calculated in Illustrative Example 16.1. This is confirmed by the MATLAB program as follows:

Solution:

```
clc
clear all
close all

%_____________________________________________________________________
% Inputs:
%   M, K
%   F = forcing function
%   t = Time period
%   u0 = initial displacement
%   v0 = initial velocity
%   w0 = initial acceleration
%_____________________________________________________________________

%%%%-GIVEN VALUES-%%%%

deltat = 0.02;  % ?t=0.02

%%%Define Mass Matrix
M = [136 0;
      0 66];

%%%Define Stiffness Matrix
 k1=30700;
 k2=44300;

 K = [k1+k2 -k2;
     -k2 k2];

%%%Define Damping Matrix
C =0;

[n,n]= size(M);

%%%Define the load vector
F = zeros(n,1); F(2) =10000;
%_____________________________________________________________________
% Initial conditions
%_____________________________________________________________________

u0 = zeros(n,1); u0(1) =0;
v0 = zeros(n,1); v0(1) =0;
w0 = inv(M)*(F-K*u0-C*v0);              % acceleration at t = 0

  t = 0;

u_t1(:,1)=u0(1,:)';
u_t2(:,1)=u0(1,:)';

v_t1(:,1)=v0(1,:)';
v_t2(:,1)=v0(2,:)';

w_t1(:,1)=w0(1,:)';
w_t2(:,1)=w0(2,:)';

%_____________________________________________________________________
% For each time step apply the Wilson-? Method to calculate u, v, and w
%_____________________________________________________________________
```

```
for i=2:100

[u,v,w,t] = STI(t, deltat, M, K, C, F, u0, v0, w0);

ti(:,i)=t(:,1)';

u_t1(:,i) = u(1,:)';
u_t2(:,i) = u(2,:)';

v_t1(:,i) = v(1,:)';
v_t2(:,i) = v(2,:)';

w_t1(:,i) = w(1,:)';
w_t2(:,i) = w(2,:)';

u0=u;                          %u_(i) for calculating u_(i+1)
v0=v;                          %v_(i) for calculating v_(i+1)
w0=w;                          %w_(i) for calculating w_(i+1)
F=F;                           %F(i) for calculating F(i+1)
t=t;                           %t(i) for calculating t(i+1)
end

figure(1)

subplot(3,2,1)
plot (ti, u_t1);
title ('1DOF'); xlabel ('Time (sec)'); ylabel ('u_1(in.)'); grid on

subplot(3,2,2)
plot (ti, u_t2);
title ('2DOF'); xlabel ('Time (sec)'); ylabel ('u_2(in.)'); grid on

subplot(3,2,3)
plot (ti, v_t1);
title ('1DOF'); xlabel ('Time (sec)'); ylabel ('v_1(in./sec)'); grid on

subplot(3,2,4)
plot (ti, v_t2);
title ('2DOF'); xlabel ('Time (sec)'); ylabel ('v_2(in./sec)'); grid on

subplot(3,2,5)
plot (ti, w_t1);
title ('1DOF'); xlabel ('Time (sec)'); ylabel ('w_1(in./sec^2)'); grid on

subplot(3,2,6)
plot (ti, w_t2);
title ('2DOF'); xlabel ('Time (sec)'); ylabel ('w_2(in./sec^2)'); grid on

u_1max= max(abs(u_t1))          %Max. displacement @ 1DOF (or 1st story)
u_2max= max(abs(u_t2))          %Max. displacement @ 2DOF (or 2nd story)

v_1max= max(abs(v_t1))          %Max. velocity @ 1DOF (or 1st story)
v_2max= max(abs(v_t2))          %Max. velocity @ 2DOF (or 2nd story)

w_1max= max(abs(w_t1))          %Max. acceleration @ 1DOF (or 1st story)
w_2max= max(abs(w_t2))          %Max. acceleration @ 2DOF (or 2nd story)
```

For each time step, the function file of STI.m is used to calculate the displacement, velocity, and acceleration using the Wilson-θ Method (Sect. 16.3).

```
function [u,v,w,t]=STI(t, deltat, M, K, C, F,u0,v0,w0)

%_____________________________________________________________________
%  The Wilson-? Method
%   t = Time period
%   u = displacement
%   v = velocity
%   w = acceleration
%_____________________________________________________________________

%%%Calculate the constants ?, a1, a2, a3 and a4 from the relationships
theta = 1.4;
tau = theta*deltat;

a1 = 3/tau; a2 = 6/tau; a3 = tau/2; a4 = 6/tau^2;            %16.3.1 (4)

K_bar = K+a1.*C+a4.*M;                                       %16.3.1.(5)

%%%For each time step (Ch 16.3.2)
F_eff0 = F+(F-F)*(theta-1)-F;                                %16.3.2.(1)
F_eff = F_eff0+(a2.*M+3*C)*v0+(3.*M+a3.*C)*w0;               %16.3.2.(2)

delta_u= inv(K_bar)*F_eff;                                   %16.3.2.(3)

delta_w =1./theta*(a4.*delta_u-a2.*v0-3*w0);                 %16.3.2.(4)&(5)

delta_v = deltat.*w0+deltat./2.*delta_w;                     %16.3.2.(6)
delta_u = deltat.*v0+deltat^2./2.*w0+deltat^2./6.*delta_w;   %16.3.2.(6)

u = u0+delta_u;                                              %16.3.2.(7)
v = v0+delta_v;
w = inv(M)*(F-K*u-C*v);
t = t+deltat;
```

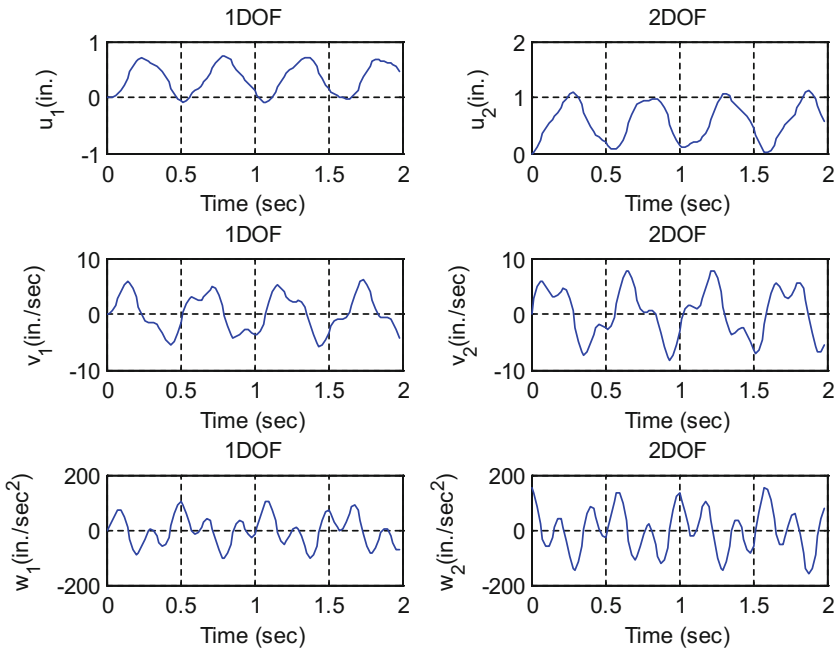

Fig. 16.4 Responses of displacement, velocity, and acceleration

16.5 The Newmark Beta Method

The Newmark beta method may be considered a generalization of the linear acceleration method. It uses a numerical parameter designated as β. The method, as originally proposed by Newmark (1959), contained in addition to β, a second parameter γ. These parameters replace the numerical coefficients $\frac{1}{2}$ and $\frac{1}{6}$ of the terms containing the incremental acceleration $\Delta\ddot{u}_i$, in Eqs. (16.20) and (16.21), respectively. Thus, replacing by γ the coefficient $\frac{1}{2}$ of $\Delta\ddot{u}_i$, in Eq. (16.20) and by β the coefficient $\frac{1}{6}$ also of $\Delta\ddot{u}_i$, in Eq. (16.20), we have

$$\Delta\dot{u}_i = \ddot{u}_i\Delta t + \gamma\Delta\ddot{u}_i\Delta_t \tag{16.25}$$

and

$$\Delta u_i = \dot{u}_i\Delta t + \frac{1}{2}\ddot{u}_i\Delta t^2 + \beta\Delta\ddot{u}_i\Delta t^2 \tag{16.26}$$

It has been found that for values of γ different than $\frac{1}{2}$, the method introduces a superfluous damping in the system. For this reason this parameter is generally set as $\gamma = \frac{1}{2}$. The solution of Eq. (16.26) for $\Delta\ddot{u}_i$, and its substitution into Eq. (16.25) after setting $\gamma = \frac{1}{2}$ yield

$$\Delta\dot{u}_i = \frac{1}{\beta\Delta t^2}\Delta u_i - \frac{1}{\beta\Delta t}\dot{u}_i - \frac{1}{2\beta}\ddot{u}_i \tag{16.27}$$

$$\Delta\dot{u}_i = \frac{1}{2\beta\Delta t}\Delta u_i - \frac{1}{2\beta}\dot{u}_i + \left(1 - \frac{1}{4\beta}\right)\Delta t\ddot{u}_i \tag{16.28}$$

Then the substitution of Eqs. (16.27) and (16.28) into the incremental equation of motion

$$M\Delta\ddot{u}_i + C_i\Delta\dot{u}_i + K_i\Delta u_i = \Delta F_i \tag{16.29}$$

results in an equation to calculate the incremental displacement Δu_i, namely

$$\bar{K}_i\Delta u_i = \Delta\bar{F}_i \tag{16.30}$$

where the effective stiffness matrix $\bar{K}_i$, and the effective incremental force vector $\Delta\bar{F}_i$ are given respectively by

$$\bar{K}_i = K_i + \frac{M}{\beta\Delta t^2} + \frac{C_i}{2\beta\Delta t} \tag{16.31}$$

and

$$\Delta\bar{F}_i = \Delta F_i + \frac{M}{\beta\Delta t}\dot{u}_i + \frac{C_i}{2\beta}\dot{u}_i + \frac{M}{2\beta}u_i - C_i\Delta t\left(1 - \frac{1}{4\beta}\right)\ddot{u}_i \tag{16.32}$$

In these equations C_i and K_i are respectively the damping and stiffness matrices with coefficients evaluated at the initial time t_i of the time step $\Delta t = t_{i+1} - t_i$.

In the implementation of the Newmark beta method, the process begins by selecting a numerical value for the parameter β. Newmark suggested a value in the range $\frac{1}{6} \leq \beta \leq \frac{1}{2}$. For $\beta = \frac{1}{6}$ the method is exactly equal to the linear acceleration method and is only conditionally stable.

For $\beta = \frac{1}{4}$ the method is equivalent to assuming that the velocity varied linearly during the time step, which would require that the mean acceleration is maintained for the interval. In this last case, that is, $\beta = \frac{1}{4}$, the Newmark beta method is unconditionally stable and it provides the satisfactory accuracy.

16.6 Elastoplastic Behavior of Framed-Structures

The dynamic analysis of beams and frames having linear elastic behavior was presented in the preceding chapters. To extend this analysis to structures whose members may be strained beyond the yield point of the material, it is necessary to develop the member stiffness matrix for the assumed elastoplastic behavior. The analysis is then carried out by a step-by-step numerical integration of the differential equations of motion. Within each short time interval Δt, the structure is assumed to behave in a linear elastic manner, but the elastic properties of the structure are changed from one interval to another as dictated by the response. Consequently, the nonlinear response is obtained as a sequence of linear responses of different elastic systems. For each successive interval, the stiffness of the structure is evaluated based on the moments in the members at the beginning of the time increment.

The changes in displacements of the linear system are computed by integration of the differential equations of motion over the finite interval and the total displacements by addition of the incremental displacement to the displacements calculated in the previous time step. The incremental displacements are also used to calculate the increment in member end forces and moments from the member stiffness equation. The magnitude of these end moments relative to the yield conditions (plastic moments) determines the characteristics of the stiffness and mass matrices to be used in the next time step.

16.7 Member Stiffness Matrix

If only bending deformations are considered, the force-displacement relationship for a uniform beam segment (Fig. 16.4) with elastic behavior (no hinges) is given by Eq. (10.20). This equation may be written in incremental quantities as follows:

$$\begin{Bmatrix} \Delta P_1 \\ \Delta P_2 \\ \Delta P_3 \\ \Delta P_4 \end{Bmatrix} = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \begin{Bmatrix} \Delta\delta_1 \\ \Delta\delta_2 \\ \Delta\delta_3 \\ \Delta\delta_4 \end{Bmatrix} \tag{16.33}$$

in which ΔP_i and $\Delta\delta_i$, are, respectively, the incremental forces and the incremental displacements at the nodal coordinates of the beam segment. When the moment at one end of the beam reaches the value of the plastic moment M_p, a hinge is formed at that end. Under the assumption of an

elastoplastic relation-between the bending moment and the angular displacement as depicted in Fig. 16.5, the section that has been transformed into a hinge cannot support a moment higher than the plastic moment M_p but it may continue to deform plastically at a constant moment M_p. The relationship reverses to an elastic behavior when the angular displacement begins to decrease as shown in Fig. 16.5. We note the complete similarity for the behavior between an elastoplastic spring (Fig. 6.5) in a single degree-of-freedom system and an elastoplastic section of a beam (Fig. 16.5).

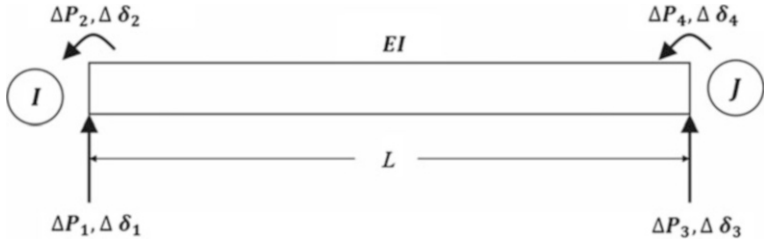

Fig. 16.5 Beam segment indicating incremental end forces and corresponding incremental displacements

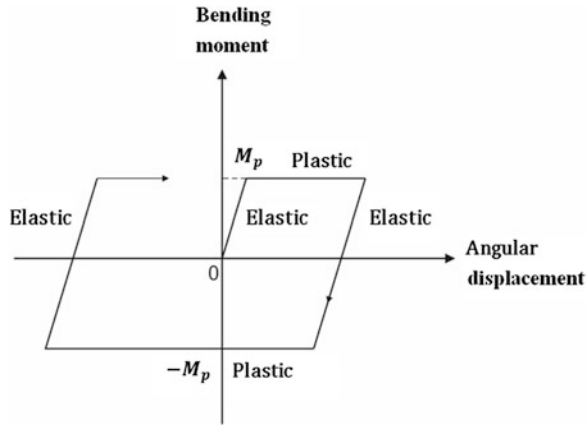

Fig. 16.6 Elastoplastic relationship between bending moment and angular displacement at a section of a beam

The stiffness matrix for a beam segment with a hinge at one end (Fig. 16.6) may be obtained by application of Eq. (10.16) which is repeated here for convenience, namely

$$k_{ij} = \int_0^L EI\, \mathrm{N}_i''(x)\mathrm{N}_j''(x)dx \tag{16.34}$$

Where $\psi_i(x)$ and $\psi_j(x)$ are displacement functions. For a uniform beam in which the formation of the plastic hinge takes place at end Θ as shown in Fig. 16.6, the deflection functions corresponding to unit displacement at one of the nodal coordinates δ_1, δ_2, δ_3, or δ_4 are given respectively by

$$N_1(x) = 1 - \frac{3x}{2L} + \frac{x^3}{2L^3} \tag{16.35a}$$

$$N_2(x) = 0 \tag{16.35b}$$

$$N_3(x) = \frac{3x}{2L} + \frac{x^3}{2L^3} \tag{16.35c}$$

$$N_4(x) = -\frac{x}{2} + \frac{x^3}{2L^2} \tag{16.35d}$$

For example, to calculate k_{11}, we substitute the second derivative $N_1''(x)$ from Eq. (16.35a) into Eq. (16.34) and obtain

$$k_{11} = EI \int_0^L \left(\frac{3x}{L^3}\right)^2 dx = \frac{3EI}{L^3} \tag{16.36}$$

Similarly, all the other stiffness coefficients for the case in which the formation of the plastic hinge takes place at end Θ of a beam segment are determined using Eq. (16.34) and the deflection functions given by Eqs. (16.35). The resulting stiffness equation in incremental form is

$$\begin{Bmatrix} \Delta P_1 \\ \Delta P_2 \\ \Delta P_3 \\ \Delta P_4 \end{Bmatrix} = \frac{EI}{L^3} \begin{bmatrix} 3 & 0 & -3 & 3L \\ 0 & 0 & 0 & 0 \\ -3 & 0 & 3 & -3L \\ 3L & 0 & -3L & 3L^2 \end{bmatrix} \begin{Bmatrix} \Delta\delta_1 \\ \Delta\delta_2 \\ \Delta\delta_3 \\ \Delta\delta_4 \end{Bmatrix} \tag{16.37}$$

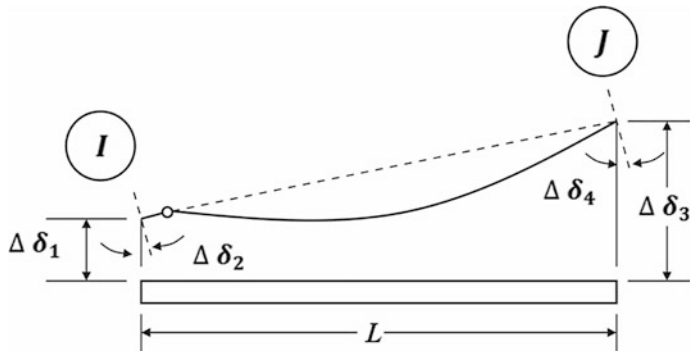

Fig. 16.7 Beam geometry with a plastic hinge at joint i

It should be pointed out that $\Delta\delta_2$ is the incremental rotation of joint Θ at the frame and not the increase in rotation at end Θ of the beam under consideration. The incremental rotation of the plastic hinge is given by the difference between $\Delta\delta_2$ and the increase in rotation of the end Θ of the member. Hinge rotation may be calculated for the various cases with formulas developed in the next section. Analogous to Eq. (16.37), the following equation gives the relationship between incremental forces and incremental displacements for a uniform beam with a hinge at end Θ (Fig. 16.7), namely,

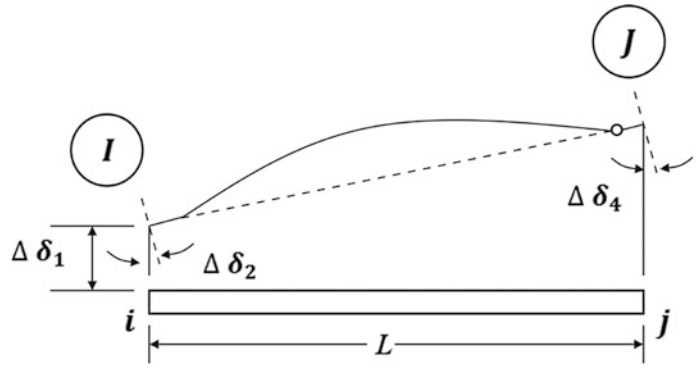

Fig. 16.8 Beam geometry with a plastic hinge at joint Θ

$$\begin{Bmatrix} \Delta P_1 \\ \Delta P_2 \\ \Delta P_3 \\ \Delta P_4 \end{Bmatrix} = \frac{EI}{L^3} \begin{bmatrix} 3 & 3L & -3 & 0 \\ 3L & 3L^2 & -3L & 0 \\ -3 & -3L & -3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \Delta\delta_1 \\ \Delta\delta_2 \\ \Delta\delta_3 \\ \Delta\delta_4 \end{Bmatrix} \tag{16.38}$$

Finally, if hinges are formed at both ends of the beam, the stiffness matrix becomes null. Hence in this case the stiffness equation is

$$\begin{Bmatrix} \Delta P_1 \\ \Delta P_2 \\ \Delta P_3 \\ \Delta P_4 \end{Bmatrix} = \frac{EI}{L^3} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \Delta\delta_1 \\ \Delta\delta_2 \\ \Delta\delta_3 \\ \Delta\delta_4 \end{Bmatrix} \tag{16.39}$$

16.8 Member Mass Matrix

The relationship between forces and accelerations at the nodal coordinates of an elastic uniform member considering flexural deformation is given by Eq. (10.34). This equation written in incremental quantities is

$$\begin{Bmatrix} \Delta P_1 \\ \Delta P_2 \\ \Delta P_3 \\ \Delta P_4 \end{Bmatrix} = \frac{\bar{m}L}{420} \begin{bmatrix} 156 & 22L & 54 & -13L \\ 22L & 4L^2 & 13L & -3L^2 \\ 54 & 13L & 156 & -22L \\ -13L & -3L^2 & -22L & 4L^2 \end{bmatrix} \begin{Bmatrix} \Delta\ddot{\delta}_1 \\ \Delta\ddot{\delta}_2 \\ \Delta\ddot{\delta}_3 \\ \Delta\ddot{\delta}_4 \end{Bmatrix} \tag{16.40}$$

where ΔP_i, and $\Delta\ddot{\delta}_i$ are, respectively, the incremental forces and the incremental accelerations at the nodal coordinates, L is the length of the member, and $\bar{m}$ is its mass per unit length. Assuming elastoplastic behavior, when the moment at an end of the beam segment reaches the magnitude of the plastic moment M_p and a hinge is formed, the consistent mass coefficients are determined from Eq. (10.33) using the appropriate deflection curves. For a uniform beam in which the formation of the plastic hinge develops at end I as shown in Fig. 16.6, the deflection functions corresponding to a unit displacement of the nodal coordinates are given by Eqs. (16.35). Analogously, the deflection functions of a beam segment with a plastic hinge at end J as shown in Fig. 16.8 for unit displacement at nodal coordinates δ_1, δ_2, δ_3, or δ_4, are respectively given by

$$N_1(x) = 1 + \frac{x^3}{2L^3} - \frac{3x^2}{2L^2}$$

$$N_2(x) = \frac{x^3}{2L^2} - \frac{3x^2}{2L} + x$$

$$N_3(x) = -\frac{x^3}{2L^3} - \frac{3x^2}{2L^2}$$

$$N_4(x) = 0 \tag{16.41}$$

The mass coefficients for a beam segment with a hinge at one end are then obtained by application of Eq. (10.33) which is repeated here, namely,

$$m_{ij} = \int_0^L \bar{\mathrm{m}} \mathrm{N}_i(x) N_j(x) dx \tag{16.42}$$

where N_i (x) and N_j (x) are the corresponding displacement functions from Eqs. (16.35) or (16.41).

Application of Eq. (16.42) and the use of displacement functions (16.34) results in the mass matrix for a beam segment with a hinge at the "j" end. The resulting mass matrix relates incremental forces and accelerations at the nodal coordinates, namely,

$$\begin{Bmatrix} \Delta P_1 \\ \Delta P_2 \\ \Delta P_3 \\ \Delta P_4 \end{Bmatrix} = \frac{\bar{m}L}{420} \begin{bmatrix} 204 & 0 & 58.5 & -16.5L \\ 0 & 0 & 0 & 0 \\ 58.5 & 0 & 99 & -36L \\ -16.5L & 0 & -36L & 8L^2 \end{bmatrix} \begin{Bmatrix} \Delta\ddot{\delta}_1 \\ \Delta\ddot{\delta}_2 \\ \Delta\ddot{\delta}_3 \\ \Delta\ddot{\delta}_4 \end{Bmatrix} \tag{16.43}$$

Analogously to Eq. (16.43), the following equation gives the relationship between incremental forces and incremental accelerations for a uniform beam segment with a hinge at the "j" end:

$$\begin{Bmatrix} \Delta P_1 \\ \Delta P_2 \\ \Delta P_3 \\ \Delta P_4 \end{Bmatrix} = \frac{\bar{m}L}{420} \begin{bmatrix} 99 & 36L & 58.5 & 0 \\ 36L & 8L^2 & 16.5L & 0 \\ 58.5 & 16.5L & 204 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \Delta\ddot{\delta}_1 \\ \Delta\ddot{\delta}_2 \\ \Delta\ddot{\delta}_3 \\ \Delta\ddot{\delta}_4 \end{Bmatrix} \tag{16.44}$$

Finally, if hinges are formed at both ends of the beam segment, the deflection curves are given by

$$\begin{aligned} N_1(x) &= -\frac{x}{L} + 1 \\ N_2(x) &= 0 \\ N_3(x) &= \frac{x}{L} \\ N_4(x) &= 0 \end{aligned} \tag{16.45}$$

and the corresponding relationship in incremental form by

$$\begin{Bmatrix} \Delta P_1 \\ \Delta P_2 \\ \Delta P_3 \\ \Delta P_4 \end{Bmatrix} = \frac{\bar{m}L}{6} \begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \Delta\ddot{\delta}_1 \\ \Delta\ddot{\delta}_2 \\ \Delta\ddot{\delta}_3 \\ \Delta\ddot{\delta}_4 \end{Bmatrix} \tag{16.46}$$

16.9 Rotation of Plastic Hinges

In the solution process, at the end of each step interval it is necessary to calculate the end moments of every beam segment to check whether or not a plastic hinge has been formed. The calculation is done using the element incremental moment-displacement relationship. It is also necessary to check if the plastic deformation associated with a hinge is compatible with the sign of the moment. The plastic hinge is free to rotate in one direction only, and in the other direction the section returns to an elastic behavior. The assumed moment rotation characteristics of the member are of the type illustrated in Fig. 16.5. The conditions implied by this model are: (1) the moment cannot exceed the plastic moment; (2) if the moment is less than the plastic moment, the hinge cannot rotate; (3) if the moment is equal to the plastic moment, then the hinge may rotate in the direction consistent with the sign of the moment; and (4) if the hinge starts to rotate in a direction inconsistent with the sign of the moment, the hinge is removed.

The incremental rotation of a plastic hinge is given by the difference between the incremental joint rotation of the frame and the increase in rotation of the end of the member at that joint. For example, with a hinge at end "i" only (Fig. 20.6), the incremental joint rotation is $\Delta\delta_2$ and the increase in rotation of this end due to rotation $\Delta\delta_4$ is $-\Delta\delta_4/2$ and that due to the displacements $\Delta\delta_1$ and $\Delta\delta_3$ is 1.5 $\left(\Delta\delta\underline{3} - \Delta\delta\right)/L$ Hence the increment in rotation $\Delta\rho_i$ of a hinge formed at end i is given by

$$\Delta\rho_i = \Delta\delta_2 + \frac{1}{2}\Delta\delta_4 - 1.5\frac{\Delta\delta_3 - \Delta\delta_1}{L} \tag{16.47}$$

Similarly, with a hinge formed at end "j" only (Fig. 16.7), the increment in rotation of this hinge is given by

$$\Delta\rho_j = \Delta\delta_4 + \frac{1}{2}\Delta\delta_2 - 1.5\frac{\Delta\delta_3 - \Delta\delta_1}{L} \tag{16.48}$$

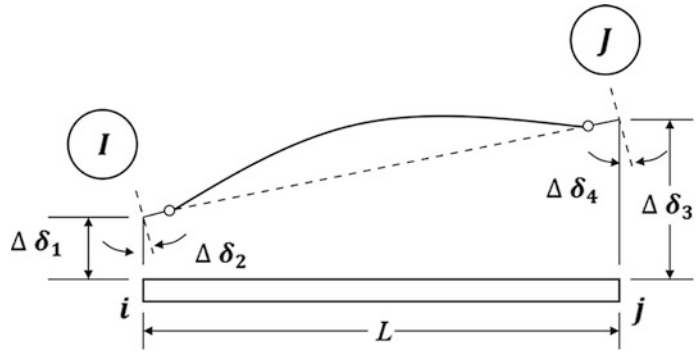

Fig. 16.9 Beam geometry with plastic hinges at both ends

Finally, with hinges formed at both ends of a beam segment (Fig. 16.8), the rotations of the hinges are given by

$$\Delta\rho_i = \Delta\delta_1 - \frac{\Delta\delta_3 - \Delta\delta_1}{L} \tag{16.49}$$

$$\Delta\rho_j = \Delta\delta_2 - \frac{\Delta\delta_3 - \Delta\delta_1}{L} \tag{16.50}$$

16.10 Calculation of Member Ductility Ratio

Nonlinear beam deformation are expressed in terms of the member ductility ratio, which is defined as the maximum total end rotation of the member to the end rotation at the elastic limit. The elastic limit rotation is the angle developed when the member is subjected to antisymmetric yield moments M_x as shown in Fig. 16.9. In this case the relationship between the end rotation and end moment is given by

$$\phi_y = \frac{M_y L}{6EI} \tag{16.51}$$

The member ductility ratio μ is then defined as

$$\mu = \frac{\phi_y + \rho_{\max}}{\phi_y} \tag{16.52}$$

which from Eq. (16.50) becomes

$$\mu = 1 + \frac{6EI}{M_y L}\rho_{\max} \tag{16.53}$$

where $\rho_{\max}$ is the maximum rotation of the plastic hinge (Fig. 16.10).

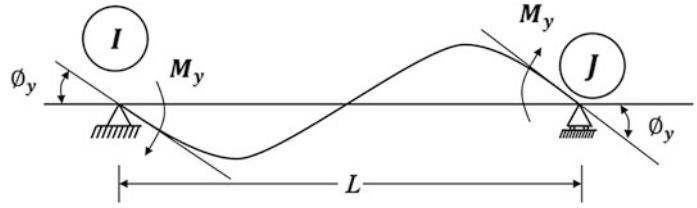

Fig. 16.10 Definition of yield rotation for beam segment

16.11 Summary

The determination of the nonlinear response of multidegree-of-freedom structures requires the numerical integration of the governing equations of motion. There are many methods available for the solution of these equations. The step-by-step linear acceleration method with a modification known as the Wilson-θ method was presented in this chapter. This method is unconditionally stable, that is, numerical errors do not tend to accumulate during the integration process regardless of the magnitude selected for the time step. The basic assumption of the Wilson-θ method is that the acceleration varies linearly over the extended interval $\tau = \theta\Delta t$ in which $\theta \geq 1.38$ for unconditional stability.

In the final sections of this chapter, stiffness and mass matrices for elastoplastic behavior of framed structures are presented. Formulas to determine the plastic rotation of hinges and to calculate the corresponding ductility ratios are also presented in this chapter.

16.12 Problems

Problem 16.1

The stiffness and the mass matrices for a certain structure modeled as a two-degree-of-freedom system are

$$[K] = \begin{bmatrix} 100 & -50 \\ -50 & 50 \end{bmatrix} (\text{Kip/in}), \qquad [M] = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \left(\text{Kip} \cdot \sec^2/\text{in}\right)$$

Use Program 19 to determine the response when the structure is acted upon by the forces.

$$\begin{Bmatrix} F_1(t) \\ F_2(t) \end{Bmatrix} = \begin{Bmatrix} 772 \\ 386 \end{Bmatrix} f(t)(\text{Kip})$$

Where $f(t)$ is given graphically in Fig. P16.1. Neglect damping in the system.

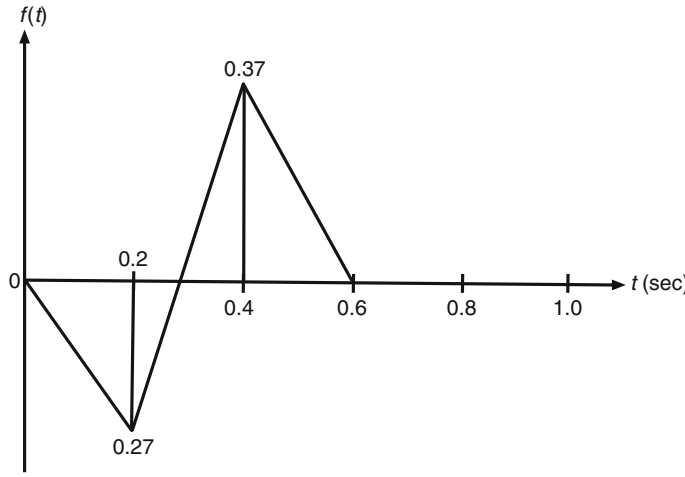

Fig. P16.1

Problem 16.2

Solve Problem 16.1 considering that the damping present in the system results in the following damping matrix:

$$[C] = \begin{bmatrix} 10 & -5 \\ -5 & 5 \end{bmatrix} (\text{Kip} \cdot \text{ sec /in})$$

Problem 16.3

Use Program 19 to determine the response of the three-story shear building subjected to the force F_3 (t) as depicted in Fig. P16.3 applied at the third level of the building. Neglect damping in the system.

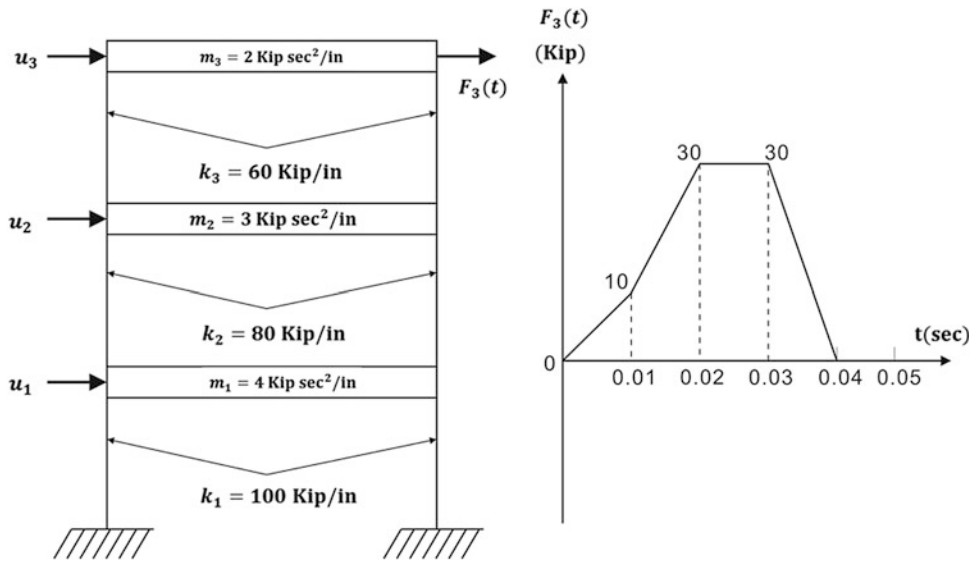

Fig. P16.3

Problem 16.4
Solve Problem 16.3 considering damping in the system of 10% in all the modes.

Problem 16.5
Use Program 19 to obtain the response in the elastic range for the structure of Problem 16.1 subjected to an acceleration at its foundation given by the function $f(t)$ shown in Fig. P16.1.

Problem 16.6
Solve Problem 16.5 considering damping in the system as indicated in Problem 16.2.

Problem 16.7
Use Program 19 to obtain the response in the elastic range of the shear building shown in Fig. P16.3 when subjected to an acceleration of its foundation given by the function $f(t)$ depicted in Fig. P16.1. Neglect damping in the system.

Part IV

Structures Modeled with Distributed Properties

Dynamic Analysis of Systems with Distributed Properties

17

The dynamic analysis of structures, modeled as lumped parameter systems with discrete coordinates, was presented in Part I for single-degree-of-freedom systems and in Parts II and III for multidegree-of-freedom systems. Modeling structures with discrete coordinates provides a practical approach for the analysis of structures subjected to dynamic loads. However, the results obtained from these discrete models can only give approximate solutions to the actual behavior of dynamic systems which have continuous distributed properties and, consequently, an infinite number of degrees of freedom.

The present chapter considers the dynamic theory of beams and rods having distributed mass and elasticity for which the governing equations of motion are partial differential equations. The integration of these equations is in general more complicated than the solution of ordinary differential equations governing discrete dynamic systems. Due to this mathematical complexity, the dynamic analysis of structures as continuous systems has limited use in practice. Nevertheless, the analysis, as continuous systems, of some simple structures provides, without much effort, results which are of great importance in assessing approximate methods based on discrete models.

17.1 Flexural Vibration of Uniform Beams

The treatment of beam flexure developed in this section is based on the simple bending theory as it is commonly used for engineering purposes. The method of analysis is known as the Bernoulli-Euler theory which assumes that a plane cross section of a beam remains a plane during flexure.

Consider in Fig. 17.1b the free body diagram of a short segment of a beam as shown in Fig 17.1a. It is of length dx and is bounded by plane faces that are perpendicular to its axis. The forces and moments which act on the element are also shown in the figure: they are the shear forces V and $V + (\partial V/\partial x)\, dx$; the bending moments M and $M + (\partial M/\partial x)\, dx$; the lateral load $p dx$; and the inertia force $(\bar{m}\, dx)\, \partial^2 y/dt^2$. In this notation $\bar{m}$ is the mass per unit length and $p = p\,(x, t)$ is the load per unit length. Partial derivatives are used to express acceleration and variations of shear and moment because these quantities are functions of two variables, position x along the beam and time t. If the deflection of the beam is small, as the theory presupposes, the inclination of the beam segment from the unloaded position is also small. Under these conditions, the equation of motion perpendicular to the x axis of the deflected beam obtained by equating to zero the sum of the forces in the free body diagram of Fig. 17.1b is

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M. Paz, Y. H. Kim, *Structural Dynamics*, https://doi.org/10.1007/978-3-319-94743-3_17

$$V - \left(V + \frac{\partial V}{\partial x}dx\right) + p(x,t)dx - \bar{m}dx\frac{\partial^2 y}{\partial t^2} = 0$$

which, upon simplification, becomes

$$\frac{\partial V}{\partial x} + \bar{m}\frac{\partial^2 y}{\partial t^2} = p(x,t) \tag{17.1}$$

From simple bending theory, we have the relationships

$$M = EI\frac{\partial^2 y}{\partial x^2} \tag{17.2}$$

and

$$V = \frac{\partial M}{\partial x} \tag{17.3}$$

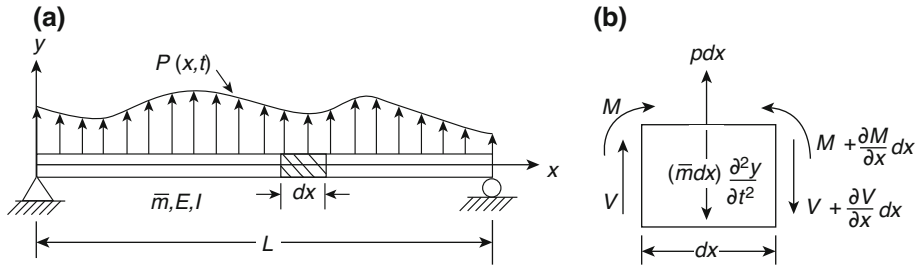


Fig. 17.1 Simple beam with distributed mass and load (a) simply supported beam, (b) free body diagram

where E is Young's modulus of elasticity and I is the moment of inertia of the cross-sectional area with respect to the neutral axis through the centroid. For a uniform beam, the combination of Eqs. (17.1), (17.2), and (17.3) results in

$$V = EI\frac{\partial^3 y}{\partial x^3} \tag{17.4}$$

and

$$EI\frac{\partial^4 y}{\partial x^4} + \bar{m}\frac{\partial^2 y}{\partial t^2} = p(x,t) \tag{17.5}$$

It is seen that Eq. (17.5) is a partial differential equation of fourth order. It is an approximate equation. Only lateral flexural deflections were considered while the deflections due to shear forces and the inertial forces caused by the rotation of the cross section (rotary inertia) were neglected. The inclusion of shear deformations and rotary inertia in the differential equation of motion considerably increases its complexity. The equation taking into consideration shear deformation and rotary inertia is known as Timoshenko's equation. The differential Eq. (17.5) also does not include the flexural effects due to the presence of forces which may be applied axially to the beam. The axial effects will be discussed in Chap. 22.

17.2 Solution of the Equation of Motion in Free Vibration

For free vibration [$p\ (x, t) = 0$], Eq. (17.5) reduces to the homogeneous differential equation

$$EI\frac{\partial^4 y}{\partial x^4} + \bar{m}\frac{\partial^2 y}{\partial t^2} = 0 \tag{17.6}$$

The solution of Eq. (17.6) can be found by the method of separation of variables. In this method, it is assumed that the solution may be expressed as the product of a function of position $\Phi\ (x)$ and a function of time *f(t),* that is,

$$\mathrm{y(x,t)} = \Phi(x)f(t) \tag{17.7}$$

The substitution of Eq. (17.7) in the differential Eq. (17.6) leads to

$$EIf(t)\frac{d^4\Phi(x)}{dx^4} + \bar{m}\Phi(x)\frac{d^2 f(t)}{dt^2} = 0 \tag{17.8}$$

This last equation may be written as

$$\frac{EI}{\bar{m}}\frac{\Phi^{IV}(x)}{\Phi(x)} = -\frac{\ddot{f}(t)}{f(t)} \tag{17.9}$$

In this notation Roman indices indicate derivatives with respect to x and overdots indicate derivatives with respect to time. Since the left-hand side of Eq. (17.9) is a function only of x while the right-hand side is a function only of t, each side of the equation must equal the same constant value; otherwise, the identity of Eq. (17.9) cannot exist. We designate the constant by ω^2 which equated separately to each side of Eq. (17.9) results in the two following differential equations:

$$\Phi^{IV}(x) - a^4\Phi(x) = 0 \tag{17.10}$$

and

$$\ddot{f}(t) + \omega^2 f(t) = 0 \tag{17.11}$$

where

$$a^4 = \frac{\bar{m}\omega^2}{EI} \tag{17.12}$$

It is particularly convenient to solve Eq. (17.12) for ω and to use the following notation, namely,

$$\omega = C\sqrt{\frac{EI}{\bar{m}L^4}} \tag{17.13}$$

in which $C = (aL)^2$.

Equation (17.11) is the familiar free-vibration equation for the undamped single degree-of-freedom system and its solution from Eq. (1.17) is

$$f(t) = A\cos\omega t + B\sin t \tag{17.14}$$

where A and B are constants of integration. Equation (17.10) can be solved by letting

$$\Phi(x) = Ce^{sx} \tag{17.15}$$

The substitution of Eq. (17.15) into Eq. (17.10) results in

$$\left(S^4 - a^4\right)Ce^{sx} = 0$$

which, for a nontrivial solution, requires that

$$S^4 - a^4 = 0 \tag{17.16}$$

The roots of Eq. (17.16) are

$$\begin{array}{ll} \mathrm{S}_1 = \mathrm{a}, & \mathrm{S}_3 = \mathrm{ai} \\ \mathrm{S}_2 = -\mathrm{a}, & \mathrm{S}_4 = -\mathrm{ai} \end{array} \tag{17.17}$$

The substitution of each of these roots into Eq. (17.15) provides a solution of Eq. (17.10). The general solution is then given by the superposition of these four possible solutions, namely,

$$\Phi(x) = C_1e^{ax} + C_2e^{-ax} + C_3e^{iax} + C_4e^{-iax} \tag{17.18}$$

where *C1, C2, C3,* and *C4* are constants of integration. The exponential functions in Eq. (17.18) may be expressed in terms of trigonometric and hyperbolic functions by means of the relationships.

$$\begin{aligned} e^{\pm ax} &= \cosh ax \pm \sinh ax \\ e^{\pm iax} &= \cos ax \pm i \sin ax \end{aligned} \tag{17.19}$$

Substitution of these relationships into Eq. (17.18) yields

$$\Phi(x) = A \sin ax + B \cos ax + C\sinh ax + D\cosh ax \tag{17.20}$$

where *A, B*, C, and *D* are new constants of integration. These four constants of integration define the shape and the amplitude of the beam in free vibration; they are evaluated by considering the boundary conditions at the ends of the beam as illustrated in the examples presented in the following section.

17.3 Natural Frequencies and Mode Shapes for Uniform Beams

17.3.1 Both Ends Simply Supported

In this case the displacements and bending moments must be zero at both ends of the beam; hence the boundary conditions for the simply supported beams are

$$\begin{array}{ll} \mathrm{y}\ (0, \mathrm{t}) = 0, & \mathrm{M}\ (0, \mathrm{t}) = 0 \\ \mathrm{y}\ (\mathrm{L}, \mathrm{t}) = 0, & \mathrm{M}\ (\mathrm{L}, \mathrm{t}) = 0 \end{array}$$

In view of Eqs. (17.2) and (17.7), these boundary conditions imply the following conditions on the shape function $\Phi(x)$.

At $x = 0$,

$$\Phi(0) = 0, \Phi''(0) = 0 \tag{17.21}$$

At x = L,

$$\Phi(L) = 0, \Phi''(L) = 0 \tag{17.22}$$

The substitution of the first two of these boundary conditions into Eq. (17.20) yields

$$\Phi(0) = A0 + B1 + C0 + D1 = 0$$
$$\Phi''(0) = a^2(-A0 - B1 + C0 + D1) = 0$$

which reduce to

$$B + D = 0$$
$$-\mathrm{B} + \mathrm{D} = 0$$

Hence

$$\mathrm{B} = \mathrm{D} = 0$$

Similarly, substituting the last two boundary conditions into Eq. (17.20) and setting $B = D = 0$ leads to

$$\Phi(L) = A \sin al + C\sinh aL = 0$$
$$\Phi''(L) = a^2(-A \sin aL + C\sinh aL) = 0 \tag{17.23}$$

which, when added, give

$$2C \sinh aL = 0$$

From this last relation, $C = 0$ since the hyperbolic sine function cannot vanish except for a zero argument. Thus Eq. (17.23) reduce to

$$A \sin aL = 0 \tag{17.24}$$

Excluding the trivial solution $(A = 0)$, we obtain the frequency equation

$$\sin aL = 0 \tag{17.25}$$

which will be satisfied for

$$a_n L = n\pi, \quad \mathrm{n} = 0, 1, 2 \ldots \tag{17.26}$$

Substitution of the roots, Eq. (17.26), into Eq. (17.13) yields

$$\omega_n = n^2\pi^2\sqrt{\frac{EI}{\bar{m}L^4}} \tag{17.27}$$

where the subscript n serves to indicate the order of the natural frequencies.

Since $B = \mathrm{C} = D = 0$, it follows that Eq. (17.20) reduces to

$$\Phi_n(x) = A \sin\frac{n\pi x}{L}$$

or simply

$$\Phi_n(x) = \sin\frac{n\pi x}{L} \tag{17.28}$$

We note that in Eq. (17.28) the constant A is absorbed by the other constants in the modal response given below by Eq. (17.29).

From Eq. (17.7) a modal shape or normal mode of vibration is given by

$$y_n(x,t) = \Phi_n(x) f_n(t)$$

or from Eqs. (17.14) and (17.28) by

$$y_n(x,t) = \sin\frac{n\pi x}{L}[A_n\cos\omega_n t + B_n\sin\omega_n t] \tag{17.29}$$

The general solution of the equation of motion in free vibration that satisfies the boundary conditions, Eqs. (17.21) and (17.22), is the sum of all the normal modes of bvibration, Eq. (17.29), namely,

$$y(x,t) = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{L}[A_n\cos\omega_n t + B_n\sin\omega_n t] \tag{17.30}$$

The constants A_n and B_n are determined, as usual, from the initial conditions. If at $t = 0$, the shape of the beam is given by

$$y(x,0) = \rho(x)$$

and the velocity by

$$\frac{\partial y(x,0)}{\partial t} = \psi(x)$$

For $0 \le x \le L$, it follows from Eq. (17.30) that

$$\sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{L} = \rho(x)$$

and

$$\sum_{n=1}^{\infty} B_n\omega_n \sin\frac{n\pi x}{L} = \psi(x)$$

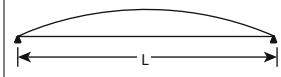
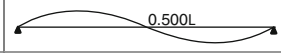
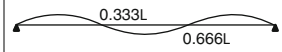
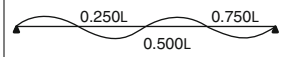
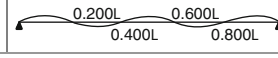
Therefore, as shown in Chap. 5, Fourier coefficients are expressed as

$$A_n = \frac{2}{L}\int_0^L \rho(x)\sin\frac{n\pi x}{L}dx$$

$$B_n = \frac{2}{\omega_n L}\int_0^L \psi(x)\sin\frac{n\pi x}{L}dx \tag{17.31}$$

The first five values for the natural frequencies and normal modes for the simply supported beam are presented in Table 17.1.

Table 17.1 Natural frequencies and normal modes for simply supported beams

Natural frequencies $\omega_n = C_n\sqrt{\frac{EI}{\bar{m}L^4}}$		Normal modes $\Phi_n = \sin\frac{n\pi x}{L}$	
n	C_n	I_n[a]	Shape
1	π^2	$4/\pi$	
2	$4\pi^2$	0	
3	$9\pi^2$	$4/3\pi$	
4	$16\pi^2$	0	
5	$25\pi^2$	$4/5\pi$	

[a] $I_n = \int_0^L \Phi_n(x)dx / \int_0^L \Phi^2(x)dx$

17.3.2 Both Ends Free (Free Beam)

The boundary conditions for a beam with both ends free are as follows.

At $x = 0$.

$$(0,\mathrm{t}) = 0 \quad \text{or} \quad \Phi''(0) = 0 \;\mathrm{M}$$

$$\mathrm{V}\;(0,\mathrm{t}) = 0 \quad \text{or} \quad \Phi'''(0) = 0 \tag{17.32}$$

At x = L,

$$\mathrm{M}\;(\mathrm{L},\mathrm{t}) = 0 \quad \text{or} \quad \Phi''(L) = 0$$

$$\mathrm{V}\;(\mathrm{L},\mathrm{t}) = 0 \quad \text{or} \quad \Phi'''(L) = 0 \tag{17.33}$$

The substitutions of these conditions in Eq. (17.20) yield

$$\Phi''(0) = a^2(-B + D) = 0$$
$$\Phi'''(0) = a^3(-A + C) = 0$$

and

$$\Phi''(L) = a^2(-A\sin\ aL - B\cos aL + C\sinh aL + D\cosh aL) = 0$$
$$\Phi'''(L) = a^3(-A\cos\ aL + B\sin\ aL + C\cosh aL + D\sinh aL) = 0$$

From the first two equations we obtain

$$\mathrm{D} = \mathrm{B}, \quad \mathrm{C} = \mathrm{A} \tag{17.34}$$

which, substituted into the last two equations, result in

$$(\sinh \mathrm{Al} - \sin \mathrm{aL})\,\mathrm{A} + (\cosh \mathrm{aL} - \cos \mathrm{aL})\,\mathrm{B} = 0$$

$$(\cosh \mathrm{aL} - \cos\ \mathrm{aL})\,\mathrm{A} + (\sinh \mathrm{aL} + \sin \mathrm{aL})\,\mathrm{B} = 0 \tag{17.35}$$

For nontrivial solution of Eq. (17.35), it is required that the determinant of the unknown coefficients A and B be equal to zero; hence

$$\begin{vmatrix} \sinh aL - \sin aL & \cosh aL - \cos aL \\ \cosh aL - \cos aL & \sinh aL + \sin aL \end{vmatrix} = 0 \tag{17.36}$$

The expansion of this determinant provides the frequency equation for the free beam, namely

$$\cos aL \cdot \cosh aL - 1 = 0 \tag{17.37}$$

The first five natural frequencies which are obtained by substituting the roots of Eq. (17.37) into Eq. (17.13) are presented in Table 17.2. The corresponding normal modes are obtained by letting $A = 1$ (normal modes are determined only to a relative magnitude), substituting in Eq. (17.35) the roots of a_n of Eq. (17.37), solving one of these equations for B, and finally introducing into Eq. (17.20) the constants C, D from Eq. (17.34) together with B. performing these operations, we obtain

$$\Phi_n(x) = \cosh a_n x + \cos a_n x - \sigma_n(\sinh a_n x + \sin a_n x) \tag{17.38}$$

where

$$\sigma_n = \frac{\cosh a_n L - \cos a_n L}{\sinh a_n L - \sin a_n L} \tag{17.39}$$

17.3.3 Both Ends Fixed

The boundary conditions for a beam with both ends fixed are as follows:

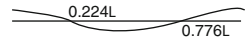
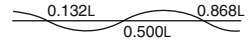
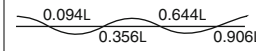
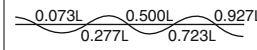
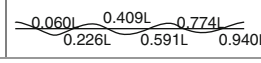
At $x = 0$,

$$\begin{aligned} \mathrm{y}\,(0,\mathrm{t}) = 0 \quad \text{or} \quad \Phi(0) = 0 \\ y'(0,\mathrm{t}) = 0 \quad \text{or} \quad \Phi'(0) = 0 \end{aligned} \tag{17.40}$$

At x = L,

$$\begin{aligned} \mathrm{y}\,(\mathrm{L},\mathrm{t}) = 0 \quad \text{or} \quad \Phi(L) = 0 \\ y'(\mathrm{L},\mathrm{t}) = 0 \quad \text{or} \quad \Phi'(L) = 0 \end{aligned} \tag{17.41}$$

Table 17.2 Natural frequencies and normal modes for free beams

Natural frequencies		Normal modes		
$\Phi_n(x) = \cosh a_n x + \cos a_n x - \sigma_n(\sinh a_n x + \sin a_n x)$				
$\omega_n = C_n\sqrt{\frac{EI}{\bar{m}L^4}}$		$\sigma_n = \frac{\cosh a_n L - \cos a_n L}{\sinh a_n L - \sin a_n L}$		
n	$C_n = (a_n L)^2$	σ_n	I_n[a]	Shape
1	22.3733	0.982502	0.8308	
2	61.6728	1.000777	0	
3	120.9034	0.999967	0.3640	
4	199.8594	1.000001	0	
5	298.5555	1.00000	0.2323	

[a] $I_n = \int_0^L \Phi_n(x)dx / \int_0^L \Phi_n^2(x)dx$

The use of the boundary conditions, Eq. (17.40), into Eq. (17.20) gives

$$\mathrm{B} + \mathrm{D} = 0 \quad \text{and} \quad \mathrm{A} - \mathrm{C} = 0$$

while conditions, Eq. (17.41), yield the homogeneous system

$$(\cos\ \mathrm{Al} - \cosh \mathrm{aL})\ \mathrm{B} + (\sin\ \mathrm{aL} - \sinh \mathrm{aL})\ \mathrm{A} = 0$$

$$-(\sin\ \mathrm{aL} + \sinh \mathrm{aL})\ \mathrm{B} + (\cos\ \mathrm{aL} - \cosh \mathrm{aL})\ \mathrm{A} = 0 \tag{17.42}$$

Equating to zero the determinant of the coefficients of this system results in the frequency equation

$$\cos \mathrm{a_n L} \cosh \mathrm{a_n L} - 1 = 0 \tag{17.43}$$

From the first of Eq. (17.42), it follows that

$$\mathrm{A} = -\frac{\cos aL - \cosh aL}{\sin aL - \sinh aL} B \tag{17.44}$$

where B is arbitrary. To each value of the natural frequency

$$\omega_n = (a_n L)^2 \sqrt{\frac{EI}{\bar{m}L^4}} \tag{17.45}$$

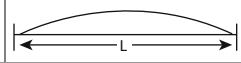
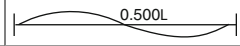
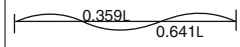
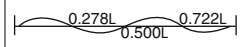
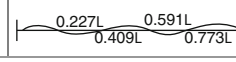
obtained by the substitution of the roots of Eq. (17.43) into Eq. (17.13), there corresponds a normal mode

$$\Phi_n(x) = \cosh a_n x - \cos a_n x - \sigma_n(\sinh a_n x - \sin a_n x) \tag{17.46}$$

$$\sigma_n = \frac{\cos a_n L - \cosh a_n L}{\sin a_n L - \sinh a_n L} \tag{17.47}$$

The first five natural frequencies calculated from Eqs. (17.43) and (17.45) and the corresponding normal modes obtained from Eq. (17.46) are presented in Table 17.3.

Table 17.3 Natural frequencies and normal modes for fixed beams

Natural frequencies		Normal modes		
$\Phi_n(x) = \cosh a_n x - \cos a_n x - \sigma_n(\sinh a_n x - \sin a_n x)$				
$\omega_n = C_n\sqrt{\frac{EI}{\bar{m}L^4}}$		$\sigma_n = \frac{\cos a_n L - \cosh a_n L}{\sin a_n L - \sinh a_n L}$		
n	$C_n = (a_n L)^2$	σ_n	I_n[a]	Shape
1	22.3733	0.982502	0.8308	
2	61.6728	1.000777	0	
3	120.9034	0.999967	0.3640	
4	199.8594	1.000001	0	
5	298.5555	1.00000	0.2323	

[a] $I_n = \int_0^L \Phi_n(x)dx / \int_0^L \Phi_n^2(x)dx$

17.3.4 One End Fixed and the Other End Free (Cantilever Beam)

At the fixed end ($x = 0$) of the cantilever beam, the deflection and the slope must be zero, and at the free end *($x = L$)* the bending moment and the shear force must be zero. Hence the boundary conditions for this beam are as follows.

At $x = 0$,

$$\begin{aligned} \mathrm{y}(0,\mathrm{t}) = 0 \quad \text{or} \quad \Phi(0) = 0 \\ y'(0,\mathrm{t}) = 0 \quad \text{or} \quad \Phi'(0) = 0 \end{aligned} \tag{17.48}$$

At $x = L$,

$$\begin{aligned} \mathrm{M}(\mathrm{L},\mathrm{t}) = 0 \qquad \text{or} \qquad \Phi''(L) = 0 \\ \mathrm{V}(\mathrm{L},\mathrm{t}) = 0 \qquad \text{or} \qquad \Phi'''(L) = 0 \end{aligned} \tag{17.49}$$

These boundary conditions when substituted into the shape equation (17.20) lead to the frequency equation.

$$\cos a_n L \cdot \cosh a_n L + 1 = 0 \tag{17.50}$$

To each root of Eq. (17.50) corresponds a natural frequency

$$\omega_n = (a_n L)^2 \sqrt{\frac{EI}{\bar{m}L}} \tag{17.51}$$

and a normal shape

$$\Phi_n(x) = (\cosh a_n x - \cos a_n x) - \sigma_n(\sinh a_n x - \sin a_n x) \tag{17.52}$$

where

$$\sigma_n = \frac{\cos a_n L + \cosh a_n L}{\sin a_n L + \sinh a_n L} \tag{17.53}$$

The first five natural frequencies and the corresponding mode shapes for cantilever beams are presented in Table 17.4.

17.3.5 One End Fixed and the Other Simply Supported

The boundary conditions for a beam with one end fixed and the other simply supported are as follows.

At $x = 0$,

$$\mathrm{y}(0,\mathrm{t}) = 0 \quad \text{or} \quad \Phi(0) = 0$$

$$y'\ (0,\mathrm{t}) = 0 \quad \text{or} \quad \Phi'(0) = 0 \tag{17.54}$$

At $x = L$,

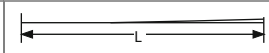
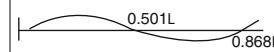
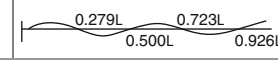
$$\mathrm{y}(\mathrm{L},\mathrm{t}) = 0 \quad \text{or} \quad \Phi(L) = 0$$

$$\mathrm{M}\ (\mathrm{L},\mathrm{t}) = 0 \quad \text{or} \quad \Phi''(L) = 0 \tag{17.55}$$

The substitution of these boundary conditions into the shape Eq. (17.20) results in the frequency equation

$$\tan \mathrm{a_n L}\text{-}\tanh \mathrm{a_n L} = 0 \tag{17.56}$$

Table 17.4 Natural frequencies and normal modes for cantilever beams

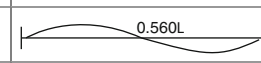
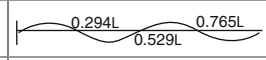
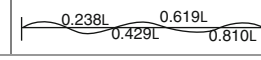
Natural frequencies		Normal modes		
$\Phi_n = (\cosh a_n x - \cos a_n x) - \sigma_n(\sinh a_n x - \sin a_n x)$				
$\omega_n = C_n\sqrt{\frac{EI}{\bar{m}L^4}}$ $\quad \sigma = \frac{\cos a_n L + \cosh a_n L}{\sin a_n L + \sinh a_n L}$				
n	$C_n = (a_n L)^2$	σ_n	I_n[a]	Shape
1	3.5160	0.734096	0.7830	
2	22.0345	1.018466	0.4340	
3	61.6972	0.999225	0.2589	
4	120.0902	1.000033	0.0017	
5	199.8600	1.000000	0.0707	

[a] $I_n = \int_0^L \Phi_n(x)dx \Big/ \int_0^L \Phi_n^2(x)dx$

$$\Phi_n(x) = (\cosh a_n x - \cos a_n x) + \sigma_n(\sinh a_n x - \sin a_n x) \tag{17.58}$$

where

Table 17.5 Natural frequencies and normal modes for fixed simply supported beams

Natural frequencies		Normal modes		
$\Phi(x) = \cosh\ a_n x - \cos\ a_n x + \sigma_n(\sinh\ a_n x - \sin\ a_n x)$				
$\omega_n = C_n\sqrt{\frac{EI}{\bar{m}L^4}}$ $\quad \sigma_n = \frac{\cos a_n L - \cosh a_n L}{\sin a_n L - \sinh a_n L}$				
n	$C_n = (a_n L)^2$	σ_n	I_n[a]	Shape
1	15.4118	1.000777	0.8600	
2	49.9648	1.000001	0.0826	
3	104.2477	1.000000	0.3345	
4	178.2697	1.000000	0.0435	
5	272.0309	1.000000	0.2076	

[a] $I_n = \int_0^L \Phi_n(x)dx / \int_0^L \Phi_n^2(x)dx$

$$\sigma_n = \frac{\cos a_n L - \cosh a_n L}{\sin a_n L - \sinh a_n L} \tag{17.59}$$

The first five natural frequencies for the fixed simply supported beam and corresponding mode shapes are presented in Table 17.5.

17.4 Orthogonality Condition Between Normal Modes

The most important property of the normal modes is that of orthogonality. It is this property which makes possible the uncoupling of the equations of motion as it has previously been shown for discrete systems. The orthogonality property for continuous systems can be demonstrated in essentially the same way as for discrete parameter systems.

Consider in Fig. 17.2 a beam subjected to the inertial forces resulting from the vibrations of two different modes, $\Phi_m(x)$ and $\Phi_n(x)$. The deflection curves for these two modes and the corresponding inertial forces are depicted in the same figure. Betti's law is applied to these two deflection patterns. Accordingly, the work done by the inertial force. f_{In}, acting on the displacements of mode m is equal to the work of the inertial forces, f_{Im}, acting on the displacements of mode n, that is

$$\int_0^L \Phi_m(x) f_{In}(x)dx = \int_0^L \Phi_n(x) f_{\mathrm{Im}}(x)dx \tag{17.60}$$

The inertial force f_{In} per unit length along the beam is equal to the mass per unit length times the acceleration. Inasmuch as the vibratory motion in a normal mode is harmonic, the amplitude of the acceleration is given by $\omega^2{}_n\Phi_n(x)$. Hence the inertial force per unit length along the beam for the nth mode is

$$f_{In} = \omega_n^2 \bar{m}(x)\Phi_n(x)$$

and for the mth mode

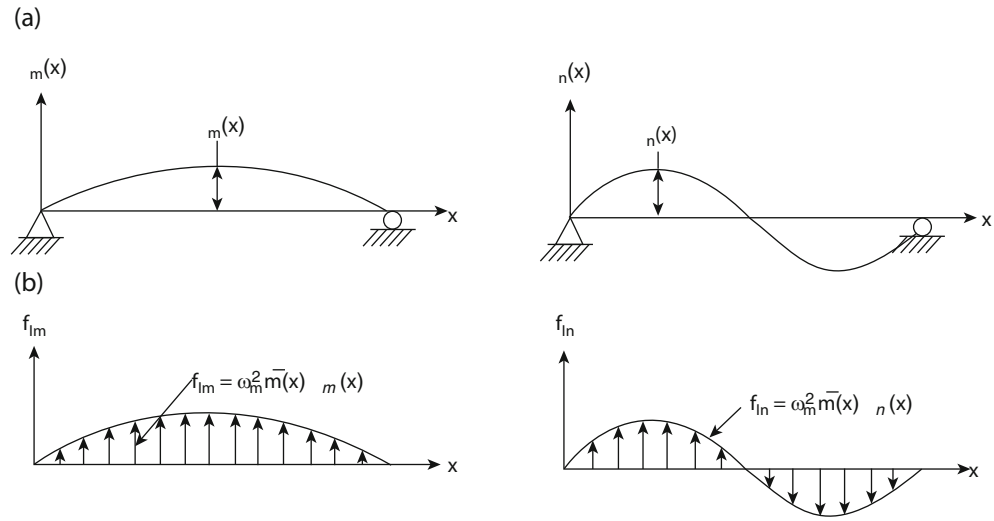


Fig. 17.2 Beam showing two modes of vibration and inertial forces. (**a**) Displacements. (**b**) Inertial forces

$$f_{\mathrm{Im}} = \omega_m^2 \bar{m}(x)\Phi_m(x) \tag{17.61}$$

Substituting these expressions in Eq. (17.60), we obtain

$$\omega_n^2 \int_0^L \Phi_m(x)\bar{m}(x)\Phi_n(x)dx = \omega_m^2 \int_0^L \Phi_n(x)\bar{m}(x)\Phi_m(x)dx$$

which may be written as

$$\left(\omega_n^2 - \omega_m^2\right) \int_0^L \Phi_m(x)\Phi_n(x)\bar{m}(x)dx = 0 \tag{17.62}$$

It follows that, for two different frequencies $\omega_n \neq \omega_m$, the normal modes must satisfy the relationship

$$\int_0^L \Phi_m(x)\Phi_n(x)\bar{m}(x)dx = 0 \tag{17.63}$$

which is equivalent to the orthogonal condition between normal modes for discrete parameter systems, Eq. (10.27).

17.5 Forced Vibration of Beams

For a uniform beam acted on by lateral forces $p\ (x,\ t)$, the equation of motion, Eq. (17.5), may be written as

$$EI\frac{\partial^4 y}{\partial x^4} = p(x,t) - \bar{m}\frac{\partial^2 y}{\partial t^2} \tag{17.64}$$

in which $p\ (x,\ t)$ is the external load per unit length along the beam. We assume that the general solution of this equation may be expressed by the summation of the products of the normal modes $\Phi_n\ (x)$ multiplied by factors $z_n\ (t)$ which are to be determined. Hence

$$y(x,t) = \sum_{n=1}^{\infty} \Phi_n(x) z_n(t) \tag{17.65}$$

The normal modes $\Phi_n(x)$ satisfy the differential Eq. (17.10), which by Eq. (17.12) may be written as

$$EI\Phi_n^{IV}(x) = \bar{m}\omega_n^2 \Phi_n(x), \qquad \text{n} = 1, 2, 3 \ldots \tag{17.66}$$

The normal modes should also satisfy the specific force boundary conditions at the ends of the beam. Substitution of Eq. (17.65) in Eq. (17.64) gives

$$EI \sum_n \Phi_n^{IV}(x) z_n(t) = p(x,t) - \bar{m} \sum_n \Phi_n(x) \ddot{z}_n(t) \tag{17.67}$$

In view of Eq. (17.66), we can write Eq. (17.67) as

$$\sum_n \bar{m}\omega_n^2 \phi_n(x) z_n(t) = p(x,t) - \bar{m} \sum_n \Phi_n(x) \ddot{z}_n(t) \tag{17.68}$$

Multiplying both sides of Eq. (17.68) by $\Phi_n(x)$ and integrating between 0 and L result in

$$\omega_m^2 z_m(t) \int_0^L \bar{m}\Phi_m^2(x)dx = \int_0^L \phi_m(x) p(x,t) - \ddot{z}_m(t) \int_0^L \bar{m}\Phi_m^2(x)dx \tag{17.69}$$

We note that all the terms that contain products of different indices ($n \neq m$) vanish from the summations in Eq. (17.68) in view of the orthogonality conditions, Eq. (17.63), between normal modes. Equation (17.69) may conveniently be written as

$$M_n \ddot{z}_n(t) + \omega_n^2 M_n z_n(t) = F_n(t), \qquad \text{n} = 1, 2, 3 \ldots, \text{m}, \ldots \tag{17.70}$$

where

$$M_n = \int_0^L \bar{m}\Phi_n^2(x)dx \tag{17.71}$$

is the modal mass, and

$$F_n(t) = \int_0^L \Phi_n(x) p(x,t)dx \tag{17.72}$$

is the modal force.

The equation of motion for the *nth* normal mode, Eq. (17.70), is completely analogous to the modal equation, Eq. (12.9), for discrete systems. Modal damping could certainly be introduced by simply adding the damping term in Eq. (17.70); hence we would obtain

$$M_n \ddot{z}_n(t) + C_n \dot{z}_n(t) + K_n z_n(t) = F_n(t) \tag{17.73}$$

which, upon dividing by M_n, gives

$$\ddot{z}_n(t) + 2\xi_n \omega_n \dot{z}_n(t) + \omega_n^2 z_n(t) = \frac{F_n(t)}{M_n} \tag{17.74}$$

where $\xi_n = c_n/c_{n,cr}$ is the modal damping ratio and $K_n = M_n \omega^2{}_n$ is the modal stiff- ness. The total response is then obtained from Eq. (17.65) as the superposition of the solution of the modal

Eq. (17.74) for as many modes as desired. Though the summation in Eq. (17.65) is over an infinite number of terms, in most structural problems only the first few modes have any significant contribution to the total response and in some cases the response is given essentially by the contribution of the first mode alone.

The modal Eq. (17.74) is completely general and applies to beams with any type of load distribution If the loads are concentrated rather than distributed, the integral in Eq. (17.72) merely becomes a summation having one term for each load. The computation of the integral in Eqs. (17.71) and (17.72) becomes tedious except for the simply supported beam because the normal shapes are rather complicated functions. Values of the ratios of these integrals needed for problems with uniform distributed load are presented in the last columns of Tables 17.1, 17.2, 17.3, 17.4, and 17.5 for some common types of beams.

Illustrative Example 17.1

Consider in Fig. 17.3 a simply supported uniform beam subjected to a concentrated constant force suddenly applied at a section *XI* units from the left support. Determine the response using modal analysis. Neglect damping.

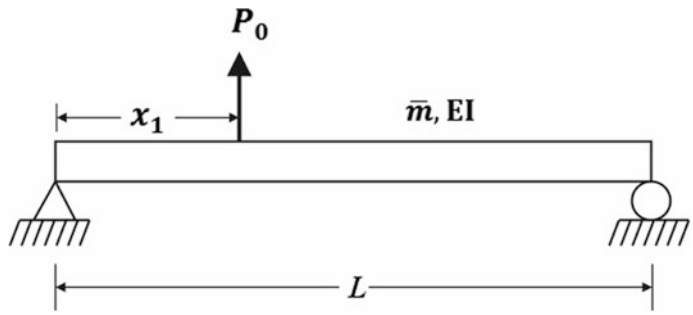


Fig. 17.3 Simply support beam subjected to a suddenly applied force

Solution:

The modal shapes of a simply supported beam by Eq. (17.28) are

$$\Phi_n = \sin\frac{n\pi x}{L}, \qquad \text{n} = 1, 2, 3 \ldots \tag{a}$$

and the modal force by Eq. (17.72)

$$F_n(t) = \int_0^L \Phi_n(x) p(x,t) dx$$

In this problem $p\ (x, t) = P_0\ at\ x = x_1$; otherwise, $p\ (\text{x}, t) = 0$. Hence

$$F_n(t) = P_0 \Phi_n(x_1)$$

or using Eq. (a), we obtain

$$F_n(t) = P_0 \sin\frac{n\pi x_1}{L} \tag{b}$$

The modal mass by Eq. (17.71) is

$$M_n = \int_0^L \bar{m}\Phi^2(x)dx = \int_0^L \bar{m}\sin^2\frac{n\pi x}{L}dx = \frac{\bar{m}L}{2} \tag{c}$$

Substituting the modal force, Eq. (b), and the modal mass, Eq. (c), into the modal Eq. (17.70) results in

$$\ddot{z}_n(t) + \omega_n^2 z_n(t) = \frac{P_0 \sin(n\pi x_1/L)}{\bar{m}L/2} \tag{d}$$

For initial conditions of zero displacement and zero velocity, the solution of Eq. (d) from Eq. (4.5) is

$$z_n = (z_{st})_n(1 - \cos\omega_n t) \tag{e}$$

in which

$$(z_{st})_n = \frac{2P_0 \sin(n\pi x_1/L)}{\omega_n^2 \bar{m}L} \tag{f}$$

so that

$$z_n = \frac{2P_0 \sin(n\pi x_1/L)}{\omega_n^2 \bar{m}L}(1 - \cos\omega_n t) \tag{g}$$

The modal deflection at any section of the beam is

$$y_n(x,t) = \Phi_n(x) z_n(t) \tag{h}$$

which, upon substitution of Eqs. (a) and (g), becomes

$$y_n(x,t) = \frac{2P_0 \sin(n\pi x_1/L)}{\omega_n^2 \bar{m}L}(1 - \cos\omega_n t)\sin\frac{n\pi x}{L} \tag{i}$$

By Eq. (17.65), the total deflection is then

$$y(x,t) = \frac{2P_0}{\bar{m}L}\sum_n \left[\frac{1}{\omega_n^2}\sin\frac{n\pi x_1}{L}(1 - \cos\omega_n t)\sin\frac{n\pi x}{L}\right] \tag{j}$$

As a special case, let us consider the force applied at midspan, i.e.,. $x_1 = L/2$ Hence Eq. (j) becomes in this case

$$y(x,t) = \frac{2P_0}{\bar{m}L}\sum_n \left[\frac{1}{\omega_n^2}\sin\frac{n\pi}{2}(1 - \cos\omega_n t)\sin\frac{n\pi x}{L}\right] \tag{17.75}$$

From the latter (due to the presence of the factor $\sin n\pi/2$) it is apparent that all the even modes do not contribute to the deflection at any point. This is true because such modes are antisymmetrical (shapes in Table 17.1) and are not excited by a symmetrical load.

It is also of interest to compare the contribution of the various modes to the deflection at midspan. This comparison will be done on the basis of maximum modal displacement disregarding the manner in which these displacements combine. The amplitudes will indicate the relative importance of the modes. The dynamic load factor $(1 - \cos\omega_n t)$ in Eq. (17.75) has a maximum value of 2 for all the modes. Furthermore, since all sines are unity for odd modes and zero for even modes, the modal contributions are simply in proportion to $1/\omega^2 n$. Hence, from Table 17.1 the maximum modal deflections are in proportion to 1, 1/81, and 1/625 for the first, third, and fifth modes, respectively. It is apparent, in this example, that the higher modes contribute very little to the midspan deflection.

Illustrative Example 17.2

Determine the maximum deflection at the midpoint of the fixed beam shown in Fig. 17.4 subjected to a harmonic load $p(x, t) = p_0$ sin 300 t lb/in uniformly distributed along the span. Consider in the analysis the first three modes contributing to the response.

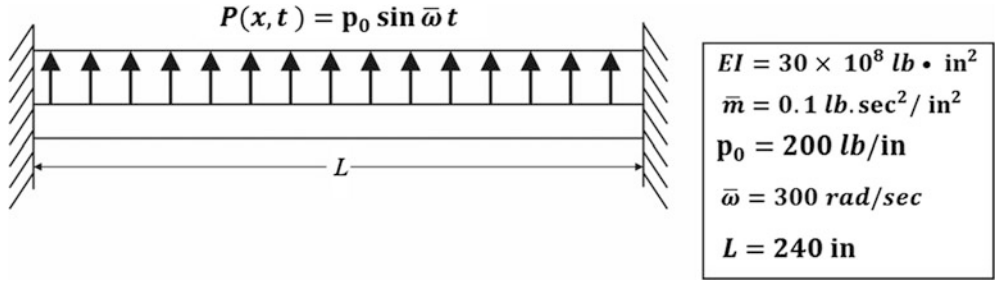


Fig. 17.4 Fixed beam with uniform harmonic load

Solution:

The natural frequencies for uniform beams are given by Eq. (17.13) as

$$\omega_n = C_n \sqrt{\frac{EI}{\bar{m}L^4}}$$

or, substituting numerical values for this example, we get

$$\omega_n = C_n \sqrt{\frac{30 \times 10^8}{0.1(240)^4}} \tag{a}$$

where the values of C_n are given for the first five modes in Table 17.3. The deflection of the beam is given by Eq. (17.65) as

$$y(x,t) = \sum_{n=1}^{\infty} \Phi_n(x) z_n(t) \tag{b}$$

in which $\Phi_n(x)$ is the modal shape defined for a fixed beam by Eq. (17.46) and $z_n(t)$ is the modal response.

The modal equation by Eq. (17.70) (neglecting damping) may be written as

$$\ddot{z}_n(t) + \omega_n^2 z_n(t) = \frac{\int_0^L p(x,t)\phi_n(x)dx}{\int_0^L \bar{m}\phi_n^2(x)dx}$$

Then, substituting numerical values to this example, we obtain

$$\ddot{z}_n(t) + \omega_n^2 z_n(t) = \frac{200 \int_0^L \phi_n(x)dx}{0.1 \int_0^L \phi_n^2(x)dx}$$

or

$$\ddot{z}_n(t) + \omega_n^2 z_n(t) = 2000 \mathrm{I_n} \sin 300t \tag{c}$$

Table 17.6 Modal response at midspan *for* the beam in Fig. 17.4

Mode	$\omega_n\left(\frac{\text{rad}}{\text{sec}}\right)$	a_nL	I_n	$z_n = \frac{2000I_n}{\omega_n^2 - \bar{\omega}^2}\,(\text{in})$	$\Phi_n\left(x = \frac{L}{2}\right)$
1	67.28	4.730	0.8380	−0.0194	1.588
2	185.45	7.853	0	0	0
3	363.56	10.996	0.3640	0.0173	−1.410
4	600.98	14.137	0	0	0
5	897.76	17.279	0.2323	0.00065	1.414

in which

$$I_n = \frac{\int_0^L \phi_n(x)dx}{\int_0^L \phi_n^2(x)dx}$$

is given for the first five modes in Table 17.3. The modal steady-state response is

$$z_n(t) = \frac{2000I_n}{\omega_n^2 - (300)^2}\sin 300\text{t} \tag{d}$$

The numerical calculations are conveniently presented in Table 17.6.

The deflections at midspan of the beam are then calculated from Eq. (b) and values in Table 17.6 as

$$y\left(\frac{L}{2}t\right) = [(1.588)(-0.0194) + (-1.410)(0.0173) + (1.414)(0.00065)]\sin 300t$$

$$y\left(\frac{L}{2}t\right) = -0.0541\sin 300\text{t}\ (\text{in})$$

17.6 Dynamic Stresses in Beams

To determine stresses in beams, we apply the following well-known relationships for bending moment M and shear force V. namely

$$M = EI\frac{\partial^2 y}{\partial x^2}$$

$$V = \frac{\partial M}{\partial x} = EI\frac{\partial^3 y}{\partial x^3}$$

Therefore, the calculation of the bending moment or the shear force requires only differentiation of the deflection function $y = y\,(x, t)$ with respect to x. For example, in the case of the simple supported beam with a concentrated load suddenly applied at its center, differentiation of the deflection function, Eq. (17.75), gives

$$M = -\frac{2\pi^2 P_0 EI}{\bar{m}L^3}\sum_n \left[\frac{n^2}{\omega_n^2}\sin\frac{n\pi}{2}(1 - \cos\omega_n t)\sin\frac{n\pi x}{L}\right] \tag{17.76}$$

$$V = -\frac{2\pi^2 P_0 EI}{\bar{m}L^4} \sum_n \left[\frac{n^3}{\omega_n^2} \sin\frac{n\pi}{2}(1 - \cos\omega_n t) \sin\frac{n\pi x}{L} \right] \tag{17.77}$$

We note that the higher modes are increasingly more important for moments than for deflections and even more so for shear force, as indicated by the factors 1, n^2, and n^3, respectively, in Eqs. (17.75), (17.76), and (17.77).

To illustrate, we compare the amplitudes for the first and third modes at their maximum values. Noting that $\omega^2{}_n$ is proportional to n^4 [Eq. (17.27)], we obtain from Eqs. (17.75), (17.76), and (17.77) the following ratios:

$$\frac{y_1}{y_3} = 3^4 = 81$$
$$\frac{M_1}{M_3} = 3^2 = 9$$
$$\frac{V_1}{V_3} = 3$$

This tendency in which higher modes have increasing importance in moment and shear calculation is generally true of beam response.

In those cases in which the first mode dominates the response, it is possible to obtain approximate deflections and stresses from static values of these quantities amplified by the dynamic load factor. For example, the maximum deflection of a simple supported beam with a concentrated force at midspan may be closely approximated by

$$y\left(x = \frac{L}{2}\right) = \frac{P_0 L^3}{48EI}(1 - \cos\omega_1 t)$$

If we consider only the first mode, the corresponding value given by Eq. (17.75) is

$$y\left(x = \frac{L}{2}\right) = \frac{2P_0}{\bar{m}L\omega_1^2}(1 - \cos\omega_1 t)b$$

Since, by Eq. (17.12). $\omega^2{}_1 = \pi EI/\bar{m}L^4$. It follows that

$$y\left(x = \frac{L}{2}\right) = \frac{2P_0 L^3}{\pi^4 EI}(1 - \cos\omega_1 t)b$$
$$= \frac{P_0 L^3}{48.7EI}(1 - \cos\omega_1 t)$$

The close agreement between these two computations is due to the fact that static deflections can also be expressed in terms of modal components, and for a beam supporting a concentrated load at mid-span the first mode dominates both static and dynamic response.

17.7 Summary

The dynamic analysis of single-span beams with distributed properties (mass and elasticity) and subjected to flexural loading was presented in this chapter. The extension of this analysis to multi-span or continuous beams and other structures, though possible, becomes increasingly complex and

impractical. The results obtained from these single-span beams are particularly important in evaluating approximate methods based on discrete models, as those presented in preceding chapters. From such evaluation, it has been found that the stiffness method of dynamic analysis in conjunction with the consistent mass formulation provides in general satisfactory results even with a rather coarse discretization of the structure.

The natural frequencies and corresponding normal modes of single-span beams with different supports are determined by solving the differential equation of motion and imposing the corresponding boundary conditions. The normal modes satisfy the orthogonality condition between any two modes m and n, namely,

$$\int_0^L \phi_m(x)\phi_n(x)\bar{m}dx = 0 \qquad (m \neq n)$$

The response of a continuous system may be determined as the superposition of modal contributions, that is

$$y(x,t) = \sum_n \phi_n(x) z_n(t)$$

where z_n (t) is the solution of n modal equation
Illustrative Example 17.2
in which

$$F_n(t) = \int_0^L \phi_n(x) p(x,t) dx$$

and

$$M_n = \int_0^L \bar{m}\phi^2{}_n(x) dx$$

The bending moment M and the shear for V at any section of a beam are calculated from the well-known relations

$$M = EI\frac{\partial^2 y}{\partial x^2}$$

$$V = EI\frac{\partial^3 y}{\partial x^3}$$

17.8 Problems

Problem 17.1
Determine the first three natural frequencies and corresponding nodal shapes of a simply supported reinforced concrete beam having a cross section 10 in wide by 24 in deep with a span of 36 ft. Assume the flexural stiffness of the beam, EI $= 3.5 \times 10^9$ lb*in^2 and weight per unit volume W $= 150$ lb/ft^3. (Neglect shear distortion and rotary inertia.)

Problem 17.2
Solve Problem 17.1 for the beam with its two ends fixed.

Problem 17.3
Solve Problem 17.1 for the beam with one end fixed and the other simply supported.

Problem 17.4
Determine the maximum deflection at the center of the simply supported beam of Problem 17.1 when a constant force of 2000 lb is suddenly applied at 9 f. from the left support.

Problem 17.5
A simply supported beam is prismatic and has the following properties: $\bar{m} = 0.3\,\text{lb}*\text{sec}^2/\text{in}$ per inch of span, $El = 10^6$ lb*in^2, and $L = 150$ in. The beam is subjected to a uniform distributed static load p_0 which is suddenly removed. Write the series expression for the resulting free vibration and deternline the amplitude of the first mode in terms of p_0.

Problem 17.6
The beam of Problem 17.5 is acted upon by a concentrated force given by $P(t) = 1000 \sin 500\ t$ lb applied at its midspan. Determine the amplitude of the steady-state motion at a quarter point from the left support in each of the first two modes. Neglect dampin.

Problem 17.7
Solve Problem 17.6 assuming 10% of critical damping in each mode. Also determine the steady-state motion at the quarter point considering the first t modes.

Problem 17.8
The cantilever beam shown in Fig. P17.8 is prismatic and has the following properties: $\bar{m} = 0.5$ lb*sec^2 /in per inch of span, $E = 30 \times 10^6$ psi, $L = 100$ in. and $I = 120$ in^4. Considering only the first mode, compute the maximum deflection and the maximum dynamic bending moment in the beam due to the load time function of Fig. P17.8b. (Chart in Fig. 4.5 may be used.)

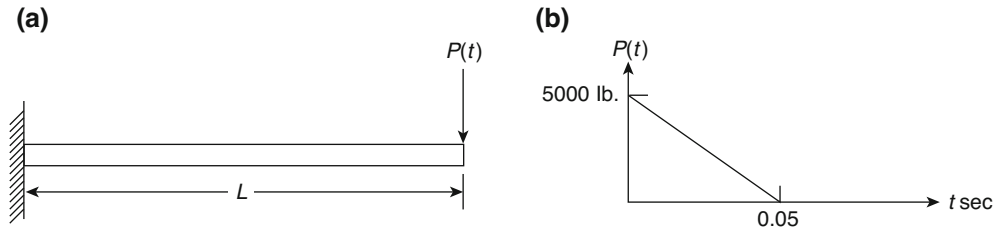


Fig. P17.8

Problem 17.9
A prismatic simply supported beam of the following properties: $L = 120$ in. $EI = 10^7$ lb*in^2, and $\bar{m} =$ 0.5 lb*sec^2/in per inch of span is loaded as shown in Fig. P17.9. Write the series expression for the deflection at the midsection of the beam.

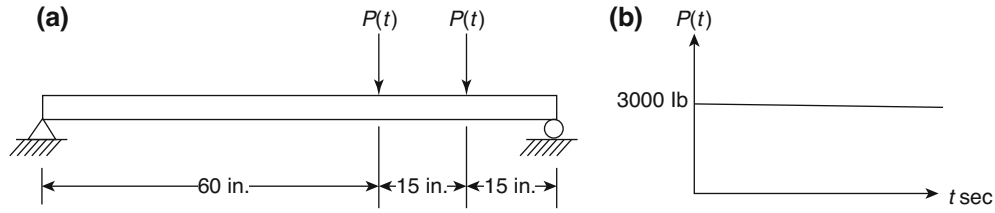


Fig. P17.9

Problem 17.10

Assuming that the forces on the beam of Problem 17.9 are applied for only a ime duration $t_d = 0.1$ sec. and considering only the first mode, determine the maximum deflection at each of the load points of the beam. (Chart in Fig. 4.4 may be used.)

Problem 17.11

A prismatic beam with its two ends fixed has the following properties: L = 180 in. EI = 30 × 108 lb*in2, $\bar{m}$ = 1 lb*sec2/in per inch of span. The beam is acted upon by a uniformly distributed impulsive force p(x, t) = 2000 sin 400 t lb during a time interval equal to half of the period of the sinusoidal load function ($t_d = \pi/400$ sec). Considering only the first mode, determine the maximum deflection at the midsection. (Chart in Fig. 5.3 may be used.)

Problem 17.12

Solve Problem 17.11 considering the first two modes.

Discretization of Continuous Systems 18

The modal superposition method of analysis was applied in the preceding chapter to some simple structures having distributed properties. The determination of the response by this method requires the evaluation of several natural frequencies and corresponding mode shapes. The calculation of these dynamic properties is rather laborious, as we have seen, even for simple structures such as one-span uniform beams. The problem becomes increasingly more complicated and unmanageable as this method of solution is applied to more complex structures. However, the analysis of such structures becomes relatively simple if for each segment or element of the structure the properties are expressed in terms of dynamic coefficients much in the same manner as done previously when static deflection functions were used as an approximation to dynamic deflections in determining stiffness, mass, and other coefficients.

In this chapter the dynamic coefficients relating harmonic forces and displacements at the nodal coordinates of a beam segment are obtained from dynamic deflection functions. These coefficients can then be used to assemble the dynamic matrix for the whole structure by the direct method as shown in the preceding chapters for assembling the system stiffness and mass matrices. Also, in the present chapter, the mathematical relationship between the dynamic coefficients based on dynamic displacement functions and the coefficients of the stiffness and consistent mass matrices derived from static displacement functions is established.

18.1 Dynamic Matrix for Flexural Effects

As in the case of static influence coefficients (stiffness coefficients, for example), the dynamic influence coefficients also relate forces and displacements at the nodal coordinates of a beam element. The difference between the dynamic and static coefficients is that the dynamic coefficients refer to nodal forces and displacements that vary harmonically while the static coefficients relate static forces and displacements at the nodal coordinates. The dynamic influence coefficient S_{ij} is then defined as the harmonic force of frequency ϖ at nodal coordinates i, due to a harmonic displacement of unit amplitude and of the same frequency at nodal coordinate j.

To determine the expressions for the various dynamic coefficients for a uniform beam element as shown in Fig. 18.1, we refer to the differential equation of motion, Eq. (17.5), which in the absence of external loads in the span, that is, $p\,(x, t) = 0$, is

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M. Paz, Y. H. Kim, *Structural Dynamics*, https://doi.org/10.1007/978-3-319-94743-3_18

$$EI\frac{\partial^4 y}{\partial x^4} + \bar{m}\frac{\partial^2 y}{\partial t^2} = 0 \tag{18.1}$$

For harmonic boundary displacements of frequency ϖ, we introduce in Eq. (18.1) the trial solution

$$y(x,t) = \Phi(x)\sin\overline{\omega}t \tag{18.2}$$

Substitution of Eq. (18.2) into Eq. (18.1) yields

$$\Phi^{IV}(x) - \bar{a}^4\Phi(x) = 0 \tag{18.3}$$

where

$$\bar{a}^4 = \frac{\bar{m}\overline{\omega}^2}{EI} \tag{18.4}$$

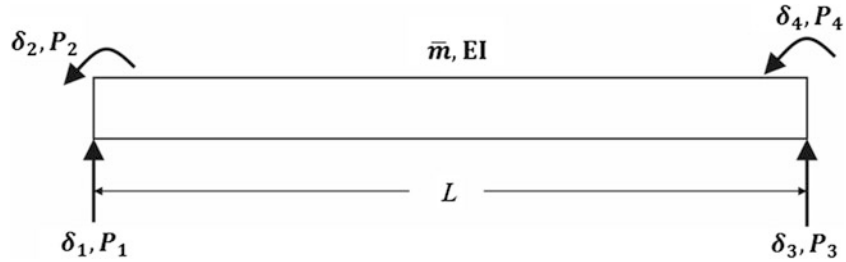

Fig. 18.1 Nodal coordinates of a flexural beam segment

We note that Eq. (18.3) is equivalent to Eq. (17.10), which is the differential equation for the shape function of a beam segment in free vibration. The difference between these two equations is that Eq. (18.3) is a function of the parameter $\bar{a}$ which, in turn, is a function of the forcing frequency ϖ while "a" in (17.10) depends on the natural frequency ω. The solution of Eq. (18.3) is of the same form as the solution of Eq. (17.10). Thus by analogy with Eq. (17.20), we can write.

$$\Phi(x) = C_1\sin\bar{a}x + C_2\cos\bar{a}x + C_3\sinh\bar{a}x + C_4\cosh\bar{a}x \tag{18.5}$$

Now, to obtain the dynamic coefficient for the beam segment, boundary conditions indicated by Eqs. (18.6) and (18.7) are imposed:

$$\begin{aligned} \Phi(0) &= \delta_1, & \Phi(L) &= \delta_3 \\ \Phi'(0) &= \delta_2, & \Phi'(L) &= \delta_4 \end{aligned} \tag{18.6}$$

Also

$$\begin{aligned} \Phi'''(0) &= \frac{P_1}{EI}, & \Phi'''(L) &= -\frac{P_3}{EI} \\ \Phi''(0) &= -\frac{P_2}{EI}, & \Phi''(L) &= \frac{P_4}{EI} \end{aligned} \tag{18.7}$$

In Eqs. (18.6), δ_1, δ_2, δ_3, and δ_4 are amplitudes of linear and angular harmonic displacements at the nodal coordinates while in Eqs. (18.7) P_1, P_2, P_3, and P_4 are the corresponding harmonic forces and moments as shown in Fig. 18.1. The substitution of the boundary conditions, Eqs. (18.6) and (18.7), into Eq. (18.5) results in

$$\begin{bmatrix}\delta_1\\ \delta_2\\ \delta_3\\ \delta_4\end{bmatrix}=\begin{bmatrix}0 & 1 & 0 & 1\\ \bar{a} & 0 & \bar{a} & 0\\ s & c & S & C\\ \bar{a}c & -\bar{a}s & \bar{a}C & \bar{a}S\end{bmatrix}\begin{bmatrix}C_1\\ C_2\\ C_3\\ C_4\end{bmatrix} \tag{18.8}$$

and

$$\begin{bmatrix}P_1\\ P_2\\ P_3\\ P_4\end{bmatrix}=EI\begin{bmatrix}-\bar{a}^3 & 0 & \bar{a}^3 & 0\\ 0 & \bar{a}^2 & 0 & -\bar{a}^2\\ \bar{a}^3c & -\bar{a}^3s & -\bar{a}^3C & -\bar{a}^3S\\ \bar{a}^2s & -\bar{a}^2c & \bar{a}^2S & \bar{a}^2C\end{bmatrix}\begin{bmatrix}C_1\\ C_2\\ C_3\\ C_4\end{bmatrix} \tag{18.9}$$

in which

$$\begin{aligned} s &= \sin\bar{a}L, & S &= \sinh\bar{a}L\\ c &= \cos\bar{a}L, & C &= \cosh\bar{a}L\end{aligned} \tag{18.10}$$

Next, Eq. (18.8) is solved for the constants of integration C_1, C_2, C_3, and C_4, which are subsequently substituted into Eq. (18.9). We thus obtain the dynamic matrix relating harmonic displacements and harmonic forces at the nodal coordinate of the beam element, namely.

$$\begin{bmatrix}P_1\\ P_2\\ P_3\\ P_4\end{bmatrix}=B\begin{bmatrix}\bar{a}^2(cS+sC) & & \text{Symmetric} & \\ \bar{a}sS & sC-cS & & \\ -\bar{a}^2(s+S) & \bar{a}(c-C) & \bar{a}^2(cS+sC) & \\ \bar{a}(C-c) & S-s & -\bar{a}sS & sC-cS\end{bmatrix}\begin{bmatrix}\delta_1\\ \delta_2\\ \delta_3\\ \delta_4\end{bmatrix} \tag{18.11}$$

where

$$B=\frac{\bar{a}EI}{1-cC} \tag{18.12}$$

We require the denominator to be different from zero, that is,

$$1-\cos\bar{a}L\cosh\bar{a}L\neq 0 \tag{18.13}$$

The element dynamic matrix in Eq. (18.11) can then be used to assemble the system dynamic matrix for a continuous beam or a plane frame in a manner entirely analogous to the assemblage of the system stiffness matrix from element stiffness matrices.

18.2 Dynamic Matrix for Axial Effects

The governing equation for axial vibration of a beam element is obtained by establishing the dynamic equilibrium of a differential element dx of the beam, as shown in Fig. 18.2. Thus

$$\begin{aligned}&\left(P+\frac{\partial P}{\partial x}dx\right)-P-\left(\bar{m}\,dx\right)\frac{\partial^2 u}{\partial t^2}=0\\ &\frac{\partial P}{\partial x}=\bar{m}\frac{\partial^2 u}{\partial t^2}\end{aligned} \tag{18.14}$$

where u is the displacement at x. The displacement at $x + dx$ will then be $u + (\partial u/\partial x)\, dx$. It is evident that the element dx in the new position has changed length by an amount $(\partial u/\partial x)\, dx$, and thus the strain is $\partial u/\partial x$. Since from Hooke's law the ratio of stress to strain is equal to the modulus of elasticity E, we can write

$$\frac{\partial u}{\partial x} = \frac{P}{AE} \tag{18.15}$$

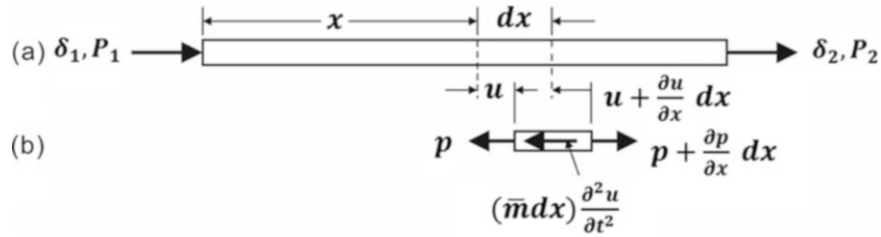

Fig. 18.2 Axial effects on a beam. (**a**) Nodal axial coordinates. (**b**) Forces acting on a differential element

where A is the cross-sectional area of the beam. Differentiating with respect to x results in

$$AE\frac{\partial^2 u}{\partial x^2} = \frac{\partial P}{\partial x} \tag{18.16}$$

and combining Eqs. (18.14) and (18.16) yields the differential equation for axial vibration of a beam segment, namely,

$$\frac{\partial^2 u}{\partial x^2} - \frac{\bar{m}}{AE}\frac{\partial^2 u}{\partial t^2} = 0 \tag{18.17}$$

A solution of Eq. (18.17) of the fond

$$u(x,t) = U(x)\sin\overline{\omega}t \tag{18.18}$$

will result in a harmonic motion of amplitude

$$U(x) = C_1 \sin bx + C_2 \cos bx \tag{18.19}$$

where

$$b = \sqrt{\frac{\bar{m}\overline{\omega}^2}{AE}} \tag{18.20}$$

and C_1 and C_2 are constants of integration.

To obtain the dynamic matrix for the axially vibrating beam segment, boundary conditions indicated by Eqs. (18.21) and (18.22) are imposed, namely.

$$U(0) = \delta_1, \qquad U(L) = \delta_2 \tag{18.21}$$

$$U'(0) = -\frac{P_1}{AE}, \qquad U'(L) = \frac{P_2}{AE} \tag{18.22}$$

where δ_1 and δ_2 are the displacements and P_1 and P_2 are the forces at the nodal coordinates of the beam segment as shown in Fig. 18.2.

Substitution of the boundary conditions, Eqs. (18.21) and (18.22), into Eq. (18.19) results in

$$\begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \sin bL & \cos bL \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} \tag{18.23}$$

and

$$\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = AEb \begin{bmatrix} -1 & 0 \\ \cos bL & -\sin bL \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} \tag{18.24}$$

Then, solving Eq. (18.23) for the constants of integration, we obtain

$$\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} -\cot bL & \operatorname{cosec} bL \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix} \tag{18.25}$$

subject to the condition

$$\sin bL \neq 0 \tag{18.26}$$

Finally, the substitution of Eq. (18.25) into Eq. (18.24) results in Eq. (18.27) relating harmonic forces and displacement at the nodal coordinates through the dynamic matrix for an axially vibrating beam segment. Thus we have.

$$\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = EAb \begin{bmatrix} \cot bL & -\operatorname{cosec} bL \\ -\operatorname{cosec} bL & \cot bL \end{bmatrix} \begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix} \tag{18.27}$$

18.3 Dynamic Matrix for Torsional Effects

The equation of motion of a beam segment in torsional vibration is similar to that of the axial vibration of beams discussed in the preceding section. Let x (Fig. 18.3) be measured along the length of the beam. Then the angle of twist for any element of length dx of the beam due to a torque T is

$$d\theta = \frac{T\,dx}{J_T G} \tag{18.28}$$

where $J_T G$ is the torsional stiffness given by the product of the torsional constant J_T (J_T is the polar moment of inertial for circular sections) and the shear modulus of elasticity G. The torque applied on the faces of the element are T and $T + (\partial T/\partial x)\,dx$ as shown in Fig. 18.3. From Eq. (18.28), the net torque is then

$$\frac{\partial T}{\partial x}dx = J_T G \frac{\partial^2 \theta}{\partial x^2}dx \tag{18.29}$$

Equating this torque to the product of the mass moment of inertia $I_{\bar{m}}$ of the element dx and the angular acceleration $\partial^2\theta/\partial t^2$, we obtain the differential equation of motion

$$J_T G \frac{\partial^2 \theta}{\partial x^2}dx = I_{\bar{m}} \frac{\partial^2 \theta}{\partial t^2}dx$$

or

$$\frac{\partial^2\theta}{\partial x^2} - \frac{I_{\bar{m}}}{J_T G}\frac{\partial^2\theta}{\partial t^2} = 0 \tag{18.30}$$

where $I_{\bar{m}}$ is the mass moment of inertia per unit length about the longitudinal axis x given by

$$I_{\bar{m}} = \bar{m}\frac{I_0}{A} \tag{18.31}$$

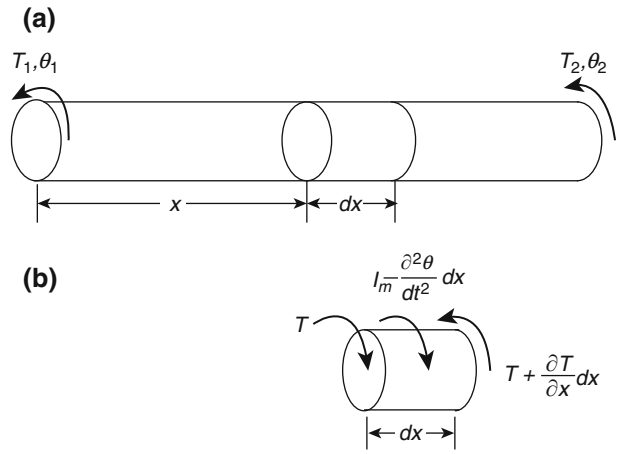

Fig. 18.3 Torsional effects on a beam. (**a**) Nodal torsional coordinates. (**b**) Moments acting on a differential element

in which I_0 is the polar moment of inertia of the cross-sectional area A.

We seek a solution of Eq. (18.30) in the form

$$\theta(x,t) = \theta(x)\sin\overline{\omega}t$$

which, upon substitution into Eq. (18.30), results in a harmonic torsional motion of amplitude

$$\theta(\mathrm{x}) = C_1 \sin cx + C_2 \cos cx \tag{18.32}$$

in which

$$c = \sqrt{\frac{I_{\bar{m}}\overline{\omega}^2}{J_T G}} \tag{18.33}$$

For a circular section, the torsional constant J_T is equal to the polar moment of inertia I_0. Thus Eq. (18.33) reduces to

$$c = \sqrt{\frac{\bar{m}\,\overline{\omega}^2}{AG}} \tag{18.34}$$

Since $I_{\bar{m}} = I_0\bar{m}/A$ as indicated by Eq. (18.31).

We note that Eq. (18.30) for torsional vibration is analogous to Eq. (18.17) for axial vibration of beam segments. It follows that by analogy to Eq. (18.27) we can write the dynamic relation between torsional moments and rotations in a beam segment. Hence

$$\begin{bmatrix} T_1 \\ T_2 \end{bmatrix} = J_T \mathrm{Gc} \begin{bmatrix} \cot cL & -\operatorname{cosec} cL \\ -\operatorname{cosec} cL & \cot cL \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} \tag{18.35}$$

18.4 Beam Flexure Including Axial-Force Effect

When a beam is subjected to a force along its longitudinal axis in addition to lateral loading, the dynamic equilibrium equation for a differential element of the beam is affected by the presence of this force. Consider the beam shown in Fig. 18.4 in which the axial force is assumed to remain constant during flexure with respect to both magnitude and direction. The dynamic equilibrium for a differential element dx of the beam (Fig. 18.4b) is established by equating to zero both the sum of the forces and the sum of the moments.

Summing forces in the y direction, we obtain

$$V + p(x,t)dx - \left(V + \frac{\partial V}{\partial x}dx\right) - (\bar{m}\ dx)\frac{\partial^2 y}{\partial t^2} = 0 \tag{18.36}$$

which, upon reduction, yields

$$\frac{\partial V}{\partial x} + \bar{m}\frac{\partial^2 y}{\partial t^2} = p(x,t) \tag{18.37}$$

The summation of moments about point 0 gives

$$M + Vdx - \left(M + \frac{\partial M}{\partial x}\right) + \frac{1}{2}\left(p(x,t) - \bar{m}\frac{\partial^2 y}{\partial t^2}\right)dx^2 - N\frac{\partial y}{\partial x}dx = 0 \tag{18.38}$$

Discarding higher order terms, we obtain for the shear force the expression

$$V = N\frac{\partial y}{\partial x} + \frac{\partial M}{\partial x} \tag{18.39}$$

Then using the familiar relationship from bending theory,

$$M = EI\frac{\partial^2 y}{\partial x^2} \tag{18.40}$$

and combining Eqs. (18.37), (18.39), and (18.40), we obtain the equation of motion of a beam segment including the effect of the axial forces, that is,

$$EI\frac{\partial^4 y}{\partial x^4} + N\frac{\partial^2 y}{\partial x^2} + \bar{m}\frac{\partial^2 y}{\partial t^2} = p(x,t) \tag{18.41}$$

A comparison of Eqs. (18.41) and (17.5) reveals that the presence of the axial force gives rise to an additional transverse force acting on the beam. As indicated previously in Sect. 17.1, in the derivation of Eq. (18.41) it has been assumed that the deflections are small and that the deflections due to shear forces or rotary inertia are negligible.

In the absence of external loads applied to the span of the beam, Eq. (18.41) reduces to

$$EI\frac{\partial^4 y}{\partial x^4} + N\frac{\partial^2 y}{\partial x^2} + \bar{m}\frac{\partial^2 y}{\partial t^2} = 0 \tag{18.42}$$

The solution of Eq. (18.42) is found as before by substituting

$$y(x,t) = \Phi(x)\sin\bar{\omega}t \tag{18.43}$$

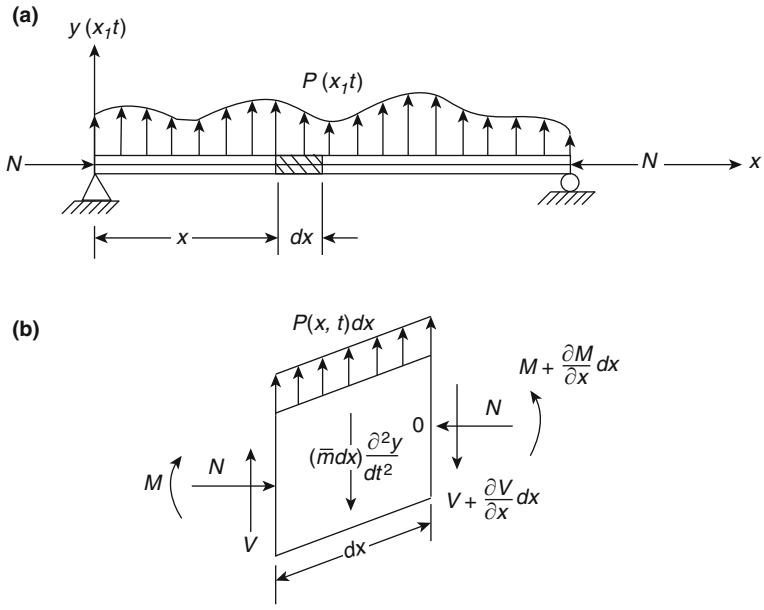

Fig. 18.4 Beam supporting constant axial force and lateral dynamic load. (**a**) Loaded beam. (**b**) Forces acting on a differential element

We thereby obtain the ordinary differential equation

$$\frac{d^4\Phi}{dx^4} + \frac{N}{EI}\frac{d^2\Phi}{dx^2} - \frac{\bar{m}\bar{\omega}^2}{EI}\Phi = 0 \tag{18.44}$$

The solution of Eq. (18.44) is

$$\Phi(x) = A\sin p_2 x + B\cos p_2 x + C\sinh p_1 x + D\cosh p_1 x \tag{18.45}$$

Where A, B, C, and D are constants of integration and

$$p_1 = \sqrt{\frac{-\alpha}{2} + \sqrt{\left(\frac{\alpha}{2}\right)^2 + \beta}}$$

$$p_2 = \sqrt{\frac{\alpha}{2} + \sqrt{\left(\frac{\alpha}{2}\right)^2 + \beta}} \tag{18.46}$$

$$\alpha = \frac{N}{EI} \tag{18.47}$$

$$\beta = \frac{\bar{m}\bar{\omega}^2}{EI} \tag{18.48}$$

To obtain the dynamic matrix (which in this case includes the effect of axial forces) for the transverse vibration of the beam element, the boundary conditions, Eqs. (18.49), are imposed, namely

$$\Phi(0) = \delta_1, \qquad \Phi(L) = \delta_3$$
$$\frac{d\Phi(0)}{dx} = \delta_2, \qquad \frac{d\Phi(L)}{dx} = \delta_4$$

$$\begin{aligned} \frac{d^3\Phi(0)}{dx^3} &= \frac{P_1}{EI} - \frac{N}{EI}\delta_2, & \frac{d^3\Phi(L)}{dx^3} &= -\frac{P_3}{EI} - \frac{N}{EI}\delta_4 \\ \frac{d^3\Phi(0)}{dx^3} &= -\frac{P_2}{EI}, & \frac{d^2\Phi(L)}{dx^2} &\frac{P_4}{EI} \end{aligned} \tag{18.49}$$

In Eqs. (18.49) δ_1, δ_3 and δ_2, δ_4 are, respectively, the transverse and angular displacements at the ends of the beam, while P_1, P_3 and P_2, P_4 are corresponding forces and moments at these nodal coordinates. The substitution into Eq. (18.45) of the boundary conditions given by Eqs. (18.49) results in a system of eight algebraic equations which upon elimination of the four constants of integration A, B, C, and D yields the dynamic matrix (including the effect of axial forces) relating harmonic forces and displacements at the nodal coordinates of a beam segment. The final result is

$$\begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{bmatrix} = \begin{bmatrix} S_{11} & & \textit{Symmetric} & \\ S_{21} & S_{22} & & \\ S_{31} & S_{32} & S_{33} & \\ S_{41} & S_{42} & S_{43} & S_{44} \end{bmatrix} \begin{bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \end{bmatrix} \tag{18.50}$$

where

$$\begin{aligned} S_{11} &= S_{33} = B\left[\left(p_1^2 p_2^3 + p_1^4 p_2\right)cS + \left(p_1 p_2^4 + p_1^3 p_2^2\right)sC\right] \\ S_{21} &= -S_{43} = B\left[\left(p_1 p_2^3 - p_1^3 p_2\right) + \left(p_1^3 p_2 - p_1 p_2^3\right)cC + 2p_2^2 p_2^2 sS\right] \\ S_{22} &= S_{44} = B\left[\left(p_2^2 p_1 + p_1^3\right)sC - \left(p_2^3 + p_1^2 p_2\right)cS\right] \\ S_{41} &= -S_{32} = B\left[\left(p_1 p_2^3 + p_1^3 p_2\right)(C - c)\right] \\ S_{31} &= B\left[\left(-p_1^2 p_2^3 - p_1^4 p_2\right)S - \left(p_1^3 p_2^2 + p_1 p_2^4\right)s\right] \\ S_{42} &= B\left[\left(p_1^2 p_2 + p_2^3\right)S - \left(p_1 p_2^2 + p_1^3\right)s\right] \end{aligned} \tag{18.51}$$

In the above, the letters s, c, S, and C denote

$$\begin{aligned} s &= \sin p_2 L, & S &= \sinh p_1 L \\ c &= \cos p_2 L, & C &= \cosh p_1 L \end{aligned}$$

and the letter B denotes

$$B = \frac{EI}{2p_1 p_2 - 2p_1 p_2 cC + \left(p_1^2 - p_2^2\right)sS} \tag{18.52}$$

Furthermore, Eq. (18.50) is subject to the condition

$$2p_1 p_2 - 2p_1 p_2 cC + \left(p_1^2 - p_2^2\right)sS \neq 0 \tag{18.53}$$

18.5 Power Series Expansion of the Dynamic Matrix for Flexural Effects

It is of interest to demonstrate that the influence coefficients of the stiffness matrix, Eq. (10.20), and of the consistent mass matrix, Eq. (10.34), may be obtained by expanding the influence coefficients of the dynamic matrix in a Taylor's series (Paz 1973). For the sake of the discussion, we consider the dynamic coefficient from the second row and first column of the dynamic matrix, Eq. (18.11),

$$S_{21} = \frac{a^2 EI \sin \bar{a}L \sinh \bar{a}L}{1 - \cos \bar{a}L \cosh \bar{a}L} \tag{18.54}$$

In the following derivation, operations with power series, including addition, subtraction, multiplication, and division, are employed. The validity of these operations and convergence of the resulting series is proved in Knopp.[1] In general, convergent power series may be added, subtracted, or multiplied and the resulting series will converge at least in the common interval of convergence of the two original series. The operation of division of two power series may be carried out formally; however, the determination of the radius of convergence of the resulting series is more complicated. It requires the use of theorems in the field of complex variables and it is related to analytical continuation. Very briefly, it can be said that the power series obtained by division of two convergent power series about a complex point Z_0 will be convergent in a circle with center Z_0 and of radius given by the closest singularity to Z_0 of the functions represented by the series in the numerator and denominator.

The known expansions in power series about the origin of trigonometric and hyperbolic functions are used in the intermediate steps in expanding the function in Eq. (18.54), namely,

$$\cos x \cosh x = 1 - \frac{x^4}{6} + \frac{x^8}{2520} - \frac{x^{12}}{7,484,400} + \dots$$
$$(1 - \cos x \cosh x)^{-1} = \frac{6}{x^4} + \frac{1}{70} + \frac{85x^4}{2,910,600} + \dots$$
$$\sin x \sinh x = x^2 - \frac{x^6}{90} + \frac{x^{10}}{113,400} - \dots$$

where $x = \bar{a}L$. Substitution of these series equations in the dynamic coefficient, Eq. (18.54), yields

$$S_{21} = \frac{\bar{a}^2 EI \sin \bar{a}L \sinh \bar{a}L}{1 - \cos \bar{a}L \cosh \bar{a}L} = \frac{6EI}{L^2} - \frac{11\bar{m}L^2\overline{\omega}^2}{210} - \frac{223\bar{m}^2L^6\overline{\omega}^4}{2,910,600EI} \tag{18.55}$$

The first term on the right-hand side of Eq. (18.55) is the stiffness coefficient k_{21} in the stiffness matrix, Eq. (10.20), and the second term, the consistent mass coefficient m_{21} in the mass matrix, Eq. (10.34). The series expansion, Eq. (18.55), is convergent in the positive real field for

$$0 < \bar{a}L < 4.73 \tag{18.56}$$

or from Eq. (18.4)

$$0 < \overline{\omega} < (4.73)^2 \sqrt{\frac{EI}{\bar{m}L^4}} \tag{18.57}$$

In Eq. (18.56) the numerical value 4.73 is an approximation of the closest singularity to the origin of the functions in the quotient expanded in Eq. (18.54).

The series expansions for all the coefficients in the dynamic matrix, Eq. (18.11), are obtained by the method explained in obtaining the expansion of the coefficient S_{21}. These series expansions are:

[1] Knopp, K., Theory and Application of Infinite Series, Blackie, London, 1963.

$$
\begin{aligned}
S_{33} = S_{11} &= \frac{12EI}{L^3} - \frac{13L\bar{m}\overline{\omega}^2}{35} - \frac{59L^5\bar{m}^2\overline{\omega}^4}{161,700EI} - \ldots \\
S_{21} = -S_{43} &= \frac{6EI}{L^2} - \frac{11L^2\bar{m}\overline{\omega}^2}{210} - \frac{223L^6\bar{m}^2\overline{\omega}^4}{2,910,600EI} - \ldots \\
S_{41} = -S_{32} &= \frac{6EI}{L^2} - \frac{13L^2\bar{m}\overline{\omega}^2}{420} + \frac{1681L^6\bar{m}^2\overline{\omega}^4}{23,284,800EI} - \ldots \\
S_{22} = -S_{44} &= \frac{4EI}{L} - \frac{L^3\bar{m}\overline{\omega}^2}{105} - \frac{71L^7\bar{m}^2\overline{\omega}^4}{4,365,800EI} - \ldots \\
S_{31} &= -\frac{12EI}{L^3} - \frac{9L\bar{m}\overline{\omega}^2}{70} - \frac{1279L^5\bar{m}^2\overline{\omega}^4}{3,880,800EI} - \ldots \\
S_{42} &= \frac{2EI}{L} - \frac{L^3\bar{m}\overline{\omega}^2}{140} - \frac{1097L^7\bar{m}^2\overline{\omega}^4}{69,854,400EI} - \ldots
\end{aligned}
\tag{18.58}
$$

18.6 Power Series Expansion of the Dynamic Matrix for Axial and for Torsional Effects

Proceeding in a manner entirely analogous to expansion of the dynamic coefficients for flexural effects, we can also expand the dynamic coefficients for axial and for torsional effects. The Taylor's series expansions, up to three terms, of the coefficients of the dynamic matrix in Eq. (18.27) (axial effects) are.

$$
\begin{aligned}
AEb\cot bL &= \frac{AE}{L} - \frac{\bar{m}\overline{\omega}^2 L}{3} - \frac{L^3\bar{m}^2\overline{\omega}^4}{45AE} - \ldots \\
-AEb\,\mathrm{cos\,ec}\,bL &= -\frac{AE}{L} - \frac{\bar{m}\overline{\omega}^2 L}{6} - \frac{7L^3\bar{m}^2\overline{\omega}^4}{300AE} - \ldots
\end{aligned}
\tag{18.59}
$$

It may be seen that the first term in each series of Eq. (18.59) is equal to the corresponding stiffness coefficient of the matrix in Eq. (11.3), and the second term to the consistent mass coefficient of the matrix in (11.26). Similarly, the Taylor's series expansions of the coefficients of the dynamic matrix for torsional effects, Eq. (18.35), are

$$
\begin{aligned}
J_T Gc\cot cL &= \frac{J_T G}{L} - \frac{LI_{\bar{m}}\overline{\omega}^2}{3} - \frac{L^3 I\, m^{-2}\overline{\omega}^4}{45GJ_T} - \ldots \\
-J_T Gc\,\mathrm{cos\,ec}\,cL &= -\frac{J_T G}{L} - \frac{LI_{\bar{m}}\overline{\omega}^2}{6} - \frac{7L^3 I\bar{m}^2\overline{\omega}^4}{300GJ_T} - \ldots
\end{aligned}
\tag{18.60}
$$

Comparing the first two terms of the above series with the stiffness and mass influence coefficients of the matrices in Eqs. (12.7) and (12.8), we find that for torsional effects the first term is also equal to the stiffness coefficient, and the second term to the consistent mass coefficient.

18.7 Power Series Expansion of the Dynamic Matrix Including the Effects of Axial Forces

The series expansions of the coefficients of the dynamic matrix, Eq. (18.50) (with axial effects), are obtained by the method described in the last two sections. Detailed derivation of these expansions are given by Paz and Dung (1975). The series expansion of the dynamic matrix, Eq. (18.50), is

$$[S] = [K] - [G_0]N - [M_0]\overline{\omega}^2 - [G_1]N^2 - [M_1]\overline{\omega}^4 - \ldots \tag{18.61}$$

where the first three matrices in this expansion [K], [G_0], and [M_0] are, respectively, the stiffness, geometric, and mass matrices which were obtained in previous chapters on the basis of static displacement functions. These matrices are given, respectively, by Eqs. (10.20), (10.45), and (10.34). The other matrices in Eq. (18.61) corresponding to higher order terms are represented as follows. The second-order mass-geometrical matrix

$$[A_1] = \frac{\bar{m}L^3}{EI}\begin{bmatrix} \frac{1}{3150} & & \text{Symmetric} & \\ \frac{L}{1360} & \frac{L^2}{3150} & & \\ -\frac{1}{3150} & \frac{L}{1680} & \frac{1}{3150} & \\ -\frac{L}{1680} & \frac{L^2}{3600} & -\frac{L}{1260} & \frac{L^2}{3150} \end{bmatrix}$$

The second-order geometrical matrix:

$$[G_1] = \frac{1}{EI}\begin{bmatrix} \frac{L}{700} & & \text{Symmetric} & \\ \frac{L^2}{1400} & \frac{11L^2}{6300} & & \\ -\frac{L}{700} & -\frac{L^2}{1400} & \frac{L}{700} & \\ \frac{L^2}{1400} & -\frac{13L^2}{12600} & -\frac{L^2}{1400} & \frac{11L^3}{6300} \end{bmatrix}$$

The second-order mass matrix:

$$[M_1] = \frac{\bar{m}^2L^3}{1000EI}\begin{bmatrix} \frac{59}{161.7} & & \text{Symmetric} & \\ \frac{223L}{2910.6} & \frac{71L^2}{4365.9} & & \\ \frac{1279}{3880.8} & \frac{1681L}{23284.8} & \frac{59}{161.7} & \\ -\frac{1681L}{23284.8} & -\frac{1097L^2}{69854.4} & -\frac{223L}{2910.6} & \frac{71L^2}{4365.9} \end{bmatrix}$$

18.8 Summary

The dynamic coefficients relating harmonic forces and displacements at the nodal coordinates of a beam segment were obtained from dynamic deflection equations. These coefficients can then be used in assembling the dynamic matrix for the entire structure by the same procedure (direct method) employed in assembling the stiffness and mass matrices for discrete systems.

In this chapter it has been demonstrated that the stiffness, consistent mass, and other influence coefficients may be obtained by expanding the dynamic influence coefficients in Taylor's series. This mathematical approach also provides higher order influence coefficients and the determination of the radius of convergence of the series expansion.

Part V

Special Topics: Fourier Analysis, Evaluation of Absolute Damping, Generalized Coordinates

Fourier Analysis and Response in the Frequency Domain

19

This chapter presents the application of Fourier series to determine: (1) the response of a system to periodic forces, and (2) the response of a system to nonperiodic forces in the frequency domain as an alternate approach to the usual analysis in the time domain. In either case, the calculations require the evaluation of integrals that, except for some relatively simple loading functions, employ numerical methods for their computation. Thus, in general, to make practical use of the Fourier method, it is necessary to replace the integrations with finite sums.

19.1 Fourier Analysis

The subject of Fourier series and Fourier analysis has extensive ramifications in its application to many fields of science and mathematics. We begin by considering a single-degree-of-freedom system under the action of a periodic loading, that is, a forcing function that repeats itself at equal intervals of time, T (the period of the function). Fourier has shown that a periodic function may be expressed as the summation of an infinite number of sine and cosine terms. Such a sum is known as a Fourier series.

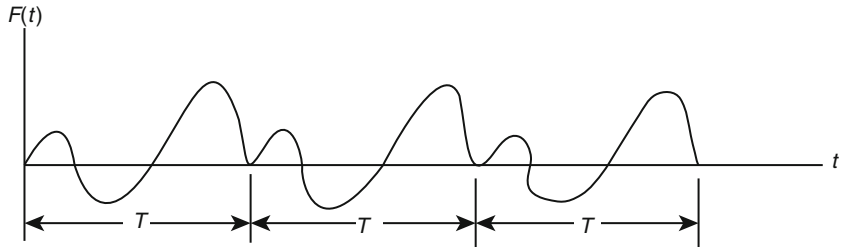

Fig. 19.1 Arbitrary periodic function

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M. Paz, Y. H. Kim, *Structural Dynamics*, https://doi.org/10.1007/978-3-319-94743-3_19

For a periodic function, such as the one shown in Fig. 19.1, the Fourier series may be written as

$$\begin{aligned} F(t) = a_0 + a_1 \cos \varpi t + a_2 \cos 2\varpi t + a_3 \cos 3\varpi t + a_4 \cos 4\varpi t + \ldots a_n \cos n\varpi t + \ldots \\ + b_1 \sin \varpi t + b_2 \sin 2\varpi t + b_3 \sin 3\varpi t + \ldots b_n \sin n\varpi t + \ldots \end{aligned} \tag{19.1}$$

or

$$F(t) = +a_0 + \sum_{n=1}^{\infty} \{a_n \cos n\varpi t + b_n \sin n\varpi t\} \tag{19.2}$$

where $\varpi = 2\pi/T$ is the frequency and T the period of the function. The evaluation of the coefficients a_o, a_n, and b_n for a given function F(t) is determined from the following expressions:

$$\begin{aligned} a_o &= \frac{1}{T}\int_{t_1}^{t_1+T} F(t)\,dt \\ a_n &= \frac{2}{T}\int_{t_1}^{t_1+T} F(t)\cos n\varpi t\,dt \\ b_n &= \frac{2}{T}\int_{t_1}^{t_1+T} F(t)\sin n\varpi t\,dt \end{aligned} \tag{19.3}$$

where t_1 in the limits of the integrals may be any value of time, but is usually equal to either –T/2 or zero. The constant a_0 equals the average of the periodic function $F(t)$.

19.2 Response to a Loading Represented by Fourier Series

The response of a single-degree-of-freedom system to a periodic force represented by its Fourier series is found as the superposition of the response to each component of the series. When the transient is omitted, the response of an undamped system to any term of the series is given by Eq. (3.9) as

$$u_n(t) = \frac{b_n/k}{1 - r_n^2} \sin n\varpi t \tag{19.4}$$

where $r_n = n\varpi/\omega$ and $\omega = \sqrt{k/m}$

Similarly, the response to any cosine term is

$$u_n(t) = \frac{a_n/k}{1 - r_n^2} \cos n\varpi t \tag{19.5}$$

The total response of an undamped, single-degree-of-freedom system may then be expressed as the superposition of the responses to all the force terms of the series, including the response a_0/k (steady-state response) to the constant force a_0. Hence we have

$$u(t) = \frac{a_0}{k} + \sum \frac{1}{1 - r_n^2}\left(\frac{a_0}{k}\cos n\varpi t + \frac{b_n}{k}\sin n\varpi t\right) \tag{19.6}$$

When the damping in the system is considered, the steady-state response for the general sine term of the series is given from Eq. (3.20) as

$$u_n(t) = \frac{b_n/k \sin(\mathrm{n}\varpi \mathrm{t} - \theta)}{\sqrt{\left(1 - \mathrm{r_n^2}\right)^2 + (2r_n\xi)^2}} \tag{19.7}$$

or

$$u_n(t) = \frac{b_n}{k} \cdot \frac{\sin \mathrm{n}\varpi \mathrm{t} \cos\theta - \cos \mathrm{n}\varpi \mathrm{t} \sin\theta}{\sqrt{\left(1 - \mathrm{r_n^2}\right)^2 + (2r_n\xi)^2}}$$

The substitution of sinθ and cosθ from Eq. (3.21) gives

$$u_s(t) = \frac{b_n}{k} \frac{\left(1 - r_n^2\right) \sin \mathrm{n}\varpi \mathrm{t} \ + 2\mathrm{r_n}\xi \cos \mathrm{n}\varpi \mathrm{t}}{\left(1 - r^2\right)^2 + (2r_n\xi)^2} \tag{19.8}$$

Similarly, for a cosine term of the series, we obtain

$$u_s(t) = \frac{a_n}{k} \frac{\left(1 - r_n^2\right) \sin \mathrm{n}\varpi \mathrm{t} \ + 2\mathrm{r_n}\xi \sin \mathrm{n}\varpi \mathrm{t}}{\left(1 - r^2\right)^2 + (2r_n\xi)^2} \tag{19.9}$$

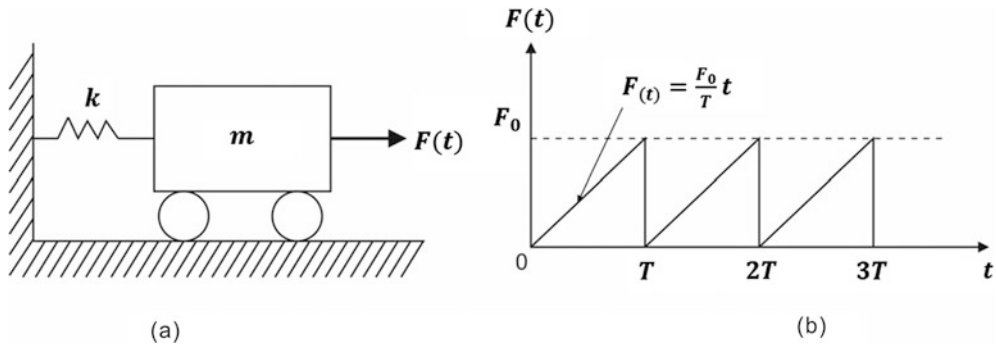

Fig. 19.2 Undamped oscillator acted upon by a periodic force

Finally, the total response is then given by the superposition of the terms expressed by Eqs. (19.8) and (19.9) in addition to the response to the constant term of the series. Therefore, the total response of a damped single-degree-of freedom system may be expressed as

$$u(t) = \frac{a_0}{k} + \frac{1}{k} \sum_{n=1}^{\infty} \left\{ \frac{a_n 2r_n\xi + b_n\left(1 - r_n^2\right)}{\left(1 - r_n^2\right)^2 + (2r_n\xi)^2} \sin n\varpi t + \frac{a_n(1 - r^2{}_n) - b_n 2r_n\xi}{\left(1 - r_n^2\right)^2 + (2r_n\xi)^2} \cos n\varpi t \right\} \tag{19.10}$$

Illustrative Example 19.1

As an application of the use of Fourier series in determining the response of a system to a periodic loading, consider the undamped simple oscillator in Fig. 19.2a which is acted upon by the periodic force shown in Fig. 19.2b.

Solution:

The first step is to determine the Fourier series expansion of F(t). The corresponding coefficients are determined from Eqs. (19.3) as follows:

$$a_0 = \frac{1}{T}\int_0^T \frac{F_0}{T} t \ \mathrm{dt} = \frac{F_0}{2}$$

$$a_n = \frac{2}{T}\int_0^T \frac{F_0}{T} t \ \cos n\varpi t \ \mathrm{dt} = 0$$

$$b_n = \frac{2}{T}\int_0^T \frac{F_0}{T} t \ \sin n\varpi t \ \mathrm{dt} = -\frac{F_0}{n\pi}$$

The response of the undamped system is then given from Eq. (19.6) as

$$U(t) = \frac{F_0}{2k} - \sum_{n=1}^{\infty} \frac{F_0 \sin n\varpi t}{n\pi k\left(1 - r_n^2\right)}$$

or in expanded form as

$$U(t) = \frac{F_0}{2k} - \frac{F_0 \sin \varpi t}{\pi k\left(1 - r_1^2\right)} - \frac{F_0 \sin 2\varpi t}{\pi k\left(1 - 4r_1^2\right)} - \frac{F_0 \sin 3\varpi t}{\pi k\left(1 - 9r_1^2\right)} - \cdots$$

where

$$r_1 = {}^{\varpi}/_{\omega}, \omega = \sqrt{k/m}, \quad and \ \varpi = 2\pi/T$$

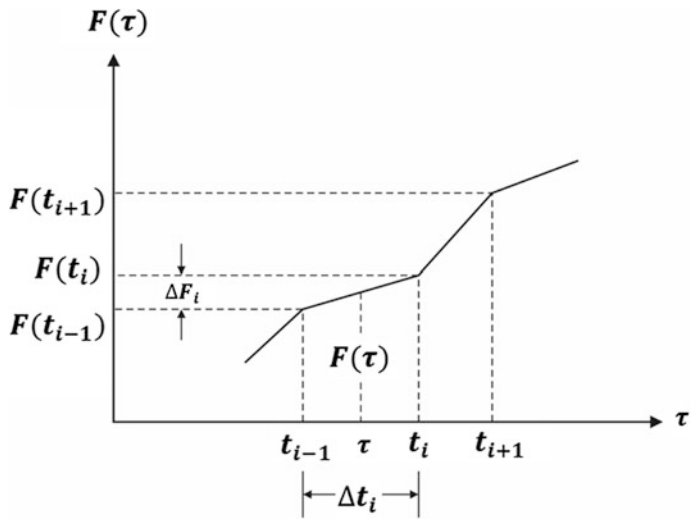

Fig. 19.3 Piecewise linear forcing function

19.3 Fourier Coefficients for Piecewise Linear Functions

Proceeding as before in the evaluation of Duhamel's integral, we can represent the forcing function by piecewise linear function as shown in Fig. 19.3.The calculation of Fourier coefficients, Eq. (19.3), is then obtained as a summation of the integrals evaluated for each linear segment of the forcing function, that is, as

$$a_0 = \frac{1}{T}\sum_{i=1}^{N}\int_{ti-1}^{ti} F(t)\,dt \tag{19.11}$$

$$a_n = \frac{2}{T}\sum_{i=1}^{N}\int_{ti-1}^{ti} F(t)\cos n\varpi t\,dt \tag{19.12}$$

$$b_n = \frac{2}{T}\sum_{i=1}^{N}\int_{ti-1}^{ti} F(t)\sin n\varpi t\,dt \tag{19.13}$$

where N is the number of segments of the piecewise forcing function. The forcing function in any interval $t_{i-1} \le t \le t_i$ is expressed by Eq. (4.20) as

$$F(t) = F(t_{i-1}) + \frac{\Delta F_i}{\Delta t_i}(t - t_{i-1}) \tag{19.14}$$

in which $\Delta F_i = F(t_i) - F(t_{i-1})$ and $\Delta t_i = t_i - t_{i-1}$. The integrals required in the expressions of a_n and b_n have been evaluated in Eqs. (4.21) and (4.22) and designated as A (t_i) and B (t_i) in the recurrent expressions (4.18) and (4.19). The use of Eqs. (4.18) through (4.22) to evaluate the coefficients a_n and b_n yields

$$\begin{aligned} a_n = \frac{2}{T}\sum_{i=1}^{N}\Big\{&\frac{1}{n\omega}\left(F(t_{i-1}) - t_{i-1}\frac{\Delta F_i}{\Delta t_i}\right)(\sin n\varpi t_i - \sin n\varpi t_{i-1}) \\ &+ \frac{\Delta F_i}{n^2\varpi^2\Delta t_i}((\cos n\varpi t_i - \cos n\varpi t_{i-1} + n\varpi(t_i \sin n\varpi t_i - t_{i-1}\sin n\varpi t_{i-1}))\Big\} \end{aligned} \tag{19.15}$$

$$\begin{aligned} b_n = \frac{2}{T}\sum_{i=1}^{N}\Big\{&\frac{1}{n\omega}\left(F(t_{i-1}) - t_{i-1}\frac{\Delta F_i}{\Delta t_i}\right)(\cos n\varpi t_{i-1} - \cos n\varpi t_i \\ &+ \frac{\Delta F_i}{n^2\varpi^2\Delta t_i}((\sin n\varpi t_i - \sin n\varpi t_{i-1}) - n\varpi(t_i \cos n\varpi t_i - t_{i-1}\cos n\varpi t_{i-1}))\Big\} \end{aligned} \tag{19.16}$$

The integral appearing in the coefficient a_o of Eq. (19.3) is readily evaluated after substituting F (t) from Eq. (19.14) into Eq. (19.11). This evaluation yields

$$a_0 = \frac{1}{T}\sum_{i=1}^{N}\{\Delta t_i(F_i + F_{i-1})/2\} \tag{19.17}$$

19.4 Exponential Form of Fourier Series

The Fourier series expression given by Eq. (19.2) may also be written in exponential form by substituting the trigonometric functions using Euler's relationships:

$$\sin n\varpi t = \frac{e^{in\varpi t} - e^{-in\varpi t}}{2i}$$

$$\cos n\varpi t = \frac{e^{in\varpi t} + e^{-in\varpi t}}{2} \tag{19.18}$$

The result of this substitution may be written as

$$F(t) = \sum_{n=-\infty}^{\infty} C_n e^{in\varpi t} \tag{19.19}$$

where

$$C_n = \frac{1}{T}\int_0^T F(t)e^{-in\varpi t}\ \mathrm{dt} \tag{19.20}$$

The interval of integration in Eq. (19.20) has been selected from zero to T for the periodic function. It should be noted that the exponential form for the Fourier series in Eq. (19.19) has the advantage of simplicity when compared to the equivalent trigonometric series, Eq. (19.2). The exponential form of the Fourier series can be used as before to determine the dynamic response of structural systems. However, a more efficient method is available for the determination of the coefficients C_n as well as for the calculation of the response for the single degree of freedom excited by the force expanded as in Eq. (19.19). This method, which is based on Fourier analysis for the discrete case, is presented in the next sections.

19.5 Discrete Fourier Analysis

When the periodic function F(t) is supplied only at N equally spaced time intervals ($\Delta t = T/N$) t_o., t_1, t_2, ... t_{N-1}, where $t_j = j\ \Delta\ t$, the integrals in Eq. (19.3) may be replaced approximately by the summations

$$\begin{aligned} a_n &= \frac{1}{T}\sum_{j=0}^{N-1} F(t_j)\cos n\varpi t_j \Delta t \\ b_n &= \frac{1}{T}\sum_{j=0}^{N-1} F(t_j)\sin n\varpi t_j \Delta t, \qquad n = 0, 1, 2, \ldots \end{aligned} \tag{19.21}$$

where $\varpi = 2\pi/T$. The above definitions for the Fourier coefficients have been slightly altered by omitting the factor 2 in the expressions for a_n and b_n. In this case Eq. (19.2) is then written as

$$F(t_i) = 2\sum_{n=1}^{\infty}\{a_n \cos n\varpi t + b_n \sin n\varpi t\} \tag{19.22}$$

If we use complex notation, Eq. (19.21) can be combined into a single form by defining

$$C_n = a_n - ib_n \tag{19.23}$$

and using Euler's relationship

$$e^{-in\varpi t_j} = \cos n\varpi t_j - i\sin n\varpi t_j \tag{19.24}$$

to obtain after substituting Eq. (19.21) into Eq. (19.23)

$$C_n = \frac{1}{T}\sum_{j=0}^{N-1} F(t_j)e^{-in\varpi t_j}\Delta t \tag{19.25}$$

Substituting $t_j = j\ \Delta\ t$, $T = N\ \Delta\ t$, and $\varpi = 2\pi/T$ into Eq. (19.25), we obtain

$$C_n = \frac{1}{N}\sum_{j=0}^{N-1} F(t_j)e^{-2\pi i(nj/N)}, \qquad n = 0, 1, 2, \ldots \tag{19.26}$$

Equation (19.26) may be considered as an approximate formula for calculating he complex Fourier coefficients in Eq. (19.20). The discrete coefficients given by Eq. (19.26) do not provide sufficient information to obtain a continuous function for F(t); however, it is a most important fact that it does allow to obtain all the discrete values of the series $\{F(t_j)\}$ exactly (Newland 1984). This fact leads to the formal definition of the discrete Fourier transform of the series $\{F(t_j)\}$, $j = 0, 1,. 2, \ldots . N{-}I$, given by

$$C_n = \frac{1}{N}\sum_{j=0}^{N-1} F(t_j)e^{-2\pi i(nj/N)}, \qquad n = 0, 1, 2, \ldots, (N-1) \tag{19.27}$$

and its inverse discrete Fourier transform by

$$F(t_j) = \sum_{n=0}^{N-1} C_n e^{2\pi i(nj/N)}, \qquad j = 0, 1, 2, \ldots, (N-1) \tag{19.28}$$

The range of the summation in Eq. (19.28) has been limited from 0 to $(N{-}1)$ in order to maintain the symmetry of transform pair Eqs. (19.27) and (19.28). It is important to realize that in the calculation of the summation indicated in Eq. (19.28), the frequencies increase with increasing index n up to $n = N/2$. It will be shown very shortly that, for $n > N/2$, the corresponding frequencies are equal to the negative of frequencies of order $N{-}n$. This fact restricts the harmonic components that may be represented in the series to a maximum of $N/2$. The frequency corresponding to this maximum order $\omega_{N/2} = (N/2)\varpi$ is known as the Nyquist frequency or sometimes as the *folding frequency*. Moreover, if there are harmonic components above $\omega_{N/2}$ in the original function, these higher components will introduce distortions in the lower harmonic components of the series. This phenomenon is called *aliasing* (Newland 1984, p. 118). In view of this fact, it is recommended that the number of intervals or sampled points N should be at least twice the highest harmonic component present in the function.

The Nyquist frequency ω_u is given in radians per second by

$$\omega_u = \frac{2\pi N/2}{T} = \frac{2\pi N/2}{N\Delta t} = \frac{\pi}{\Delta t}\left(\frac{rad}{\sec}\right) \tag{19.29}$$

and in cycles per second by

$$f_u = \frac{\omega_u}{2\pi} = \frac{1}{2\Delta t}(cps) \tag{19.30}$$

As a matter of interest, Example 19.4 is presented later in this chapter to illustrate the importance of choosing the number of sampling points N for the excitation function sufficiently large to avoid spurious results due to aliasing.

Having represented an arbitrary discrete function by a finite sum, we may hen also obtain as a discrete function the response of a simple oscillator excited by the harmonic components of the

loading function. Again, only the steady-state response will be considered. The introduction of the unit exponential forcing function $E_n = e^{i\omega_n t}$ into the equation of motion, Eq. (3.13), leads to

$$m\ddot{u} + c\dot{u} + ku = e^{i\omega_n t} \tag{19.31}$$

which has a steady-state solution of the form

$$u(t) = H(\omega_n)e^{i\omega_n t} \tag{19.32}$$

When Eq. (19.32) is introduced into Eq. (19.31), it is found that the function $H(\omega_n)$, which will be designated as the *complex frequency response* function, takes the form

$$H(\omega_n) = \frac{1}{k - m\omega_n^2 + ic\omega_n} \tag{19.33}$$

Upon introducing the frequency ratio

$$r_n = \frac{\omega_n}{\omega}$$

and the damping ratio

$$\xi = \frac{c}{c_{cr}} = \frac{c}{2\sqrt{km}}$$

Eq. (19.33) becomes

$$H(\omega_n) = \frac{1}{k\left(1 + r_n^2 + 2ir_n\xi\right)}$$

Therefore, the response $y_n(t_j)$ at time $t_j = j\ \Delta\ t$ to a harmonic force component of amplitude C_n indicated in Eq. (19.28) is given by

$$u_n(t_j) = \frac{C_n e^{2\pi i(nj/N)}}{k\left(1 - r_n^2 + 2ir_n\xi\right)} \tag{19.34}$$

and the total response due to the N harmonic force components by

$$u_n(t_j) = \sum_{n=0}^{N-1} \frac{C_n e^{2\pi i(nj/N)}}{k\left(1 - r_n^2 + 2ir_n\xi\right)} \tag{19.35}$$

where C_n is expressed in discrete form by Eq. (19.27). In the determination of the response $y(t_i)$ using Eq. (19.35), it is necessary to bear in mind that in Eq. (19.28) the force component of the frequency of order n is equal to the negative of the component of the frequency of order N−n. This fact may be verified by substituting −(N−n) for n in the exponential factor of Eq. (19.28). In this case we obtain,

$$e^{-2\pi i[(N-n)j/N]} = e^{-2\pi ij}e^{2\pi i(nj/N)} = e^{2\pi i(nj/N)} \tag{19.36}$$

since $e^{-2\pi ij} = \cos 2\pi j - \sin 2\pi j = 1$ for all integer values of j. Equation (19.36) together with Eq. (19.28) shows that harmonic components of the force corresponding to frequencies of orders n and −(N−n) have the same value. As a consequence of this fact, $r_n = \omega_n/\omega$, where $\omega = \sqrt{k/m}$ should be evaluated (selecting N as an even number) as

$$\omega_n = n\varpi \quad \text{for} \qquad \mathrm{n} \leq \mathrm{N}/2$$

and

$$\omega_n = -(N-n)\varpi \quad \text{for} \qquad \mathrm{n} > \mathrm{N}/2$$

where the frequency corresponding to n = N/2, as already mentioned, is the highest frequency that can be considered in the discrete Fourier series.

The evaluation of the sums necessary to determine the response using the discrete Fourier transform is greatly simplified by the fact that the exponential functions involved are harmonic and extend over a range of N^2 as demonstrated in the next section.

19.6 Fast Fourier Transform

A numerical technique is available that is efficient for computer determination f the response in the frequency domain. This method is known as the fast *Fourier transform* (FFT) (Cooley et al. 1965). The corresponding computer program is reproduced as a subroutine of computer Program 4. The response in frequency domain of a single-degree-of-freedom system to a general force is given by Eq. (19.35) and the coefficients required are computed from Eq. (19.27). It can be seen that either Eq. (19.35) or Eq. (19.27) may be represented, except for sign in the exponent by the exponential function as

$$A(j) = \sum_{n=0}^{N-1} A^{(0)}(n) W_N^{jn} \tag{19.37}$$

where

$$W_n = e^{2\pi i/N} \tag{19.38}$$

The evaluation of the sum in Eq. (19.37) will be most efficient if the number of time increments N into which the period T is divided is a power of 2, that is,

$$\mathrm{N} = 2^{\mathrm{M}} \tag{19.39}$$

where M is an integer. In this case, the integers j and n can be expressed in binary form. For the purpose of illustration, we will consider a very simple case where the load period is divided into only eight time increments, that is, N = 8, M = 3. In this case, the indices in Eqs. (19.27) and (19.35) will have the binary representation

$$\begin{aligned} \mathrm{j} &= \mathrm{j}_0 + 2\mathrm{j}_1 + 4\mathrm{j}_2 \\ \mathrm{n} &= \mathrm{n}_0 + 2\mathrm{n}_1 + 4\mathrm{n}_2 \end{aligned} \tag{19.40}$$

and Eq. (19.37) may be written as

$$A(j) = \sum_{n_2=0}^{1} \sum_{n_1=0}^{1} \sum_{n_{0m}=0}^{1} A^{(0)}(n) W_8^{(j_0+2j_1+4j_2)(n_0+2n_1+4n_2)} \tag{19.41}$$

The exponential factor can be written as

$$W_8^{jn} = W_8^{8(j_1n_2+2j_2n+j_2n_1)} W_8^{4n_2j_0} W_8^{2n_1(2j_1+j_0)} W_8^{n_0(4j_2+2j_1+j_0)}$$

We note that the first factor on the right-hand side is unity since from Eq. (19.38)

$$W_8^{8I} = e^{2\pi i(8/8)I} = \cos 2\pi i + \sin 2\pi i = 1$$

where $I = j_1n_2 + 2j_2n_2 + j_2n_1$ is an integer. Therefore, only the remaining three factors need to be considered in the summations. These summations may be performed conveniently in sequence by introducing a new notation to indicate the successive steps in the summation process. Thus the first step can be indicated by

$$A^{(1)}(j_0, n_1, n_0) = \sum_{n_2=0}^{1} A^{(0)}(n_2, n_1, n_0) W_8^{4n_2j_0}$$

where $A^{(0)}(n_2,n_1,n_0) = A^{(0)}(n)$ in Eq. (19.37). Similarly, the second step is

$$A^{(2)}(j_0, j_1, n_0) = \sum_{n_1=0}^{1} A^{(1)}(j_2, n_1, n_0) W_8^{2n_1(2j_1+j_0)}$$

and the third step (final step for $M = 3$) is

$$A^{(3)}(j_0, j_1, j_2) = \sum_{n_0=0}^{1} A^{(2)}(j_0, j_1, n_0) W_8^{n_0(4j_2+2j_1+j_0)}$$

The final result $A^{(3)}(j_0,j_1,j_2)$ is equal to A(j) in Eq. (19.37) or (19.41). This process, indicated for $N = 8$, can readily be extended to any integer $N = 2M$. The method is particularly efficient because the results of one step are immediately used in the next step, thus reducing storage requirements and also because the exponential takes the value of unity in the first factor of the summation. The reduction in computational time that results from this formulation is significant when the time interval is divided into a large number of increments. The comparative times required for computing the Fourier series by a conventional program and by the fast Fourier transform algorithm are illustrated in Fig. 19.4. It is seen here how, for large values of N, one can rapidly consume so much computer time as to make the conventional method unfeasible.

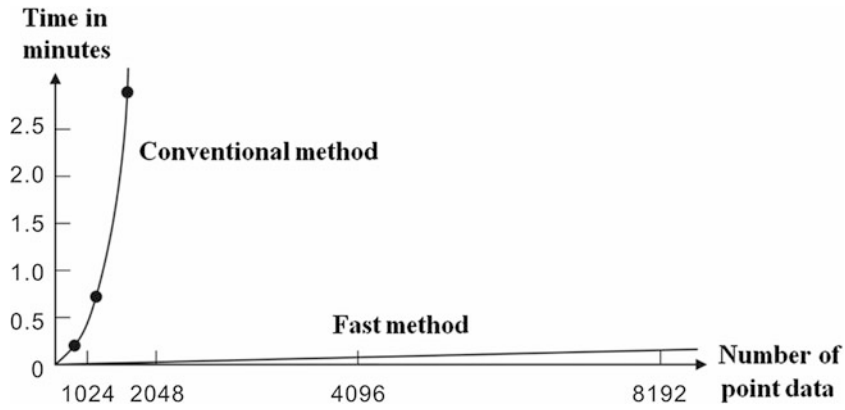

Fig. 19.4 Time required for Fourier transform using conventional and fast method. (From Cooley, J. W., Lewis, P. A. W., and Welch, P. D. (1969), IEEE Trans. Education, E-12 (1))

19.7 Response in the Frequency Domain Using MATLAB

The MATLAB program presented in this chapter calculates the response in the frequency domain for a damped single-degree-of-freedom system. The excitation is input as a discrete function of time. The program output prints the displacement history of the steady-state motion of the response. The main body of this program performs the tasks of calculating, using the FFT algorithm, the coefficients C_n in Eq. (19.27), and the function $F(t_j)$ in Eq. (19.28), and the response $u(t_j)$ in Eq. (19.35).

Illustrative Example 19.2
Determine the response of the tower shown in Fig. 19.5a subjected to the impulsive load of duration 0.64 sec as shown in Fig. 19.5b.

Assume damping equal to 10% of the critical damping.

Solution:
Problem Data:

Mass:	$m = 38{,}600/386 = 100$ (lb.sec^2/in.)
Spring constant:	$k = 100{,}000$ (lb/in.)
Damping coefficient:	$c = 2\ \xi\sqrt{km} = 632$(lb. sec /in.)

Select M such that $2^M = 8$; $M = 3$

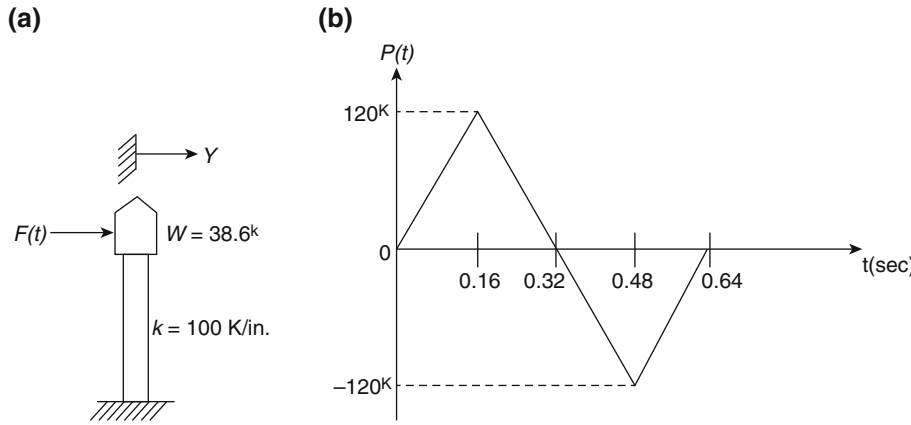

Fig. 19.5 Idealized structure and loading for Illustrative Example 19.2

Excitation function:

Time (sec)	Force (lb)
0.00	0
0.16	120,000
0.48	120,000
0.64	0

The MATLAB file to calculate total response is presented below (Fig. 19.6).

```
close all
clear all
clc

%%%%-GIVEN VALUES-%%%%
m=100;                    %Mass (lb.sec^2/in.)
k =100000;                %Stiffness (lb/in.)
xi =0.1;                  %Damping ratio
omega = sqrt(k/m);        %Natural frequency
c=2*m*omega*xi;           %Damping coefficient. (lb.sec/in.)

T = 0.64;                 %Time period, T(sec)
omega_bar = 2*pi/T;       %Excitation frequency (rad/sec)
M= 3;                     %Select M, M=3
N = 2^M;                  %The number of time increments N

t=0:0.08:0.64;            %Time ranging from 0 to 0.64 sec with deltat = 0.08 sec
Dt = t(2)-t(1);           %Deltat = 0.08
tt= length(t);            %Total number of calculation

for i= 1:tt-1
%%%Define the function of N harmonic force
   if t(i)<=0.16
      F(i) = 120000*t(i)/0.16;
   elseif t(i) <=0.48
      F(i) =-750000*(t(i)-0.16)+120000;
   else
      F(i)=min(0, 750000*(t(i)-0.64)) ;
   end
 %%%Define the discrete Fourier transform of the series
Cn=fft(F/N);                                %Eq.19.27

 %%%Calculate frequency ratio, r_n
    if i<=N/2
        omega_n(i) = (i-1)*omega_bar;
    else
        omega_n(i) = -(N-(i-1))*omega_bar;
    end

    rn(i)=omega_n(i)/omega;                 %Frequency ratio, r_n


end

%%%Total response due to the N harmonic force
uu=Cn./(k*(1-rn.^2+2*xi.*sqrt(-1)*rn));         %Eq.19.35

%%%FFT Discrete Fourier transform (Built-in-MATLAB function)
u=fft(uu')

%%%Reponse
figure(1)
t = 0:Dt:(0.64-Dt);
plot(t',real(u))
xlabel ('Time (sec)'); ylabel ('Displacement(in.)'); grid on
```

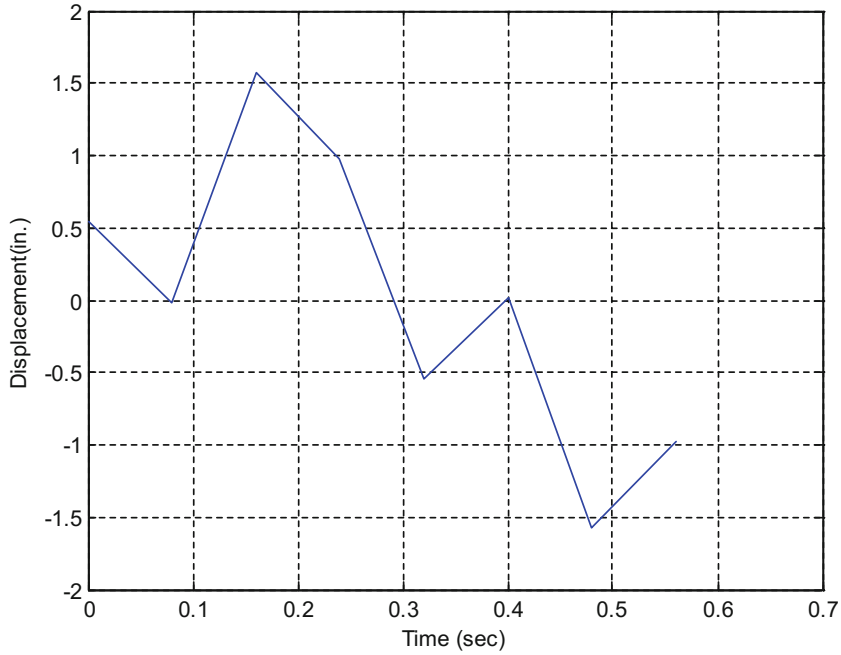

Fig. 19.6 Response of Illustrative Example 19.2

Illustrative Example 19.3

Determine the response of the simple oscillator shown in Fig. 19.7a when subjected to the forcing function depicted in Fig. 19.7b. Use M = 4 for the exponent in N = 2M. Assume 15% of the critical damping.

Solution:

Problem Data:

Mass:	m = 100/386 = 0.259 (Kip. sec^2/in)
Spring constant:	k = 200 Kip/in.
Damping coefficient:	$c = 2\xi\sqrt{km}$
	$c = 2 \times 0.15\sqrt{200 \times 0.259} = 2.159$(Kip. sec /in}
Exponent of $N = 2^M$	M = 4
Gravitational index	G = 0 (force on the mass)

Excitation function:

Time (sec)	Force (Kip)
0.00	0
0.10	–10
0.20	–8
0.40	0
0.45	6
0.60	0
1.00	0

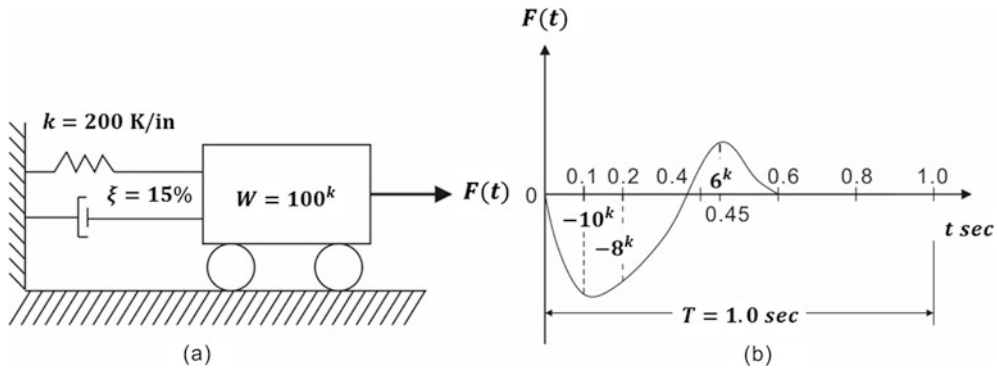

Fig. 19.7 Simple oscillator and loading for Illustrative Example 19.3

The MATLAB file to calculate total response is presented below (Fig. 19.8).

```
close all
clear all
clc

%%%%-GIVEN VALUES-%%%%
m=0.259;                %Mass (lb.sec^2/in.)
k =200;                 %Stiffness (lb/in.)
xi =0.15;               %Damping ratio
omega = sqrt(k/m);      %Natural frequency
c=2*m*omega*xi;         %Damping coefficient. (lb.sec/in.)

T = 1;                  %Time period, T(sec)
omega_bar = 2*pi/T;     %Excitation frequency (rad/sec)
M= 4;                   %Select M, M=4
N = 2^M;                %The number of time increments N

Dt = T/N;               %Deltat
t=0:Dt:T;               %Time ranging from 0 to T sec with deltat
tt= length(t);          %Total number of calculation
 F = zeros(1,N);        %Setting up N harmonic force

for i= 1:tt-1
 %%%Define the function of N harmonic force
  F(2)=-6.25; F(3)=-9.5; F(4)=-8.25; F(5)=-6;
  F(6)=-3.5; F(7)=-1; F(8)=4.5; F(9)=4.0;
  F(10)=1.5;

 %%%Define the discrete Fourier transform of the series
Cn=fft(F/N);                                %Eq.19.27

 %%%Calculate frequency ratio, r_n
    if i<=N/2
        omega_n(i) = (i-1)*omega_bar;
    else
        omega_n(i) = -(N-(i-1))*omega_bar;
    end

    rn(i)=omega_n(i)/omega;                 %Frequency ratio, r_n


end
```

```
%%%Total response due to the N harmonic force
uu=Cn./(k*(1-rn.^2+2*xi.*sqrt(-1)*rn))      %Eq.19.35

%%%FFT Discrete Fourier transform (Built-in-MATLAB function)
u=fft(uu')

%%%Reponse
figure(1)
t = 0:Dt:(1-Dt);
plot(t',real(u))
xlabel ('Time (sec)'); ylabel ('Displacement(in.)'); grid on
```

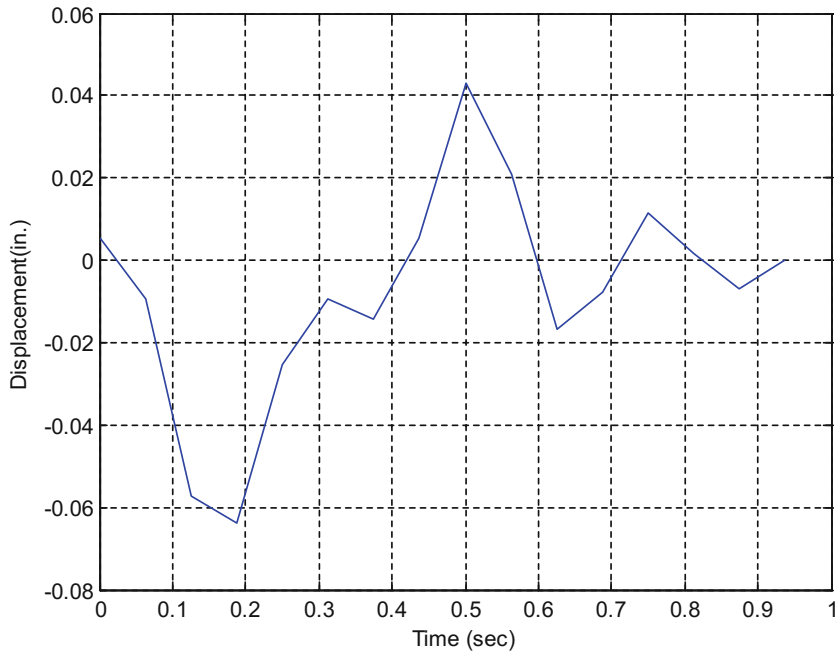

Fig. 19.8 Response of Illustrative Example 19.3

Illustrative Example 19.4

Consider a single-degree-of-freedom undamped system in which k = 200 lb/in, m = 100 lb.sec^2/in subjected to a force expressed as

$$P(t) = \sum_{n=1}^{16} 100 \cos 2\pi nt \qquad (a)$$

Determine the steady-state response of the system using MATLAB with M = 3, 4, 5, and 6 corresponding to N = 8, 16, 32, and 64 sampled points. Then discuss the results in relation to the limitations imposed by the Nyquist frequency.

Solution:

The fundamental frequency of the excitation function, Eq. (a), is $\omega_1 = 2\,\pi$ and its period $T = \omega_1/2\,\pi = 1$ sec. Since the highest component in Eq. (a) is of order $\omega_{16} = 16\,\omega_1$, to avoid aliasing, the number of sampled points should be at least twice that order, that is, the minimum number of sampled points should be N = 32.

The applied force is calculated in the MATLAB program. The results given by the MATLAB for this example are conveniently arranged in two tables: Table 19.1, giving the displacement response to

the excitation having all 16 harmonic components as prescribed for this problem; and Table 19.2, showing the displacement response to a reduced number of harmonic terms in the excitation function.

For this example, in which the exciting force is supplied in 16 harmonic components, the response given in Table 19.1 corresponding to N = 32 or N = 64 may be considered the exact solution. A comparison of the response shown for sample points N = 8 or N = 16 with the exact solution (N = 32) dramatically demonstrates the risk of not choosing N sufficiently large enough so that none of the frequencies of the components in the exciting force exceed the Nyquist frequency. The response obtained for N = 8 or N = 16 gives spurious numerical results.

As the demonstration purpose, this MATLAB program is presented for the response for the N value of 8 and 16 harmonic force. To produce the Table 19.1, you need to update the line for the M = 3 and the iteration from j = 1:16.

```
clear all
close all
clc

%%%%-GIVEN VALUES-%%%%
m=100;                    %Mass (lb.sec^2/in.)
k=200;                    %Stiffness (lb/in.)
omega = sqrt(k/m);        %Natural frequency
c=0;                      %Damping coefficient. (lb.sec/in.)
xi =c/(2*m*omega);        %Damping ratio

T = 1.0;                  %Time period, T(sec)
omega_bar = 2*pi/T;       %Excitation frequency (rad/sec)
M= 3;                     %Select M, M=3
N = 2^M;                  %The number of time increments N

Dt=T/N;                   %Deltat
t=0:Dt:1.0;               %Time ranging from 0 to T sec with deltat
tt= length(t);            %Total number of calculation


for i= 1:tt-1
   %%%Define the function of 16 harmonic force
   for j=1:16
        F3(j,i)=100*cos((2*pi()*j)*t(i));
   end

   F3s= sum(F3);

   %%%Calculate frequency ratio, r_n
    if i<=N/2
        omega_n(i) = (i-1)*omega_bar;
    else
        omega_n(i) = -(N-(i-1))*omega_bar;
    end

    rn(i)=omega_n(i)/omega;                %Frequency ratio, r_n

end

%%%Define the discrete Fourier transform of the series
Cn3 =fft(F3s/N);

%%%Total response due to the N harmonic force
uu=Cn3./(k*(1-rn.^2+2*xi.*sqrt(-1)*rn));   %Eq.19.35

%%%FFT Discrete Fourier transform (Built-in-MATLAB function)
u=fft(uu')

%%%Reponse
figure(1)
t = 0:Dt:(1-Dt);
plot(t',real(u))
xlabel ('Time (sec)'); ylabel ('Displacement(in.)'); grid on
```

Table 19.1 Displacement Response for Example 19.4 (Excitation Having 16 Harmonics)

	Number of Sampling Points for the Excitation			
Time(sec)	$N = 8$	$N = 16$	$N = 32$	$N = 64$
0	0.8531	0.4201	−0.0416	−0.041.6
0.125	0.9357	0.4698	−0.0153	−0.0153
0.250	1.022	0.5107	0.0052	0.0052
0.375	1.071	0.5358	0.0178	0.0178
0.500	1.089	0.5443	0.0221	0.0221
0.625	1.071	0.5358	0.0178	0.0178
0.750	1.022	0.5107	0.0052	0.0052
0.875	0.9357	0.4698	−0.0153	−0.0153
1.000	0.8531	0.4201	−0.0416	−0.0416

Results in Table 19.2, which were obtained using N = 8 sampled points, also verify that when the exciting force contains harmonic components higher than the Nyquist frequency which corresponds, in this case, to $N_y = 4$, the results are again spurious.

A final comment is in order. The example presented, having equal amplitude for all the components of the exciting force, serves to emphasize the importance of choosing the number of sampling points N sufficiently large to avoid aliasing. In practical situations normally the higher harmonics have much smaller amplitude than that of the fundamental or lower frequencies. Consequently, the distortion in the response might not be as dramatic as shown in Tables 19.1 and 19.2.

Table 19.2 Displacement Response for Example 19.4 (Excitation Force Sampled at N = 8 Points)

Number of Sampling Points for the Excitation			
Time(sec)	$N = 16$	$N = 8$	$N = 4$
0	−0.0375	0.4246	0.8531
0.125	−0.0153	0.4679	0.9357
0.250	0.0048	0.5112	1.0220
0.375	0.0184	0.5353	1.0710
0.500	0.0215	0.5446	1.0890
0.625	0.0184	0.5353	1.0710
0.750	0.0048	0.5112	1.0220
0.875	−0.0153	0.4679	0.9357
1.000	−0.0375	0.4246	0.8531

19.8 Summary

In general, any periodic function may be expanded into a Fourier series, Eq. (19.1), whose terms are sine and cosine functions of successive multiples of the fundamental frequency. The coefficients of these functions may be calculated by integrating over a period the product of the periodic function multiplied by a sine or cosine function, Eq. (19.3). The response of the dynamic system is then obtained as the superposition of the response for each term of the Fourier series expansion of the excitation function. The extension of the Fourier series to non-periodic functions results in integrals

which are known as Fourier transforms. The discrete form of these transforms, Eqs. (19.27) and (19.28), permits their use in numerical applications. An extremely efficient algorithm known as the Fast Fourier Transform (FFT) can save as much as 99% of the computer time otherwise consumed in the evaluation of Fourier complex coefficients for the excitation function and for the response of a dynamic system.

19.9 Problems

Problem 19.1
Determine the first three terms of the Fourier series expansion for the time varying force shown in Fig. P19.1.

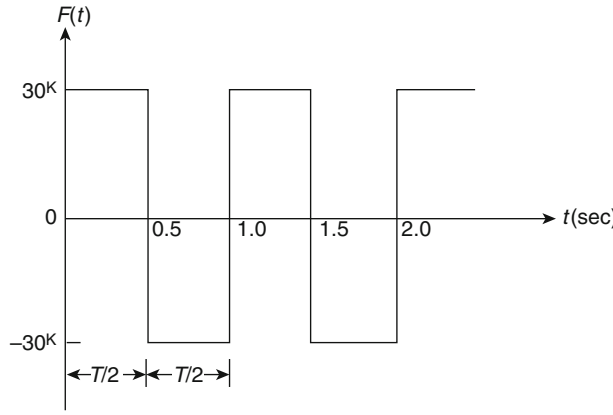

Fig. P19.1

Problem 19.2
Determine the steady-state response for the damped spring-mass system shown in Fig. P19.2 that is acted upon by the forcing function of Problem 19.1.

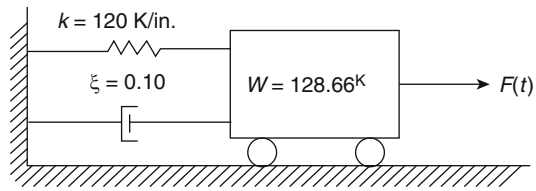

Fig. P19.2

Problem 19.3
The spring-mass system of Fig. P19.2 is acted upon by the time-varying force shown in Fig.P19.3. Assume that the force is periodic of period T = 1 sec. Determine the steady-state response of the system by applying Fourier series expansion of F(t).

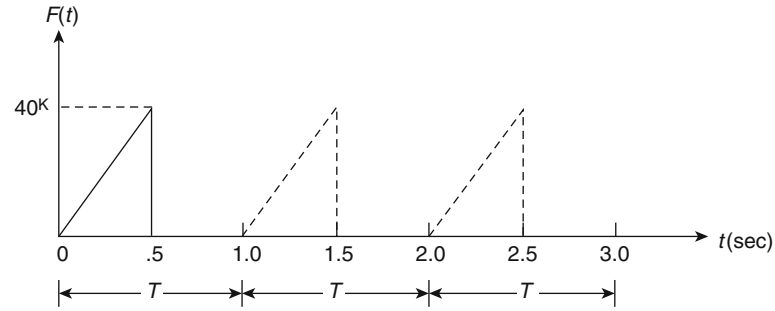

Fig. P19.3

Problem 19.4

The cantilever beam shown in Fig. P19.4a carries a concentrated weight at its free end and it is subjected to a periodic acceleration at its support which is the rectified sine function of period T = 0.4 sec and amplitude $\ddot{u}_0 = 180$ in/sec^2 as shown in Fig. P19.4b. Determine: (a) the Fourier series expansion of the forcing function and (b) the steady-state response considering only three terms of the series. Neglect damping in the system and assume the beam massless.

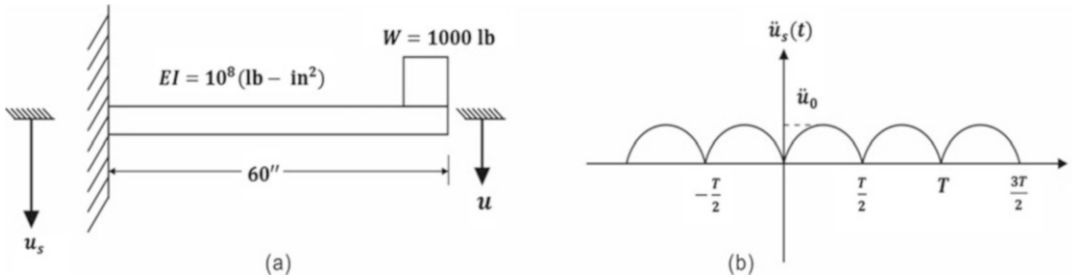

Fig. P19.4

Problem 19.5

Solve Problem 19.4 using Program 4. Take 16 Fourier terms. Input the values of the excitation functions at intervals of 0.025 sec.

Problem 19.6

Solve Problem 19.4 in the frequency domain using Program 4. Take the exponent of $N = 2^M$, M = 4. Input the effective force, $F_{eff} = -m\ddot{u}_s(t)$ calculated for every 0.025 sec.

Problem 19.7

Repeat Problem 19.6 assuming 20% of critical damping.

Problem 19.8

The forcing function shown in Fig. P19.8a is assumed to be periodic in the extended interval T = 1.4 sec. Use Program 4 to determine the first eight Fourier coefficients and the steady-state response of a structure modeled by the undamped oscillator shown in Fig. P19.8b.

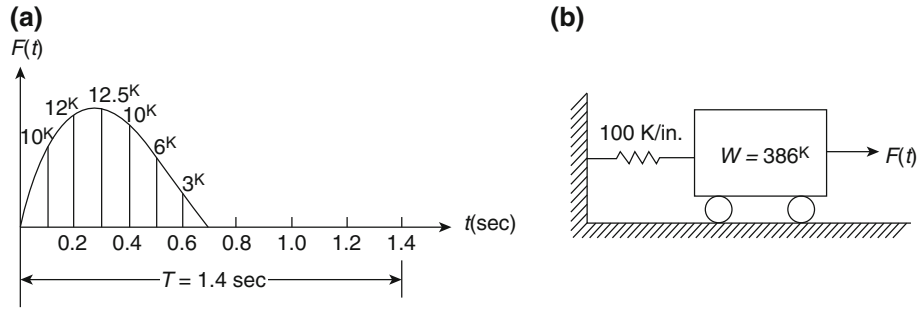

Fig. P19.8

Problem 19.9

Solve Problem 19.8 in the frequency domain using Program 4. Take M = 4 for the exponent in N = 2 M.

Problem 19.10

Use Program 4 to determine: (I) the Fourier series expansion of the forcing function shown in Fig. P19.10a and (2) the steady-state response calculated in the frequency domain for the spring-mass system shown in Fig. P19.10b. Assume 15% of the critical damping. Take M = 3 for the exponent in $N = 2^M$ and compare results with those obtained in the solution of Illustrative Example 19.3.

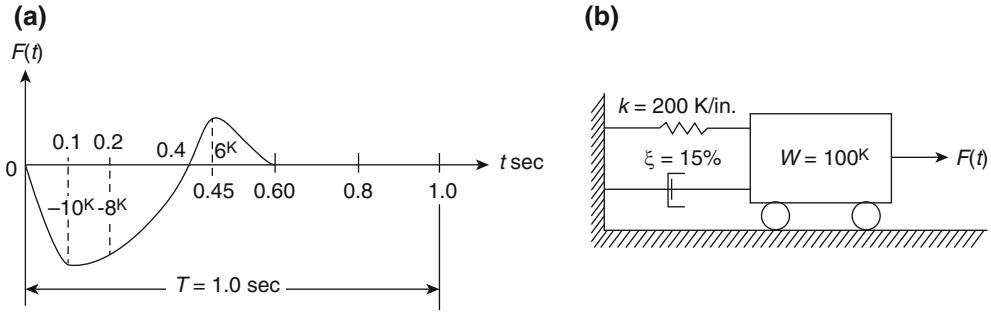

Fig. P19.10

Problem 19.11

Solve Problem 19.10 in the frequency domain using Program 4. Take M = 5 for the exponent in $N = 2^M$.Compare results with those in the solution of Example 19.3.

Problem 19.12

Consider the system shown in Fig. P19.12 and its loading with assumed period T = 2 sec. Determine: (a) the first four terms of the Fourier series expansion for the forcing fonction in terms of P_o; (b) the first four terms of the Fourier series expansion for the response.

Problem 19.13
Using Program 4, determine the response of the system ande its load in Problem 19.12. Take 32 terms of Fourier series and input the force at intervals of 0.10 sec.

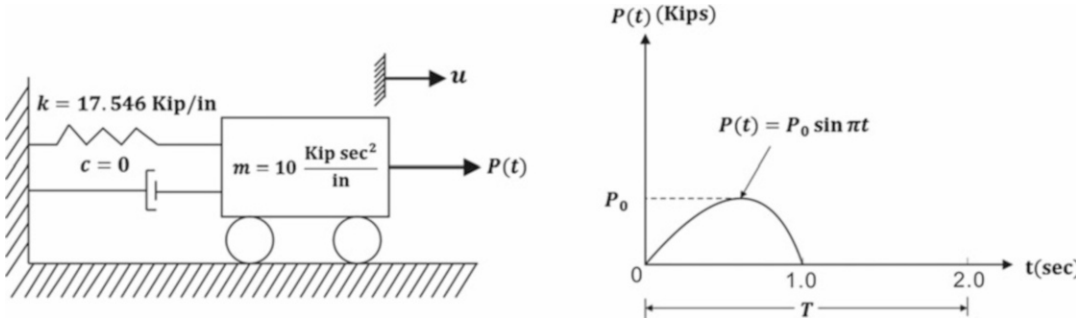

Fig. P19.12

Problem 19.14
Repeat Problem 19.13 assuming that the system has 20% of the critical damping.

Problem 19.15
Obtain the close solution for the system in Problem 19.12 by considering the half-circle sinusoidal excitation as the superposition of two sinusoidal functions: $P_1 = P_0 \sin \pi t$ starting at $= 0$ and P_2 and $P_0 \sin \pi(t-1)$ starting at $t = 1$ sec as shown in Fig. P19.15.

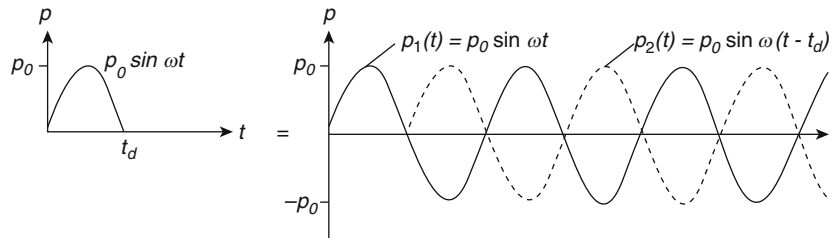

Fig. P19.15

Problem 19.16
A single-degree-of-freedom system having a natural period of 0.8 sec and stiffness of 5000 lb fin subjected to an impulse of duration 0.5 sec which varies as shown in Fig. P19.16. Compute the response with an extension of 1.1 sec for which the value of force is zero. Use Program 4 to obtain: (a) the discrete transform of the forcing function and of the response, (b) the displacement response, and (c) the applied force calculated using the inverse discrete transform. Neglect damping and discretize the forcing function using a time step $\Delta t = 0.1$ sec.

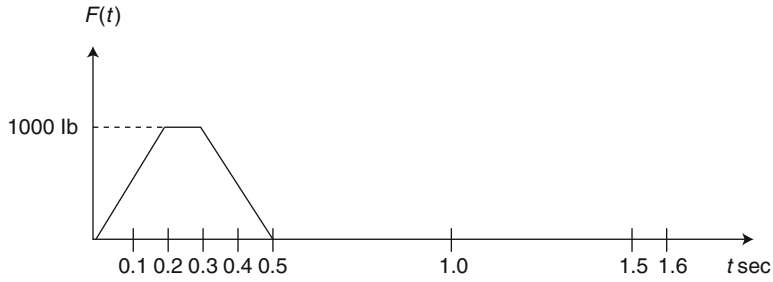

Fig. P19.16

Problem 19.17
Solve Problem 19.16 assuming a 10% of the critical damping in the system.

Problem 19.18
The water tower shown in Fig. P19.8a is subjected to impulsive acceleration of its base that varies as half the sine function shown in Fig. P19.18b. Use Program 4 to determine: (a) the discrete Fourier coefficients for the excitation and for the response, (b) the relative displacement of the tower with respect to the ground displacement, and (c) the excitation obtained by the inverse discrete transform. Use an extended excitation of total duration 1.6 sec and time step Δ t = 0.1 sec. Neglect damping.

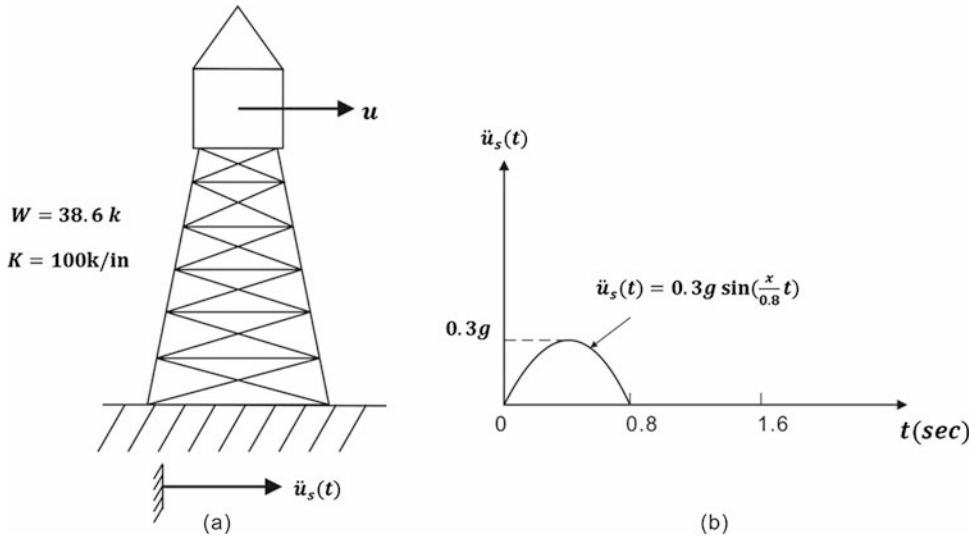

Fig. P19.18

Problem 19.19
Repeat Problem 19.18 assuming that the damping in the system is 5% of the critical.

Problem 19.20

Solve Problem 19.18 for an acceleration at the base of tower that varies as a symmetrical triangular load as shown in Fig. P19.20.

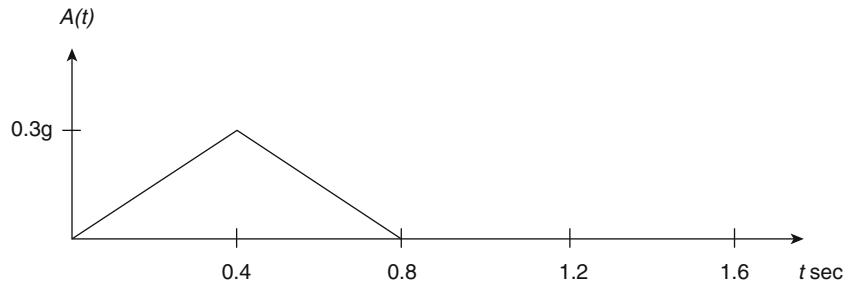

Fig. P19.20

Evaluation of Absolute Damping from Modal Damping Ratios

20

In the previous chapters, we determined the natural frequencies and modal shapes for undamped structures when modeled as shear buildings. We also determined the response of these structures using the modal superposition method. In this method, as we have seen, the differential equations of motion are uncoupled by means of a transformation of coordinates that incorporates the orthogonality property of the modal shapes.

The consideration of damping in the dynamic analysis of structures complicates the problem. Not only will the differential equations of motion have additional terms due to damping forces, but the uncoupling of the equations will be possible only by imposing some restrictions or conditions on the functional expression for the damping coefficients.

The damping normally present in structures is relatively small and practically does not affect "the calculation of natural frequencies and modal shapes of the system. Hence, the effect of damping is neglected in determining the natural frequencies and" modal shapes of the structural systems. Therefore, in practice, the eigenproblem for the damped structure is solved by using the same methods employed for undamped structures.

20.1 Equations for Damped Shear Building

For a viscously damped shear building, such as the three-story building shown in Fig. 20.1, the equations of motion obtained by summing forces in the corresponding free body diagrams are

$$\begin{aligned} m_1\ddot{u}_1 + c_1\dot{u}_1 + k_1u_1 - c_2(\dot{u}_2 - \dot{u}_1) - k_2(u_2 - u_1) &= F_1(t) \\ m_2\ddot{u}_2 + c_2(\dot{u}_2 - \dot{u}_1) + k_2(u_2 - u_1) - c_3(\dot{u}_3 - \dot{u}_2) - k_3(u_3 - u_2) &= F_2(t) \\ m_3\ddot{u}_3 + c_3(\dot{u}_3 - \dot{u}_2) + k_3(u_3 - u_2) &= F_3(t) \end{aligned} \tag{20.1}$$

These equations may be conveniently written in matrix notation as

$$[M]\{\ddot{u}\} + [C]\{\dot{u}\} + [K]\{u\} = \{F(t)\} \tag{20.2}$$

where the matrices and vectors are as previously defined, except for the damping matrix $[C]$, which is given by

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M. Paz, Y. H. Kim, *Structural Dynamics*, https://doi.org/10.1007/978-3-319-94743-3_20

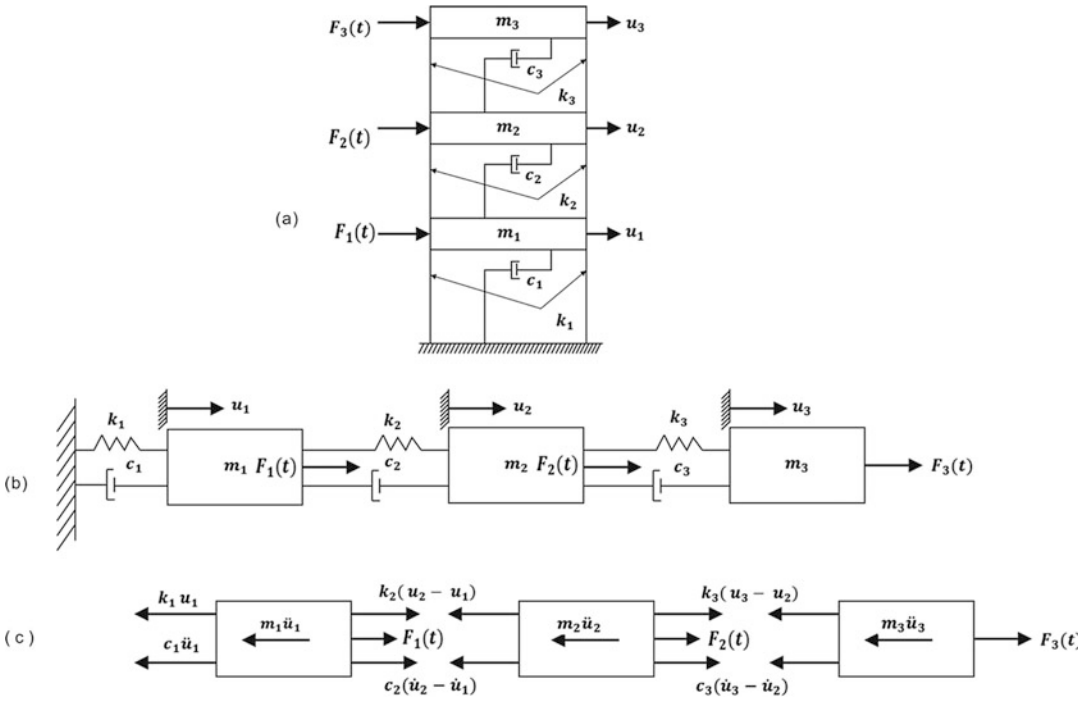

Fig. 20.1 (**a**) Damped shear building. (**b**) Mathematical model. (**c**) Free body diagram

$$[C] = \begin{bmatrix} c_1 + c_2 & -c_2 & 0 \\ -c_2 & c_2 + c_3 & -c_3 \\ 0 & -c_3 & c_3 \end{bmatrix} \tag{20.3}$$

In the next section, we shall establish the conditions under the damped equations of motion may be transformed to an uncoupled set of independent equations.

20.2 Uncoupled Damped Equations

To solve the differential equations of motion, Eq. (20.2), we seek to uncouple these equations. We, therefore, introduce the transformation of coordinates

$$\{u\} = [\Phi]\{z\} \tag{20.4}$$

where [Φ] is the modal matrix obtained in the solution of the undamped free-vibration system. The substitution of Eq. (20.4) and its derivatives into Eq. (20.2) leads to

$$[M][\Phi]\{\ddot{z}\} + [C][\Phi]\{\dot{z}\} + [K][\Phi]\{z\} = \{F(t)\} \tag{20.5}$$

Premultiplying Eq. (20.5) by the transpose of the nth modal vector $\{\phi\}_n^T$ yield

$$\{\phi\}_n^T[M][\Phi]\{\ddot{z}\} + \{\phi\}_n^T[C][\Phi]\{\dot{z}\} + \{\phi\}_n^T[K][\Phi]\{z\} = \{\phi\}_n^T\{F(t)\} \tag{20.6}$$

It is noted that the orthogonality property of the modal shapes,

$$\begin{aligned}\{\phi\}_n^T[M]\{\phi\}_m &= 0, m \neq n \\ \{\phi\}_n^T[K]\{\phi\}_m &= 0\end{aligned} \tag{20.7}$$

causes all components except the nth mode in the first and third terms of Eq. (20.6) to vanish. A similar reduction is *assumed* to apply to the damping term in Eq. (20.6), that is, if it is assumed that

$$\{\phi\}_n^T[C]\{\phi\}_m = 0, \quad n \neq m \tag{20.8}$$

then the coefficient of the damping term in Eq. (20.6) will reduce to

$$\{\phi\}_n^T[C]\{\phi\}_n$$

In this case Eq. (20.6) may be written as

$$M_n\ddot{z}_n + C_n\dot{z}_n + K_n z_n = F_n(t)$$

or alternatively as

$$\ddot{z}_n + 2\xi_n\omega_n\dot{z}_n + \omega_n^2 z_n = F_n(t)/M_n \tag{20.9}$$

in which case

$$M_n = \{\phi\}_n^T[M]\{\phi\}_n \tag{20.10a}$$

$$K_n = \{\phi\}_n^T[K]\{\phi\}_n = \omega_n^2 M_n \tag{20.10b}$$

$$C_n = \{\phi\}_n^T[C]\{\phi\}_n = 2\xi\omega_n M_n \tag{20.10c}$$

$$F_n(t) = \{\phi\}_n^T\{F(t)\} \tag{20.10d}$$

The normalization discussed previously (Sect. 7.2)

$$\{\phi\}_n^T[M]\{\phi\}_n = 1 \tag{20.11}$$

will result in

$$M_n = 1$$

so that Eq. (20.9) reduces to

$$\ddot{z}_n + 2\xi_n\omega_n\dot{z}_n + \omega_n^2 z_n = F_n(t) \tag{20.12}$$

which is a set of N uncoupled differential equations ($n = 1,2, \ldots, N$).

20.3 Conditions for Damping Uncoupling

In the derivation of the uncoupled damped Eq. (20.12), it has been assumed that the normal coordinate transformation, Eq. (20.4), that serves to uncouple the inertial and elastic forces also uncouples the damping forces. It is of interest to consider the conditions under which this uncoupling will occur, that is, the form of the damping matrix $[C]$ to which Eq. (20.8) applies.

When the damping matrix is of the form

$$[C] = a_0[M] + a_1[K] \tag{20.13}$$

in which a_0 and a_1 are arbitrary proportionality factors, the orthogonality condition will be satisfied. This may be demonstrated by applying the orthogonality condition to Eq. (20.13), that is, premultiplying both sides of this equation by the transpose of the nth mode $\{\phi\}_n^T$ and postmultiplying by the modal matrix $[\Phi]$. We obtain

$$\{\phi\}_n^T[C][\Phi] = a_0\{\phi\}_n^T[M][\Phi] + a_1\{\phi\}_n^T[K][\Phi] \tag{20.14}$$

The orthogonality conditions, Eq. (20.7), then reduce Eq. (20.14) to

$$\{\phi\}_n^T[C][\Phi] = a_0\{\phi\}_n^T[M]\{\phi\}_n + a_1\{\phi\}_n^T[K]\{\phi\}_n$$

or, by Eqs. (20.10), to

$$\{\phi\}_n^T[C][\Phi]_n = a_0M_n + a_1M_n\omega_n^2$$

$$\{\phi\}_n^T[C][\Phi]_n = \left(a_0 + a_1\omega_n^2\right)M_n$$

which shows that, when the damping matrix is of the form of Eq. (20.13), the damping forces are also uncoupled with the transformation Eq. (20.4). However, it can be shown that there are other matrices formed from the mass and stiffness matrices which also satisfy the orthogonality condition. In general, damping matrix may be of the form

$$[C] = [M]\sum_i a_i([M])^{-1}[K])^i \tag{20.15}$$

where i can be anywhere in the range $-\infty < i < \infty$ and the summation may include as many terms as desired. The damping matrix, Eq. (20.13), can obviously be obtained as a special case of Eq. (20.15). By taking two terms corresponding to $i = 0$ and $i = 1$ in Eq. (20.15), we obtain the damping matrix expressed by Eq. (20.13). With this form of the damping matrix it is possible to compute the damping coefficients necessary to provide uncoupling of a system having any desired damping ratios in any specified number of modes. For any mode n, the modal damping is given by Eq. (20.10c), that is

$$C_n = \{\phi\}_n^T[C]\{\phi\}_n = 2\xi_n\omega_nM_n$$

If $[C]$ as given by Eq. (20.15) is substituted in the expression for C_n, we obtain

$$C_n = \{\phi\}_n^T[M]\sum_i a_i\left([M]^{-1}[K]\right)^i\{\phi\}_n \tag{20.16}$$

Now, using Eq. (7.24) $\left(\mathrm{K}\{\phi\}_n = \omega_n^2M\{\phi\}_n\right)$ and performing several algebraic operations, we can show (Clough and Penzien 1975, p. 195) that the damping coefficient associated with any mode n may be written as

$$C_n = \sum_i a_i\omega_n^{2i}M_n = 2\xi\omega_nM_n \tag{20.17}$$

from which

$$\xi_n = \frac{1}{2\omega_n} \sum_i a_i \omega_n^{2i} \tag{20.18}$$

Equation (20.18) may be used to determine the constants a_i, for any desired values of modal damping ratios corresponding to any specified numbers of modes. For example, to evaluate these constants specifying the first four modal damping ratios ξ_1, ξ_2, ξ_3, ξ_4 we may choose $i = 1, 2, 3, 4$. In this case Eq. (20.18) gives the following system of equations:

$$\begin{Bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{Bmatrix} = \frac{1}{2} \begin{bmatrix} \omega_1 & \omega_1^3 & \omega_1^5 & \omega_1^7 \\ \omega_2 & \omega_2^3 & \omega_2^5 & \omega_2^7 \\ \omega_3 & \omega_3^3 & \omega_3^5 & \omega_3^7 \\ \omega_4 & \omega_4^3 & \omega_4^5 & \omega_4^7 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{Bmatrix} \tag{20.19}$$

In general, Eq. (20.19) may be written symbolically as

$$\{\xi\} = \frac{1}{2}[Q]\{a\} \tag{20.20}$$

where $[Q]$ is a square matrix having different powers of the natural frequencies. The solution of Eq. (20.20) gives the constants $\{a\}$ as

$$\{a\} = 2[Q]^{-1}\{\xi\} \tag{20.21}$$

Finally the damping matrix is obtained after the substitution of Eqs. (20.21) into (20.15).

It is interesting to observe from Eq. (20.18) that in the special case when the damping matrix is proportional to the mass $\{C\} = a_0\,[M]$ $(i = 0)$, the damping ratios are inversely proportional to the natural frequencies; thus the higher modes of the structures will be given very little damping. Analogously, when the damping is proportional to the stiffness matrix ($[C] = a_1\,[K]$), the damping ratios are directly proportional to the corresponding natural frequencies, as can be seen from Eq. (20.18) evaluated for $i = 1$; and in this case the higher modes of the structure will be very heavily damped.

Illustrative Example 20.1

Determine the absolute damping coefficients for the structure presented in Example 7.1. Assume 10% of the critical damping for each mode.

Solution:

From Example 7.1, we have the following information.

Natural frequencies:

$$\begin{aligned} \omega_1 &= 11.83\,\text{rad/sec} \\ \omega_2 &= 32.89\,\text{rad/sec} \end{aligned} \tag{a}$$

Modal matrix:

$$[\Phi] = \begin{bmatrix} 1.00 & 1.00 \\ 1.26 & -1.63 \end{bmatrix} \tag{b}$$

Mass matrix:

$$[M] = \begin{bmatrix} 136 & 0 \\ 0 & 66 \end{bmatrix}$$

Stiffness matrix:

$$[K] = \begin{bmatrix} 75,000 & -44,300 \\ -44,300 & 44,300 \end{bmatrix}$$

Using Eqs. (20.18) and (20.19) with $i = 0, 1$ to calculate the constants a_i needed in Eq. (20.15), we obtain the following system of equations:

$$\begin{Bmatrix} 0.1 \\ 0.1 \end{Bmatrix} = \frac{1}{2} \begin{bmatrix} 11.83 & (11.83)^3 \\ 32.89 & (32.89)^3 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix}$$

Solving this system of equations gives

$$a_1 = 0.01851$$
$$a_2 = -0.00001146$$

We also calculate

$$[M]^{-1} = \begin{bmatrix} 0.007353 & 0 \\ 0 & 0.01515 \end{bmatrix}$$

and

$$[M]^{-1}[K] = \begin{bmatrix} 551475 & -325738 \\ -671145 & 671145 \end{bmatrix}$$

Then

$$\sum_{i=1}^{2} a_i \left([M]^{-1}[K] \right)^i = 0.0185 \begin{bmatrix} 551475 & -325738 \\ -671145 & 671145 \end{bmatrix} - 0.00001146 \begin{bmatrix} 551.475 & -325.738 \\ -671.145 & 671.145 \end{bmatrix}^2 = \begin{bmatrix} 4.2172 & -1.4654 \\ -3.0193 & 4.7556 \end{bmatrix}$$

Finally, substituting this matrix into Eq. (20.15) yields the damping matrix as

$$[C] = \begin{bmatrix} 136 & 0 \\ 0 & 66 \end{bmatrix} \begin{bmatrix} 4.2172 & -1.4654 \\ -3.0193 & 4.7556 \end{bmatrix} = \begin{bmatrix} 573.5 & -199.3 \\ -199.3 & 313.9 \end{bmatrix}$$

There is yet a second method for evaluating the damping matrix corresponding to any set of specified modal damping ratio. The method may be explained starting with the relationship

$$[A] = [\Phi]^T [C][\Phi] \tag{20.22}$$

where the matrix $[A]$ is defined as

$$[A] = \begin{bmatrix} 2\xi_1\omega_1 M_1 & 0 & 0 & - \\ 0 & 2\xi_2\omega_2 M_2 & 0 & - \\ 0 & 0 & 2\xi_3\omega_3 M_3 & - \\ - & - & - & - \end{bmatrix} \tag{20.23}$$

in which the modal masses M_1, M_2, M_3, ... are equal to one if the modal matrix $\{\Phi\}$ has been normalized. It is evident that the damping matrix $[C]$ may be evaluated by postmultiplying and premultiplying Eq. (20.22) by the inverse of the modal matrix $[\Phi]^{-1}$ and its inverse transpose $[\Phi]^{-T}$, such that

$$[C] = [\Phi]^{-T} [A][\Phi]^{-1} \tag{20.24}$$

Therefore, for any specified set of modal damping ratios $\{\xi\}$, matrix $[A]$ can be evaluated from Eq. (20.23) and the damping matrix $[C]$ from Eq. (20.24). However, in practice, the inversion of the modal matrix is a large computational effort. Instead, taking advantage of orthogonality properties of the mode shapes, we can deduce the following expression for the system damping matrix:

$$[C] = [M]\left(\sum_{n=1}^{N} \frac{2\xi_n\omega_n}{M_n}\{\Phi\}_n\{\Phi\}_n^T\right)[M] \tag{20.25}$$

Equation (20.25) may be obtained from the condition of orthogonality of the normal modes given by Eq. (7.35) as

$$[I] = [\Phi]^{-T}[M][\Phi] \tag{20.26}$$

Postmultiplying Eq. (20.26) by $[\Phi]^{-1}$ we obtain

$$[\Phi]^{-1} = [\Phi]^{-T}[M] \tag{20.27}$$

Applying the transpose operation to Eq. (20.27) results in

$$[\Phi]^{-T} = [M][\Phi] \tag{20.28}$$

in which $[M] = [M]^T$ since the mass matrix $[M]$ is a symmetric matrix.

Finally, the substitution into Eq. (20.24) of Eqs. (20.27) and (20.28) gives

$$[C] = [M][\Phi][A][\Phi]^T[M]$$

which results in Eq. (20.25) after substituting matrix [A] from Eq. (20.23). The damping matrix $[C]$ obtained from Eq. (20.25) will satisfy the orthogonality property and, therefore, the damping term in the differential Eq. (20.2) will be uncoupled with the same transformation, Eq. (20.4), which serves to uncouple the inertial and elastic forces.

It is of interest to note in Eq. (20.25) that the contribution to the damping matrix of each mode is proportional to the modal damping ratio; thus any undamped mode will contribute nothing to the damping matrix.

We should mention at this point the circumstances under which it will be desirable to evaluate the elements of the damping matrix, as Eq. (20.15) or Eq. (20.25). It has been stated that absolute structural damping is a rather difficult quantity to determine or even to estimate. However, modal damping ratios may be estimated on the basis of past experience. This past experience includes laboratory determination of damping in different materials, as well as damping values obtained from vibration tests in existing buildings and other structures. Numerical values for damping ratios in structures are generally in the range of 1–10%. These values depend on the type of structure, materials utilized, nonstructural elements, etc. They also depend on the period and the magnitude of vibration. It has also been observed that damping ratios corresponding to higher modes have increasing values. Figure 20.2 shows the values of damping ratios measured in existing buildings as reported by H. Aoyama (1980) [in Wakabayashi (1986)]. It may be observed from this figure that experimental values for damping scatter over a wide range, and that it is difficult to give definite recommendations. The scatter observed in Fig. 20.2 is typical for experiments conducted to determine damping. On this basis, the obvious conclusion should be that the assumption of viscous damping to represent damping does not describe the real mechanism of energy dissipation in structural dynamics. However, for analytical expediency and also because of the uncertainties involved in attempting other formulation of damping, we still accept the assumption of viscous damping. At this time, the best recommendation that can be given in regard to damping is to use conservative values, 1–2% for steel buildings and 3–5% for reinforced concrete buildings for the fundamental frequency, and to assume damping ratios for the higher modes to increase in proportion to the natural frequencies. Thus, on the basis of giving some consideration to the type of structure, we assign numerical values to the damping ratios in all the modes of interest These values are then used directly in the modal equations or they are used to determine the damping matrix that is needed when the dynamic response is obtained by some analytical method other than modal superposition, e.g., the time history response of a linear or nonlinear system.

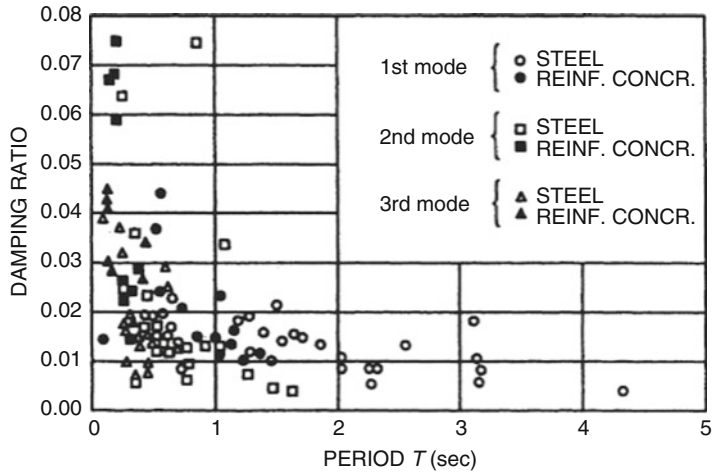

Fig. 20.2 Damping ratios measured in existing buildings. [H. Aoyama in Wakabayashi (1986)]

Illustrative Example 20.2
Determine the damping matrix of Example 20.1 using the method based on Eq. (20.22).

Solution:
From Example 7.2 we have the natural frequencies

$$\omega_1 = 11.83 \text{ rad/ sec}$$
$$\omega_2 = 32.89 \text{ rad/ sec}$$

and the mass matrix

$$[M] = \begin{bmatrix} 136 & 0 \\ 0 & 66 \end{bmatrix}$$

To determine $[C]$, we could use either Eq. (20.24) or (20.25). From Example 7.2, the normalized modal matrix is

$$[\Phi] = \begin{bmatrix} 0.06437 & 0.567 \\ 0.0813 & -0.0924 \end{bmatrix}$$

and its inverse by Eq. (20.27) is

$$[\Phi]^{-1} = \begin{bmatrix} 8.752 & 5.370 \\ 7.700 & -6.097 \end{bmatrix}$$

Substituting into Eq. (20.23), we obtain

$$2\xi_1\omega_1 M_1 = (2)(0.1)(11.83)(1) = 2.366$$
$$2\xi_2\omega_2 M_2 = (2)(0.1)(32.89)(1) = 6.578$$

Then by Eq. (20.24)

$$[C] = \begin{bmatrix} 8.752 & 7.700 \\ 5.370 & -6.097 \end{bmatrix} \begin{bmatrix} 2.366 & 0 \\ 0 & 6.578 \end{bmatrix} \begin{bmatrix} 8.752 & 7.700 \\ 5.370 & -6.097 \end{bmatrix}$$
$$[C] = \begin{bmatrix} 572 & -198 \\ -198 & 313 \end{bmatrix}$$

which in this case of equal damping ratios in all the models checks with the damping matrix obtained in Example 20.1 for the same structure using Eq. (20.15).

The MATLAB program can be used to determine damping matrix.

```
clc
clear all
close all
%_____________________________________________________________________
% Inputs:
%   M, K
%   F = forcing function
%   t = Time period
%   u0 = initial displacement
%   v0 = initial velocity
%_____________________________________________________________________

%%%%-GIVEN VALUES-%%%%

%%%Time for Response
t = 0:0.01:10;

%%%Define Mass Matrix
M = [136 0; 0 66]

%%%Define Stiffness Matrix
k1=30700;
k2=44300;

K = [k1+k2 -k2;
    -k2 k2];

%%%Determine #s of DOFs
[n,n]= size(M);
%_____________________________________________________________________
% Solve the eignevalue problem and normalized eigenvectors
%_____________________________________________________________________

%%%Solve for eigenvalues (D) and eigenvectors (a)
[a, D] = eig(K, M)

%%%Natural Frequencies
[omegas,k] = sort(sqrt(diag(D)));

%%%Eigenvectors
a = a(:,k)

%%%Natural Periods
T = 2*pi./omegas;

%%%aMa = {a}'*[M]*(a}
aMa = diag(a'*M*a)

%%%Normalized modal matrix
nom_phi = (a)*inv(sqrt(diag(aMa)))
%_____________________________________________________________________
% Damping matrix using the proportional damping matrix
% [C] = a0[M]+a1[K] (Eq. 8.55)
% zetas = damping ratios
%_____________________________________________________________________

%%%Define damping ratio for each mode
zetas = zeros(n,1);
zetas(1)=0.1;
zetas(2)=0.1;
Q= [omegas(1) omegas(1)^3;
    omegas(2) omegas(2)^3];                    %Eq.20.20

a =2*inv(Q)*zetas;                             %Eq.20.21

aMK = a(1)*(inv(M)*K)+a(2)*(inv(M)*K)^2;       %Sum(a(M^-1)(K))^i)

C = M*aMK                                      %Eq.20.15

nom_C = nom_phi'*(C)*nom_phi;                  %Eq.20.25
zetas = diag((1/2)*nom_C*inv(diag(omegas)));
```

The MATLAB program prints out the damping matrix of the illustrative Example 20.2.

$$[C] = \begin{bmatrix} 573.5 & -199.3 \\ -199.3 & 313.9 \end{bmatrix}$$

20.4 Summary

The most common method of taking into account the dissipation of energy in structural dynamics is to assume in the mathematical model the presence of damping forces of magnitudes that are proportional to the relative velocity and of directions opposite to the motion. This type of damping is known as viscous damping because it is the kind of damping that will be developed by motion in an ideal viscous fluid. The inclusion of this type of damping in the equations does not alter the linearity of the differential equations of motion. Since the amount of damping commonly present in structural systems is relatively small, its effect is neglected in the calculation of natural frequencies and mode shapes. However, to uncouple the damped differential equations of motion, it is necessary to impose some restrictions on the values of damping coefficients in the system. These restrictions are of no consequence owing to the fact that in practice it is easier to determine or to estimate modal damping ratios rather than absolute damping coefficients. In addition, when solving the equations of motion by the modal superposition method, only damping ratios are required. When the solution is sought by other methods, the absolute value of the damping coefficients may be calculated from modal damping ratios by one of the two methods presented in this chapter.

20.5 Problems

Problem 20.1
The stiffness and mass matrices for a certain two-degree-of-freedom structure are

$$[K] = \begin{bmatrix} 400 & -200 \\ -200 & 200 \end{bmatrix}, \quad [M] = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

Determine the damping matrix for this system corresponding to 20% of the critical damping for the first mode and 10% for the second mode. Use the method based on Eqs. (20.16) and (20.17).

Problem 20.2
Repeat Problem 20.1 using the method based on Eqs. (20.22) and (20.25).

Problem 20.3
The natural frequencies and corresponding normal modes (arranged in the modal matrix) for the three-story shear building shown in Fig. P20.3 are $\omega_1 = 9.31$ rad/sec, $\omega_2 = 20.94$ rad/sec, $\omega_3 = 29.00$ rad/sec, and

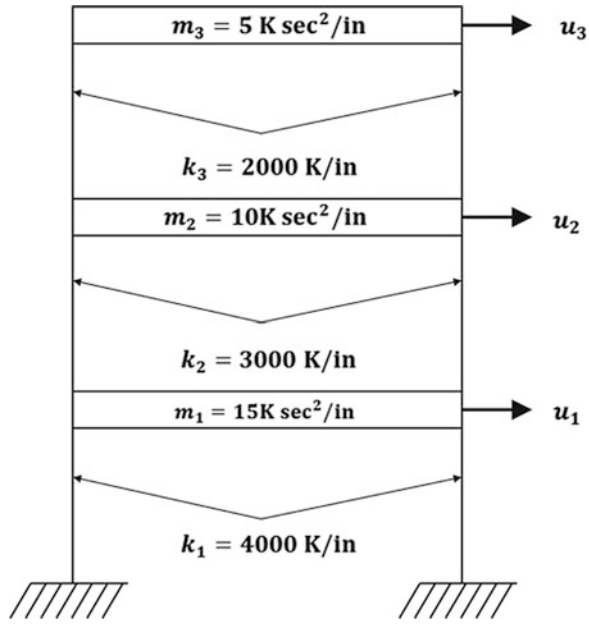

Fig. P20.3

$$[\Phi] = \begin{bmatrix} 0.1114 & -0.1968 & -0.1245 \\ 0.2117 & -0.0277 & 0.2333 \\ 0.2703 & 0.2868 & -0.2114 \end{bmatrix}$$

Determine the damping matrix for the system corresponding to damping ratios of 10% for all the modes.

Problem 20.4
Repeat Problem 20.3 for damping ratios of 20% for all the modes.

Problem 20.5
Repeat Problem 20.3 for the following value of modal damping ratios:

$$\xi_1 = 0.2, \quad \xi_2 = 0.1, \quad \xi_3 = 0.0$$

Problem 20.6
Use Program 7 and 8 to model and determine the natural frequencies and normal modes for the five-story shear building shown in Fig. P20.6; then use Program 11 to determine the damping matrix corresponding to an 8% damping ratio in all the modes.

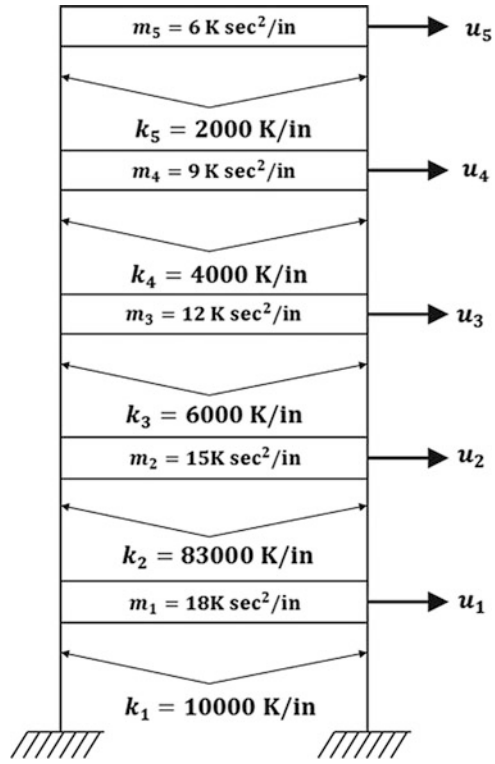

Fig. P20.6

Problem 20.7

Repeat Problem 20.6 for the following values of the modal damping ratios:

$$\xi_1 = 0.20, \quad \xi_2 = 0.15, \quad \xi_3 = 0.10, \quad \xi_4 = 0.05, \quad \xi_5 = 0,$$

Check for updates

21 Generalized Coordinates and Rayleigh's Method

In the preceding chapters we concentrated our efforts in obtaining the response to dynamic loads of structures modeled by the simple oscillator, that is, structures that may be analyzed as a damped or undamped spring-mass system. Our plan in the present chapter is to discuss the conditions under which a structural system consisting of multiple interconnected rigid bodies or having distributed mass and elasticity can still be modeled as a one-degree of-freedom system. We begin by presenting an alternative method to the direct application of Newton's Law of Motion, the principle of virtual work.

21.1 Principle of Virtual Work

An alternative approach to the direct method employed thus far for the formulation of the equations of motion is the use of the principle of virtual work. This principle is particularly useful for relatively complex structural systems which contain many interconnected parts. The principle of virtual work was I originally stated for a system in equilibrium. Nevertheless, the principle can readily be applied to dynamic systems by the simple recourse to D'Alembert's Principle, which establishes dynamic equilibrium by the inclusion of the inertial forces in the system.

The principle of virtual work may be stated as follows: For a system that is in equilibrium, the work done by all the forces during an assumed displacement (virtual displacement) that is compatible with the system constraints is equal to zero. In general the equations of motion are obtained by introducing virtual displacements corresponding to each degree of freedom and equating the resulting work done to zero.

To illustrate the application of the principle of virtual work to obtain the equation of motion for a single-degree-of-freedom system, let us consider the damped oscillator shown in Fig. 21.1a and its corresponding free body diagram in Fig. 21.1b. Since the inertial force has been included among the external forces, the system is in "equilibrium" (dynamic equilibrium). Consequently, the principle of virtual work is applicable. If a virtual displacement δu is assumed to have taken place, the total work done by the forces shown in Fig. 21.1b is equal to zero, that is,

$$m\ddot{u}\delta u + c\dot{u}\,\delta u + ku\delta u - F(t)\delta u = 0$$

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M. Paz, Y. H. Kim, *Structural Dynamics*, https://doi.org/10.1007/978-3-319-94743-3_21

or

$$\{m\ddot{u} + c\dot{u} + ku - F(t)\}\delta u = 0 \tag{21.1}$$

Since δu is arbitrarily selected as not equal to zero, the other factor in Eq. (21.1) must equal zero. Hence,

$$m\ddot{u} + c\dot{u} + ku - F(t) = 0 \tag{21.2}$$

Thus we obtained in Eq. (21.2) the differential equation for the motion of the damped oscillator.

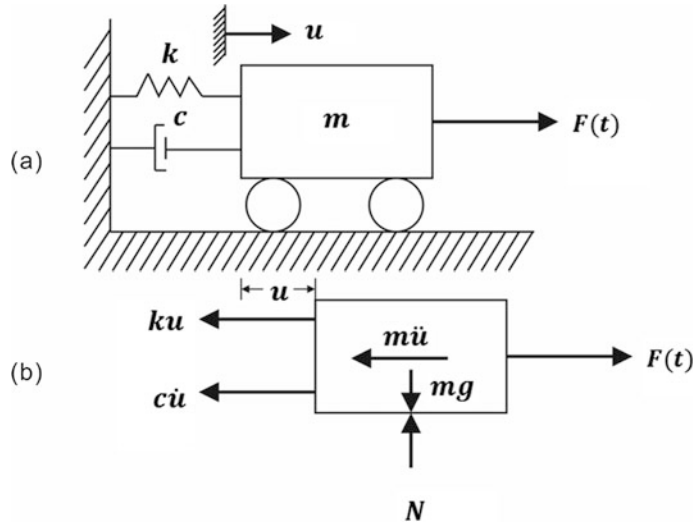

Fig. 21.1 Damped simple oscillator undergoing virtual displacement δu

21.2 Generalized Single-Degree-of-Freedom System–Rigid Body

Most frequently the configuration of a dynamic system is specified by coordinates indicating the linear or angular positions of elements of the system. However, coordinates do not necessarily have to correspond directly to displacements; they may in general be any independent quantities that are sufficient in number to specify the position of all parts of the system. These coordinates are usually called generalized coordinates and their number is equal to the number of degrees of freedom of the system.

The example of the rigid-body system shown in Fig. 21.2 consists of a rigid bar with distributed mass supporting a circular plate at one end. The bar is supported by springs and dampers in addition to a single frictionless support. Dynamic excitation is provided by a transverse load $F(x,\ t)$ varying linearly on the portion AB of the bar. Our purpose is to obtain the differential equation of motion and to identify the corresponding expressions for the parameters of the simple oscillator representing this system.

Since the bar is rigid, the system in Fig. 21.2 has only one degree of freedom, and, therefore, its dynamic response can be expressed with one equation of motion. The generalized coordinate could be selected as the vertical displacement of any point such as A, B, or C along the bar, or may be taken as the angular position of the bar. This last coordinate designated $\theta\ (t)$ is selected as the generalized coordinate of the system. The corresponding free body diagram showing all the forces including the inertial forces and the inertial moments is shown in Fig. 21.3. In evaluating the displacements of the different forces, it is assumed that the displacements of the system are small and, therefore, vertical displacements are simply equal to the product of the distance to support D multiplied by the angular displacement $\theta = \theta(t)$.

The displacements resulting at the points of application of the forces in Fig. 21.3 due to a virtual displacement $\delta\theta$ are indicated in this figure. By the principle of virtual work, the total work done by the forces during this virtual displacement is equal to zero. Hence

$$\delta\theta\left[I_0\ddot{\theta} + I_1\ddot{\theta} + 4L^3\bar{m}\ddot{\theta} + mL^2\ddot{\theta} + cL^2\dot{\theta} + 4kL^2\theta - \frac{7}{6}L^2 f(t)\right] = 0$$

or, since $\delta\theta$ *is* arbitrarily set not equal to zero, it follows that

$$\left(I_0 + I_1 + 4L^3\bar{m} + mL^2\right)\ddot{\theta} + cL^2\dot{\theta} + 4kL^2\dot{\theta} - \frac{7}{6}L^2 f(t) = 0, \tag{21.3}$$

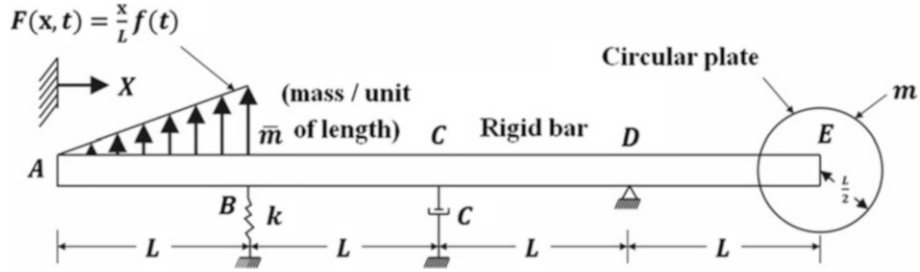

Fig. 21.2 Example of single-degree-of-freedom rigid system

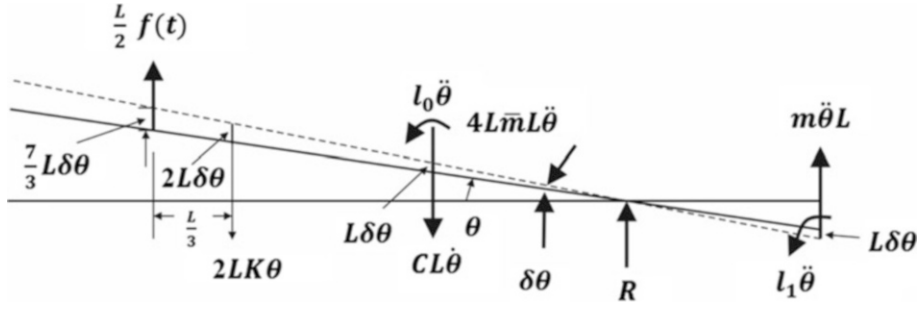

Fig. 21.3 Displacements and resultant forces for system in Fig. 21.2

where

$$I_0 = \frac{1}{12}(4\bar{m}L)(4L)^2 = \text{mass moment of inertia of the rod}$$

$$I_0 = \frac{1}{2}m\left(\frac{L}{2}\right)^2 = \text{mass moment of inertia of the circular plate}$$

The differential Eq. (21.3) governing the motion of this system may conveniently be written as

$$I^*\ddot{\theta} + C^*\dot{\theta} + K^*\theta = F^*(t) \tag{21.4}$$

where I^*, C^*, K^*, and $F^*(t)$ are, respectively, the generalized inertia, generalized damping, generalized stiffness, and generalized force for this system. These quantities are given in Eq. (21.3) by the factors corresponding to the acceleration, velocity, displacement, and force terms, namely,

$$\begin{aligned} I^* &= I_0 + I_1 + 4\bar{m}L^3 + mL^2 \\ C^* &= cL^2 \\ K^* &= 4kL^2 \\ F^*(t) &= \frac{7}{6}L^2 f(t) \end{aligned}$$

Illustrative Example 21.1

For the system shown in Fig. 21.4. determine the generalized physical properties M^*. C^*. K^* and generalized loading $F^*(t)$. Let $U(t)$ at the point A_2 in Fig. 1.4 be the generalized coordinate of the system.

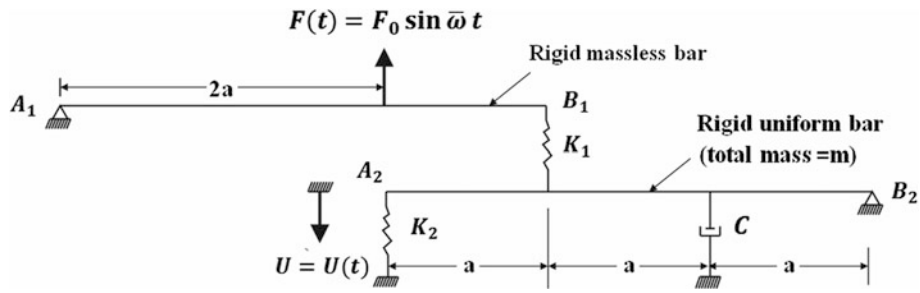

Fig. 21.4 System for Illustrative Example 21.1

Solution:

The free body diagram for the system is depicted in Fig. 21.5 which shows all the forces on the two bars of the system including the inertial force and the inertial moment. The generalized coordinate is $U(t)$ and the displacement of my point in the system should be expressed in terms of this coordinate; nevertheless, for convenience, we select also the auxiliary coordinate $U_1(t)$ as indicated in Fig. 21.5.

The summation of the moments about point A_1 of all the forces acting on bar A_1 -B_1. and the summation of moments about B_2 of the forces on bar A_2 -B_2. Give the following equations:

$$k_1\left(\frac{2}{3}U - U_1\right)3a = 2aF_0 \sin \overline{\omega}t \tag{21.5}$$

$$\frac{I_0}{3a}\ddot{U} + \frac{3}{4}ma\ddot{U} + \frac{a}{3}c\dot{U} + k_1\left(\frac{2}{3}U - U_1\right)2a + 3ak_2U = 0 \tag{21.6}$$

Substituting U_1 from Eq. (21.5) into Eq. (21.6), we obtain the differential equation for the motion of the system in terms of the generalized coordinate $U(t)$, namely

$$M^*\ddot{U}(t) + C^*\dot{U}(t) + K^*U(t) = F^*(t)$$

where the generalized quantities are given by

$$M^* = \frac{I_0}{3a^2} + \frac{3m}{4}$$
$$C^* = \frac{c}{3}$$
$$K^* = 3k_2$$

and

$$F^*(t) = -\frac{4}{3}F_0 \sin \varpi t$$

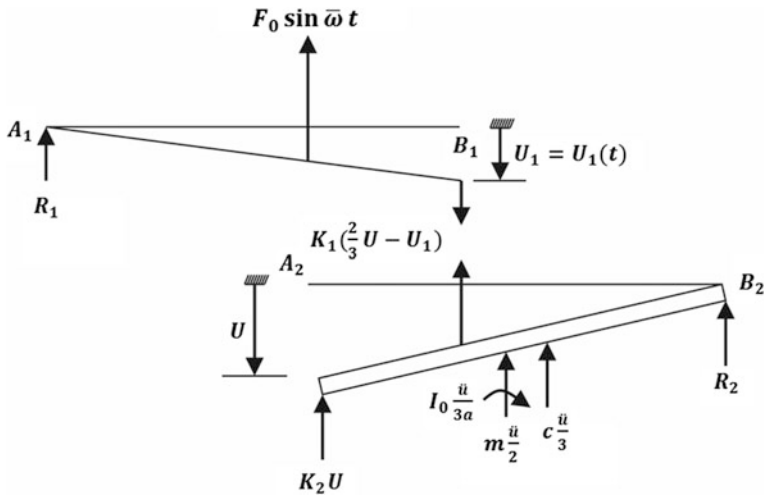

Fig. 21.5 Displacements and resultant forces for Example 21.1

21.3 Generalized Single-Degree-of-Freedom System–Distributed Elasticity

The example presented in the preceding section had only one degree of freedom in spite of the complexity of the various parts of the system because the two bars were interconnected through a spring and one of the bars was massless so that only one coordinate sufficed to completely specify the motion. If the bars were not rigid, but could deform in flexure, the system would have an infinite

number of degrees of freedom. However, a single-degree-of-freedom analysis could still be made, provided that only a single shape could be developed during motion, that is, provided that the knowledge of the displacement of a single point in the system determines the displacement of the entire system.

As an illustration of this method for approximating the analysis of a system with an infinite number of degrees of freedom with a single degree of freedom, consider the cantilever beam shown in Fig. 21.6. In this illustration, the physical properties of the beam are the flexural stiffness $EI(x)$ and its mass per unit of length $m(x)$. It is assumed that the beam is subjected to an arbitrary distributed forcing function $p(x, t)$ and to an axial compressive force N.

In order to approximate the motion of this system with a single coordinate, it is necessary to assume that the beam deflects during its motion in a prescribed shape. Let ϕ (x) be the function describing this shape and, as a generalized coordinate, $U(t)$ the function describing the displacement of the motion corresponding to the free end of the beam. Therefore, the displacement at any point x along the beam is

$$u(x,\ \ t) = \phi(x)U(t) \tag{21.7}$$

where ϕ $(L) = 1$.

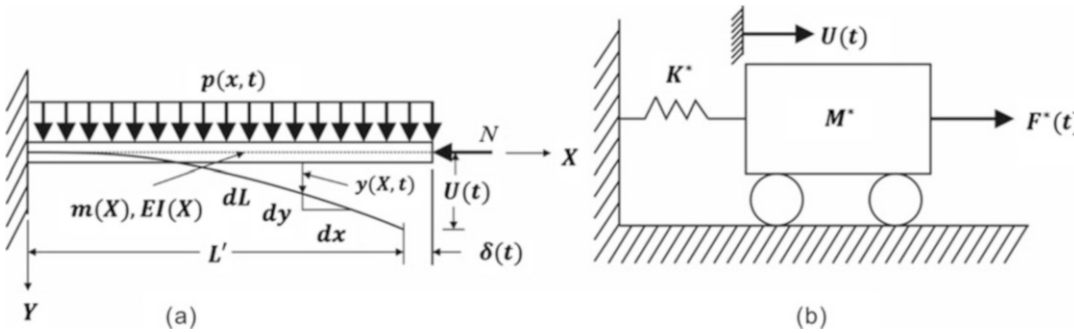

Fig. 21.6 Single-degree-of-freedom continuous system

The equivalent one-degree-of-freedom system (Fig. 21.6b) may be defined simply as the system for which the kinetic energy, potential energy (strain energy), and work done by the external forces have at all times the same values in the two systems.

The kinetic energy T of the beam in Fig. 21.6 vibrating in the pattern indicated by Eq. (21.7) is

$$T = \int_0^L \frac{1}{2} m(x)\{\phi(x)U(t)\}^2 dx \tag{21.8}$$

Equating this expression for the kinetic energy of the continuous system to the kinetic energy of the equivalent single-degree-of-freedom system $\frac{1}{2}M^*U^2(t)$ and solving the resulting equation for the generalized mass, we obtain

$$M^* = \int_0^L m(x)\ \ \phi^2(x)dx \tag{21.9}$$

The flexural strain energy V of a prismatic beam may be determined as the work done by the bending moment $M(x)$ undergoing an angular displacement $d\theta$. This angular displacement is obtained from the well-known formula for the flexural curvature of a beam, namely

$$\frac{d^2u}{dx^2} = \frac{d\theta}{dx} = \frac{M(x)}{EI} \tag{21.10}$$

or

$$d\theta = \frac{M(x)}{EI}dx \tag{21.11}$$

since $du/dx = \theta$, where θ, being assumed small, is taken as the slope of the elastic curve. Consequently, the strain energy is given by

$$V = \int_0^L \frac{1}{2}M(x)d\theta \tag{21.12}$$

The factor $\frac{1}{2}$ is required for the correct evaluation of the work done by the flexural moment increasing from zero to its final value $M(x)$ [average value $M(x)/2$]. Now, utilizing Eqs. (21.10 and 21.11) in Eq. (21.12), we obtain

$$V = \int_0^L \frac{1}{2}EI(x)\left(\frac{d^2u}{dx^2}\right)^2 dx \tag{21.13}$$

Finally, equating the potential energy, Eq. (21.13), for the continuous system to the potential energy of the equivalent system and using Eq. (21.7) results in

$$\frac{1}{2}K^*U(t)^2 = \int_0^L \frac{1}{2}EI(x)\{\phi''(x)U(t)\}^2 dx$$

or

$$K^* = \int_0^L EI(x)\{\phi''(x)\}^2 dx \tag{21.14}$$

where

$$\phi''(x) = \frac{d^2u}{dx^2}$$

The generalized force $F^*(t)$ may be found from the virtual displacement $\delta U(t)$ of the generalized coordinate $U(t)$ upon equating the work performed by the external forces in the structure to the work done by the generalized force: in the equivalent single-degree-of-freedom system. The work of the distributed external force $p(x, t)$ due to this virtual displacement is given by

$$W = \int_0^L p(x,t)\delta u dx$$

Substituting $\delta u = \phi(x)\ \delta U$ from Eq. (21.7) gives

$$W = \int_0^L p(x,t)\phi(x)\delta U dx \tag{21.15}$$

The work of the generalized force $F^*(t)$ in the equivalent system corresponding to the virtual displacement δU of the generalized coordinate is

$$W^* = F^*(t)\delta U \tag{21.16}$$

Equating Eq. (21.15) with Eq. (21.16) and canceling the factor $\delta\, u$, which is taken to be different from zero, we obtain the generalized force as

$$\mathrm{F}^*(\mathrm{t}) = \int_0^L p(x,t)\phi(x)dx \tag{21.17}$$

Similarly, to determine the generalized damping coefficient, assume a virtual displacement and equate the work of the damping forces in the physical system with the work of the damping force in the equivalent single-degree of-freedom system. Hence

$$C^*\dot{U}\,\delta U = \int_0^L c(x)u\,\delta u dx$$

where $c(x)$ is the distributed damping coefficient per unit length along the beam. Substituting $\delta u = \phi(x)\delta u$ and $\dot{u} = \phi(x)\dot{U}$ from Eq. (21.7) and canceling the common factors, we obtain

$$C^* = \int_0^L c(x)[\phi(x)]^2 dx \tag{21.18}$$

which is the expression for the generalized damping coefficient.

To calculate the potential energy of the axial force N which is unchanged during the vibration of the beam and consequently is a conservative force, it is necessary to evaluate the horizontal component of the motion $\delta(t)$ of the free end of the beam as indicated in Fig. 21.6. For this purpose, we consider a differential element of length dL along the beam as shown in Fig. 21.6a. The length of this element may be expressed as

$$dL = \left(dx^2 + du^2\right)^{1/2}$$

or

$$dL = \left(1 + (du/dx)^2\right)^{1/2} dx \tag{21.19}$$

Now, integrating over the horizontal projection of the length of beam (L') and expanding in series the binomial expression, we obtain

$$\begin{aligned} L &= \int_0^{L'} \left(1 + \left(\tfrac{du}{dx}\right)^2\right)^{1/2} dx \\ &= \int_0^{L'} \left\{1 + \frac{1}{2}\left(\frac{du}{dx}\right)^2 - \frac{1}{8}\left(\frac{du}{dx}\right)^4 + \ldots\right\} dx \end{aligned}$$

Retaining only the first two terms of the series results in

$$L = L' + \int_0^L \frac{1}{2}\left(\frac{du}{dx}\right)^2 dx \tag{21.20}$$

or

$$\delta(t) = L - L' = \int_0^L \frac{1}{2}\left(\frac{du}{dx}\right)^2 dx \tag{21.21}$$

The reader should realize that Eqs. (21.20 and 21.21) involve approximations since the series was truncated and the upper limit of the integral in the final expression was conveniently set equal to the initial length of the beam L instead of to its horizontal component L'.

Now we define a new stiffness coefficient to be called the generalized geometric stiffness K^*_G as the stiffness of the equivalent system required to store the same potential energy as the potential energy stored by the normal force N, that is,

$$\frac{1}{2}K^*_G U(t)^2 = N\delta(t)$$

Substituting $\delta(t)$ from Eq. (21.21) and the derivative du/dx from Eq. (21.7), we have

$$\frac{1}{2}K^*_G U(t)^2 = \frac{1}{2}N\int_0^L \left\{U(t)\frac{d\phi}{dx}\right\}^2 dx$$

or

$$K^*_G = N\int_0^L \left(\frac{d\phi}{dx}\right)^2 dx \tag{21.22}$$

Equations (21.9, 21.14, 21.17, 21.18 and 21.22) give, respectively, the generalized expression for the mass, stiffness, force, damping, and geometric stiffness for a beam with distributed properties and load, modeling it as a simple oscillator.

For the case of an axial compressive force, the potential energy in the beam decreases with a loss of stiffness in the beam. The opposite is true for a tensile axial force, which results in an increase of the flexural stiffness of the beam. Customarily, the geometric stiffness is determined for a compressive axial force. Consequently, the combined generalized stiffness K^*_G is then given by

$$K^*_c = K^* - K^*_G \tag{21.23}$$

Finally, the differential equation for the equivalent system may be written as

$$M^*\ddot{U}(t) + C^*U(t) + K^*_C U(t) = F^*(t) \tag{21.24}$$

The critical buckling load N_{cr} is defined as the axial compressive load that reduces the combined stiffness to zero, that is,

$$K^*_c = K^{*-}K^*_G = 0$$

The substitutian of K^* and K^*_G from Eqs. (21.14 and 21.22) gives

$$\int_0^L EI\left(\frac{d^2\phi}{dx^2}\right)^2 dx - N_{cr}\int\left(\frac{d\phi}{dx}\right)^2 dx = 0$$

and solving for the critical buckling load, we obtain

$$N_{cr} = \frac{\int_0^L EI\left(d^2\phi/dx^2\right)^2 dx}{\int_0^L \left(d\phi/dx\right)^2 dx} \tag{21.25}$$

Once the generalized stiffness K^* and generalized mass M^* have been determined, the system can be analyzed by any of the methods presented in the preceding chapters for single-degree-of-freedom systems. In particular, the square of the natural frequency, ω^2, is given from Eqs. (21.9 and 21.14) by

$$\omega^2 = \frac{K^*}{M^*} = \frac{\int_0^L EI(x)\phi''(x)^2 dx}{\int_0^L m(x)\phi^2(x)dx} \tag{21.26}$$

The displacement $U(t)$ of the generalized single-degree-of-freedom system is found as the solution of the differential equation

$$M^*U(1) + C^*U(1) + K^*U(1) = F^*(1) \tag{21.27}$$

and the displacement $u(x, t)$ at location x and time t is then calculated by Eq. (21.7).

21.4 Shear Forces and Bending Moments

The internal forces-shear forces and bending moments-associated with the displacements $u(x, t)$ may be obtained by loading the structure with the equivalent forces that would produce the dynamic displacements calculated. From elementary beam theory, the rate of loading $p(x, t)$ per unit of length along abeam is related to the lateral displacement $u = u(x, t)$ by the differential equation

$$p(x,t) = \frac{\partial}{\partial x^2}\left[EI(x)\frac{\partial^2 u}{\partial x^2}\right] \tag{21.28}$$

or substituting $u(x, t) = \phi(x)\cdot U(t)$ from Eq. (21.7) by

$$p(x,t) = \frac{\partial^2}{\partial x^2}\left[EI(x)\frac{\partial^2 \Phi}{\partial x^2}\right]U(t) \tag{21.29}$$

The equivalent force $p(x, i)$, calculated using Eq. (21.29), depending on derivatives of the shape function, will in general, give internal forces that are less accurate than the displacements, because the derivatives of approximate shape functions tends to increase errors in the shape function.

A better approach to estimate the equivalent forces is to compute the inertial forces that resulted in the calculated displacements. These inertial forces are given by

$$p(x,t) = m(x)\ddot{u}(x,t) \tag{21.30}$$

or substituting $\ddot{U}(x, t)$ from Eq. (21.7) by

$$p(x,t) = m(x)\phi(x)\ddot{U}(t) \tag{21.31}$$

At any time t the equivalent load $p(x, t)$ given by Eq. (21.31) is applied to the structure as a static load and the internal forces (shear forces and bending moments), as well as the stresses resulting from these internal forces, are determined.

To provide an example of the determination of the equivalent one degree of freedom for a system with distributed mass and stiffness, consider the water tower in Fig. 21.7 to have uniformly distributed mass iii and stiffness EI along its length with a concentrated mass $M = \bar{m}L$ at the top. The tower is subjected to an earthquake ground motion excitation of acceleration $a_g(t)$ and to an axial compressive load due to the weight of its distributed mass and concentrated mass at the top. Neglect damping in the system. Assume that during the motion the shape of the tower is given by

$$\phi(x) = 1 - \cos\frac{\pi x}{2L} \tag{21.32}$$

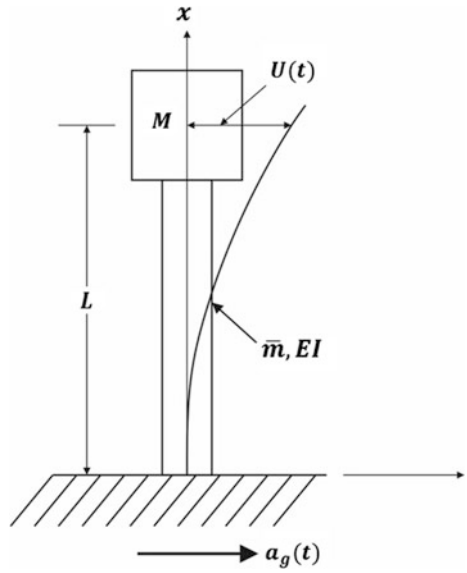

Fig. 21.7 Water tower with distributed properties for Illustrative Example 21.2

Selecting the lateral displacement $u(t)$ at the top of the tower as the generalized coordinate as shown in Fig. 21.7, we obtain for the displacement at any point

$$U(x,t) = U(t)\phi(x) = U(t)\left(1 - \cos\frac{\pi x}{2L}\right) \tag{21.33}$$

The generalized mass and the generalized stiffness of the tower are computed, respectively, from Eqs. (21.9 and 21.14) as

$$M^* = \bar{m}L + \bar{m}\int_0^L \left(1 - \cos\frac{\pi x}{2L}\right)^2 dx$$

$$M^* = \frac{\bar{m}L}{2\pi}(5\pi - 8) \tag{21.34}$$

and

$$K^* = \int_0^L EI\left(\frac{\pi}{2L}\right)^4 \cos^2\frac{\pi x}{2L}dx$$

$$K^* = \frac{\pi^4 EI}{32L^3} \tag{21.35}$$

The axial force is due to the weight of the tower above a particular section, including the concentrated weight at the top, and may be expressed as

$$N(x) = \bar{m}\ Lg\left(2 - \frac{x}{L}\right) \tag{21.36}$$

where g is the gravitational acceleration. Since the normal force in this case is a function of x, it is necessary in using Eq. (21.22) to include $N(x)$ under the integral sign. The geometric stiffness coefficient K^*_G is then given by

$$K^*_G = \int_0^L \bar{m}Lg\left(2 - \frac{x}{L}\right)\left(\frac{\pi}{2L}\right)^2 \sin^2\frac{\pi x}{2L}dx$$

which upon integration yields

$$K^*_G = \frac{\bar{m}g}{16}\left(3\pi^2 - 4\right) \tag{21.37}$$

Consequently, the combined stiffness from Eqs. (21.35 and 21.37) is

$$K^*_c = K^* - K^*_G = \frac{\pi^4 EI}{32L^3} - \frac{\bar{m}g}{16}\left(3\pi^2 - 4\right) \tag{21.38}$$

By setting $K^*_c = 0$, we obtain

$$\frac{\pi^4 EI}{32L^3} - \frac{\bar{m}g}{16}\left(3\pi^2 - 4\right) = 0$$

which gives the critical load

$$\left(\bar{m}g\right)_{cr} = \frac{\pi^4 EI}{2(3\pi^3 - 4)L^3} \tag{21.39}$$

The equation of motion in terms of the relative motion $u = u(t) - u_g(t)$ is given by Eq. (3.50) for the undamped system as

$$M^*\ddot{u}_r + K^*_c u_r = F^*_{eff}(t) \tag{21.40}$$

where M^* is given by Eq. (21.34), K^*_c by Eq. (21.38), and the effective force by Eq. (21.17) for the effective distributed force and by $-\ \bar{m}\ La_g(t)$ for the effective concentrated force at the top of the tower. Hence

$$F^*_{eff}(t) = \int_0^L p_{eff}(x,t)\phi(x)dx - \bar{m}La_g(t)$$

where $p_{eff}(x,\ t) = -\bar{m}\ a_g(t)$ is the effective distributed force. Then

$$F^*_{eff}(t) = \int_0^L -\bar{m}a_g(t)\phi(x)dx - \bar{m}La_g(t)$$

Substitution of $\phi(x)$ from Eq. (21.32) into the last equation yields upon integration

$$F^*_{eff}(t) = -\frac{2\bar{m}a_g(t)L}{\pi}(\pi - 1) \tag{21.41}$$

Illustrative Example 21.2

As a numerical example of calculating the response of a system with distributed properties, consider the water tower shown in Fig. 21.7 excited by a sinusoidal ground acceleration $a_g(t) = 20\sin 6.36\ t$ (in/sec^2).

Model the structure by assuming the shape given by Eq. (21.32) and determine the response.

Solution:

The numerical values for this example are:

$$\bar{m} = 0.1\ \text{Kip} \cdot \sec^2/\text{in per unit of length}$$
$$EI = 1.2\ \ 10^{13}\ \text{Kip} \cdot \text{in}^2$$
$$L = 100\ \text{ft} = 1200\ \text{in}$$
$$\varpi = 6.36\ \text{rad/sec}$$

From Eq. (21.34) the generalized mass is

$$M^* = \frac{0.1 \times 1200}{2\pi}(5\pi - 8) = 147.21\ \text{Kip} \cdot \sec^2/\text{in}$$

and from Eq. (21.38) the generalized combined stiffness is

$$K^*_c = \frac{\pi^4 1.2 10^{13}}{32 \times (1200)^3} - \frac{0.1 \times 386}{16}\left(3\pi^2 - 4\right) = 21,077\ \text{Kip/in}$$

The natural frequency is

$$\omega = \sqrt{K^*_c/M^*} = 11.96\ \text{rad/sec}$$

and the frequency ratio

$$r = \frac{\varpi}{\omega} = \frac{6.36}{11.96} = 0.532$$

From Eq. (21.41) the effective force is

$$F^*_{eff} = -\frac{2(0.1)(1200)(\pi - 1)}{\pi}20.0\sin 6.36t$$

or

$$F^*_{eff} = -3272\sin 6.36t(\text{Kip})$$

Hence the steady-state response (neglecting damping) in terms of relative motion is given from Eq. (3.9) as

$$u_r = \frac{F^*_{eff}/k^*}{1-r^2}\sin\varpi t$$

$$= -\frac{3272/21,077}{1-(0.532)^2}\sin 6.36t$$

$$= -0.217\sin 6.36t \text{ in} \qquad \text{(Ans)}$$

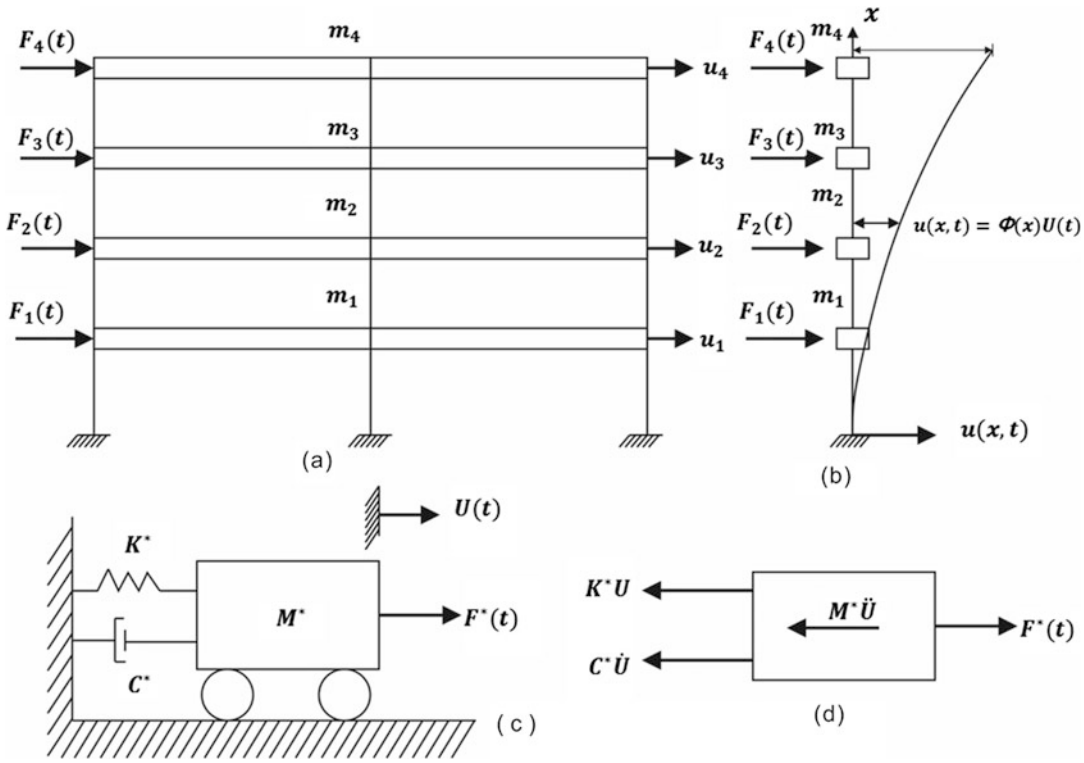

Fig. 21.8 (**a**) Multistory building subjected to lateral forces $F_i(t)$. (**b**) Column modeling the building showing assumed lateral displacement function $u(x,t) = \phi(x)U(t)$. (**c**) Equivalent single-degree-of-freedom system. (**d**) Free body diagram

21.5 Generalized Equation of Motion for a Multistory Building

Consider a multistory building such as the model for the four-story building shown in Fig. 21.8a subjected to lateral dynamic forces $F_i(t)$ at the different levels of the building. The mass of the building is assumed to be concentrated at the various levels (floors and roof) which are assumed to be rigid in their own planes; thus only horizontal displacements are possible in such a building.

To model this structure as a single degree of freedom (Fig. 21.8c), the lateral displacement shape u (x, t) is defined in terms of a single generalized coordinate $U(t)$ as

$$u(x,t) = \phi(x)U(t) \tag{21.42}$$

The generalized coordinate $f(t)$ in Eq. (21.42) is selected as the lateral displacement at the top level of the building which requires that shape function be assigned a unit value at that level; that is, $\phi(H) = 1.0$ where H is the height of the building.

The equation of motion for the generalized single-degree-of-freedom system in Fig. 21.8c is obtained by equating to zero the sum of forces in the corresponding free-body diagram (Fig. 21.8d), that is

$$M^*\ddot{U} + C^*\dot{U} + K^*U = F^*(t) \tag{21.43}$$

where M^*, C^*, K^*, and $F^*(t)$ are, respectively, the generalized mass, generalized damping, generalized stiffness, and generalized force, and are given for a discrete system, modeled in Fig. 21.8b, by

$$M^* = \sum_{i=1}^{N} m_i \Phi_i^2 \tag{21.44a}$$

$$C^* = \sum_{i=1}^{N} c_i \Delta\Phi_i^2 \tag{21.44b}$$

$$K^* = \sum_{i=1}^{N} k_i \Delta\Phi_i^2 \tag{21.44c}$$

$$F^*(t) = \sum_{i=1}^{N} F_i(t)\Phi_i \tag{21.44d}$$

where the upper index N in the summations is equal to the number of stories or levels in the building.

The various expressions for the equivalent parameters in Eq. (21.44) are obtained by equating the kinetic energy, potential energy, and the virtual work done by the damping forces and by external forces in the actual structure with the corresponding expressions for the generalized single-degree-of-freedom system.

In Eq. (21.44), the relative displacement $\Delta\phi_i$ between two consecutive levels of the building is given by

$$\Delta\phi_i = \phi_i - \phi_{i-1} \tag{21.45}$$

with $\phi_0 = 0$ at the ground level.

As shown in Fig. 21.8b, m_i and $F_i(t)$ are, respectively, the mass and the external force at level i of the building, while k_i and c_i are the stiffness and damping coefficients corresponding to the ith story.

It is convenient to express the generalized damping coefficient C^* of Eq. (21.44b) in terms of generalized damping ratio ξ^*; thus by Eq. (2.7)

$$C^* = \sum_{i=1}^{N} c_i \Delta\Phi_i^2 = \xi^* C^*_{cr} = 2\xi^* \ M^*\omega \tag{21.46}$$

in which w is the natural frequency calculated for the generalized system as

$$\omega = \sqrt{\frac{K^*}{M^*}} \tag{21.47}$$

The substitution of the generalized damping coefficient, ξ^*, from Eq. (21.46) and of $K^* = \omega^2 M^*$ from Eq. (21.47) into the differential equation of motion, Eq. (21.43), results in

$$\ddot{U} + 2\xi^* \omega \dot{U} + \omega^2 U = \frac{F^*(t)}{M^*} \tag{21.48}$$

When the excitation of the building is due to an acceleration function, $a(t) = U_0(t)$, acting at the base of the building, the displacements $u_i(t)$ are conveniently measured relative to the motion of the base $u_0(t)$, that is,

$$u_{ri}(t) = u_i(t) - u_0(t) \tag{21.49}$$

Also, in this case, the effective forces $F_{eff,i}$ at the various levels of the building are given from Eq. (4.39) as

$$F_{eff,i} = -m_i \ddot{u}_0(t) \tag{21.50}$$

The differential equation for the generalized single-degree-of-freedom system excited at its base is then written as

$$M^*_{\ddot{U}_r^*}(t) + C^*_{\dot{U}_r}(t) + K^* U_r(t) = F^*_{eff}(t) \tag{21.51}$$

where the generalized coordinate for the relative displacement, $U_r(t)$ is

$$U_r(t) = U(t) - U_{r0}(t) \tag{21.52}$$

The generalized effective force $F^*_{eff}(t)$ is calculated by Eqs. (21.44d and 21.50) as

$$F^*_{eff} = -\ddot{U}_0(t) \sum_{i=1}^{N} m_i \Phi_i \tag{21.53}$$

The generalized equation of motion may then be expressed as

$$\ddot{U}_r(t) + 2\xi^* \omega \dot{U}_r(t) + \omega^2 U_r(t) = \Gamma^* \ddot{U}_{r0}(t) \tag{21.54}$$

in which the coefficient Γ^* is the generalized participation factor given by

$$\Gamma^* = -\frac{\sum_{i=1}^{N} m_i \Phi_i}{\sum_{i=1}^{N} m_i \Phi_i^2} \tag{21.55}$$

21.6 Shape Function

The use of generalized coordinates transforms a multidegree-of-freedom system into an equivalent single-degree-of-freedom system. The shape function describing the deformed structure could be any arbitrary function that satisfies the boundary conditions. However, in practical applications, the success of this approach will depend on how close the assumed shape function approximates the

actual displacements of the dynamic system. For structural buildings, selection of the shape function is most appropriate by considering the aspect ratio of the structure, which is defined as the ratio of the building height to the dimension of the base. The recommended shape functions for high-rise, mid-rise, and low-rise buildings are summarized in Fig. 21.9. Most seismic building codes use the straight-line shape which is shown for the mid-rise building. The displacements in the structure are calculated using Eq. (21.7) after the dynamic response is obtained in terms of the generalized coordinate.

Illustrative Example 21.3

A four-story reinforced concrete framed building has the dimensions shown in Fig. 6.10. The sizes of the exterior columns (nine each on lines *A* and *C*) are 12 in × 20 in. and the interior columns (nine on line *B*) are 12 in × 24 in for the bottom two stories, and, respectively, 12 in × 16 in and 12 in × 20 in for the highest two stories. The height between floors is 12 ft. The dead load per unit area of the floor (floor slab, beam, half the weight of columns above and below the floor, partition walls, etc.) is estimated to be 140 psf. The design live load is taken as 25% of an assumed live load of 125 psf. Determine the generalized mass, generalized stiffness, generalized damping (for damping ratio $\xi^* = 0.1$), and the fundamental period for lateral vibration perpendicular to the long axis of the building. Assume the following shape functions: (a) $\phi(x) = x/H$ and (b) $\phi(x) \sin(\pi x/2H)$ where H is the height of the building (Fig. 21.10).

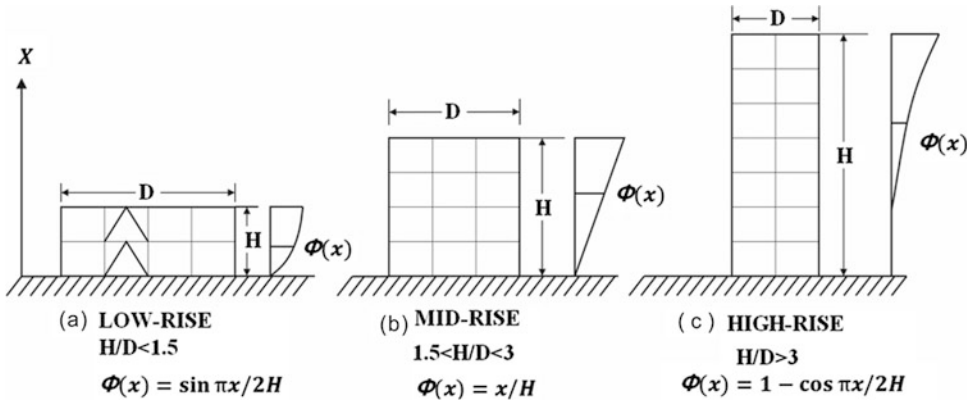

Fig. 21.9 Possible shape functions based on aspect ratio (Naeim 1989, p. 100)

Solution:

1. Effective weight at various floors:

No live load needs to be considered on the roof. Hence, the effective weight at all floors, except at the roof, will be 140 + 0.25 × 125 = 171.25 psf, and the effective weight for the roof will be 140 psf. The plan area is 48 ft. × 96 ft. =4608 ft^2. Hence, the weights of various levels are:

$W_1 = W_2 = W_3 = 4608 \times 0.17125 = 789.1$ Kips.
$W_4 = 4608 \times 0.140 = 645.1$ Kips.

The total seismic design weight of the building is then
$W = 789.1 \times 3 + 645.1 = 3012.4$ Kips

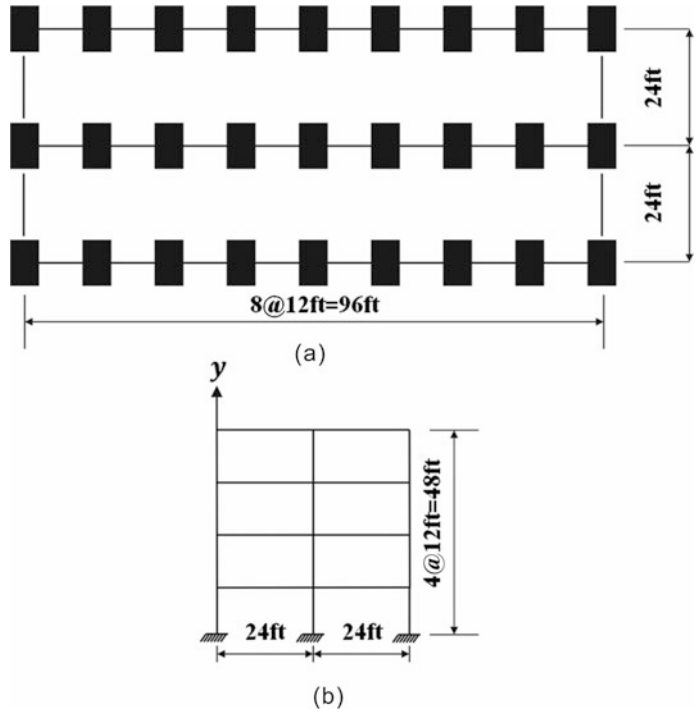

Fig. 21.10 Plan and elevation for a four-story building for Example 21.3; (**a**) Plan. (**b**) Elevation

2. Story lateral stiffness:
It will be assumed that horizontal beam-and-floor diaphragms are rigid compared to the columns of the building in order to simplify the hand calculation. In this case, the stiffness between two consecutive levels is given by

$$k = \frac{12EI}{L^3}$$

where
$L = 12$ ft. (distance between two floors)
$E = 3 \times 10^3$ ksi (modulus of elasticity of concrete)
$I = \frac{1}{12} 12 \times 20^3 = 8000$ in^4 (moment of inertia for the concrete section for columns 12 in $\times$ 20 in)

Therefore, for these columns,

$$k = \frac{12 \times 3 \times 10^3 \times 8000}{144^3} = 96.450 \text{ Kip/in}$$

Similarly, for columns 12 in $\times$ 24 in,

$$I = 13{,}824 \text{ in}^4 \text{ and } k = 166.667 \text{ Kip/in}$$

The total stiffness for the first and second stories is then

$$K_1 = K_2 = 18 \times 96.45 + 9 \times 166.67 = 3236 \text{ Kip/in}$$

Similarly, for the columns 12 in × 16 in. of the top and third stories,

$$I = \frac{1}{12}12 \times 16^3 = 4096 \ \text{in}^4, \ \ k = \frac{12 \times 3 \times 10^3 \times 4096}{144^3} = 49.4 \ \text{Kip/in}$$

and for the columns 12 in × 20 in,

$$I = \frac{1}{12}12 \times 20^3 = 8000 \ \text{in}^4, \ \ k = \frac{12 \times 3 \times 10^3 \times 8000}{144^3} = 96.5 \ \text{Kip/in}$$

Hence, total stiffness for the third or fourth stories is

$$K_3 = K_4 = 18 \times 49.4 + 9 \times 96.5 = 1757.7 \ \text{Kip/in}$$

3. *Generalized mass and stiffness:*
   (a) Assuming $\phi\,(x) = x/H$,

Table 21.1 shows the necessary calculations to obtain using Eqs. (21.44a and 21.44c) the generalized mass M^* and the generalized stiffness K^* for this example assuming $\phi(x) = x/H$.

The natural frequency is then calculated from Eq. (21.47) as

$$\omega = \sqrt{\frac{K^*}{M^*}} = \sqrt{\frac{625.250}{3.559}} = 13.25 rad/\sec, \ \ \text{then} \ \ T_a = \frac{2\pi}{\omega} = 0.47 \sec$$

and the generalized critical damping C^*_{cr} and the absolute generalized damping C^*, respectivaly, by Eqs. (2.6 and 2.19) as

$$C*_{cr} = 2\sqrt{K^*M^*} = 2\sqrt{(625.250)(3.559)} = 94.345(lb \cdot \sec/in)$$

and

$$C^* = \xi^* C^*_{cr} = (0.1)(94.345) = 9.43(lb \cdot \sec/in)$$

Table 21.1 Calculation of M^* and K^* Assuming $\phi\,(x) = x/H$

Level	k_i(Kip/in)	m_i(Kip · sec^2/in)	ϕ_i	$\Delta\phi_i$	$m_i\phi_i^2$(Kip sec^2/in)	$k_i\Delta\phi_i^2$ (Kip/in)
4		1.671	1.000		1.671	
	1758			0.250		109.875
3		2.044	0.750		1.150	
	1758			0.250		109.875
2		2.044	0.500		0.610	
	3236			0.250		202.250
1		2.044	0.250		0.128	
	3236			0.250		202.250
					$M^* = 3.559$	$K^* = 625.250$

(b) Assuming $\phi\,(x) = \sin(\pi\, x/2H)$.

Table 21.2 shows the necessary calculations to obtain using Eqs. (21.44a and 21.44c) the generalized mass $M*$ and the generalized stiffness $K*$ for this example assuming $\phi\,(x) = \sin(\pi\, x/2H)$.

Table 21.2 Calculation of M^* and K^* Assuming $\phi(x) = \sin(\pi x/2H)$

Level	k_i(Kip/in)	m_i(Kip·sec^2/in)	ϕ_i	$\Delta\phi_i$	$m_i\phi_i^2$(Kip·sec^2/in)	$k_i\Delta\phi_i^2$(Kip/in)
4		1.671	1.000		1.671	
	1758			0.076		10.154
3		2.040	0.924		1.745	
	1758			0.217		82.782
		2.040	0.707		1.022	
	3236			0.324		339.702
1		2.040	0.383		0.300	
	3236			0.383		476.686
					$M^* = 4.738$	$K^* = 909.324$

The natural frequency is then calculated from Eq. (21.47) as

$$\omega = \sqrt{\frac{K^*}{M^*}} = \sqrt{\frac{909.324}{4.738}} = 13.85 rad/\sec \quad \text{then} \quad T_b = \frac{2\pi}{\omega} = 0.45\sec$$

and the generalized critical damping C^*_{cr} and the absolute generalized damping C^*, respectively, by Eqs. (21.48 and 21.49) as

$$C^*_{cr} = 2\sqrt{K^* M^*} = 2\sqrt{(909.324)(4.738)} = 131.276(lb \cdot \sec/in)$$

and

$$C^* = \xi^* C^*_{cr} = (0.1)(131.276) = 13.13(lb \cdot \sec/in)$$

For this example, either of the two assumed shape functions results in essentially the same value for the fundamental period. However, assuming $\phi(x) = x/H$ is a slightly better approximation[1] to the true deflected shape than is $\phi(x) = \sin(\pi x/2H)$ because $T_a > T_b$.

21.7 Rayleigh's Method

In the preceding sections of this chapter the differential equation for a vibrating system was obtained by application of the principle of virtual work as an alternative method of considering the dynamic equilibrium of the system. However, the differential equation of motion for an undamped system in free vibration may also be obtained with the application of the *Principle of Conservation of Energy*. This principle may be stated as follows: If no external forces are acting on the system and there is no dissipation of energy due to damping, then the total energy of the system must remain constant during motion and consequently its derivative with respect to time must be equal to zero.

To illustrate the application of the Principle of Conservation of Energy in obtaining the differential equation of motion, consider the spring-mass system shown in Fig. 21.11. The total energy in this

[1] For an assumed displacement shape closer to the actual displacement shape, the structure will vibrate closer to free condition with less imposed constrains, thus with the stiffness reduced and a longer period.

case consists of the sum of the kinetic energy of the mass and the potential energy of the spring. In this case the kinetic energy T is given by.

$$T = \frac{1}{2} m\dot{u}^2 \tag{21.56}$$

where $\dot{u}$ is the instantaneous velocity of the mass.

The force in the spring, when displaced y units from the equilibrium position, is ku and the work done by this force on the mass for an additional displacement du is $-ku\,du$. This work is negative because the force ku acting on the mass is opposite to the incremental displacement du given in the positive direction of coordinate u. However, by definition, the potential energy is the value of this work but with opposite sign. It follows then that the total potential energy (PE) in the spring for a final displacement u will be

$$(PE) = \int_0^y kudu = \frac{1}{2} ku^2 \tag{21.57}$$

Adding Eqs. (21.56 and 21.57), and setting this sum equal to a constant, will give

$$\frac{1}{2} m\dot{u}^2 + \frac{1}{2} ku^2 = C_0 \tag{21.58}$$

Differentiation with respect to time yields

$$m\dot{u}\,\ddot{u} + ku\dot{u} = 0$$

Since $\dot{y}$ cannot be zero for all values of t, it follows that

$$m\ddot{u} + ku = 0 \tag{21.59}$$

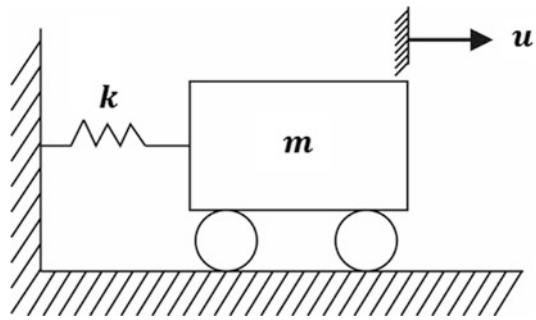

Fig. 21.11 Spring-mass system in free vibration

This equation is identical with Eq. (1.11) of Chap. 1 obtained by application of Newton's Law of Motion. Used in this manner, the energy method has no particular advantage over the equilibrium method. However, in many practical problems it is only the natural frequency that is desired. Consider again the simple oscillator of Fig. 21.11, and assume that the motion is harmonic. This assumption leads to the equation of motion of the form

$$u = C \sin(\omega t + \alpha) \tag{21.60}$$

and velocity

$$\dot{u} = \omega C \cos(\omega t + \alpha) \tag{21.61}$$

where C is the maximum displacement and ωC the maximum velocity. Then, at the neutral position ($u = 0$), there will be no force in the spring and the potential energy is zero. Consequently, the entire energy is then kinetic energy and

$$T_{\max} = \frac{1}{2} m(\omega C)^2 \tag{21.62}$$

At the maximum displacement the velocity of the mass is zero and all the energy is then potential energy, thus

$$(PE)_{\max} = \frac{1}{2} k C^2 \tag{21.63}$$

The energy in the system changes gradually over one-quarter of the cycle from purely kinetic energy, as given by Eq. (21.62), to purely potential energy, as given by Eq. (21.63). If no energy has been added or lost during the quarter cycle, the two expressions for this energy must be equal. Thus

$$\frac{1}{2} m \omega^2 C^2 = \frac{1}{2} k C^2 \tag{21.64}$$

Canceling common factors and solving Eq. (21.64) will give

$$\omega = \sqrt{\frac{k}{m}} \tag{21.65}$$

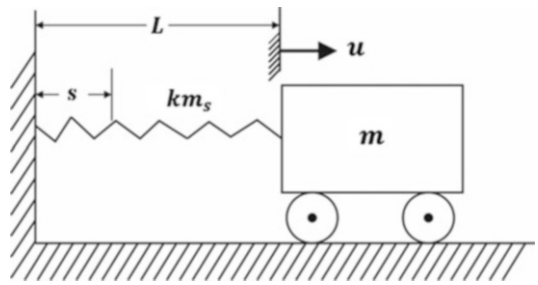

Fig. 21.12 Spring-mass system with heavy spring

which is the natural frequency for the simple oscillator obtained previously from the differential equation of motion. This method, in which the natural frequency is obtained by equating maximum kinetic energy with maximum potential energy, is known as Rayleigh's Method.

Illustrative Example 21.4

In the previous calculations on the spring-mass system, the mass of the spring was assumed to be so small that its effect on the natural frequency could be neglected. A better approximation to the true value of the natural frequency may be obtained using Rayleigh's Method. The distributed mass of the spring could easily be considered in the calculation by simply assuming that the deflection of the spring along its length is linear. In this case, consider in Fig. 21.12 the spring-mass system for which the spring has a length L and a total mass m_s. Use Rayleigh's Method to determine the fraction of the spring mass that should be added to the vibrating mass.

Solution:
The displacement of an arbitrary section of the spring at a distance s from the support will now be assumed to be $u_r = su/L$. Assuming that the motion of the mass m is harmonic and given by Eq. (21.60), we obtain

$$u_r = \frac{s}{L}u = \frac{s}{L}C\sin(\omega t + \alpha) \tag{21.66}$$

The potential energy of the uniformly stretched spring is given by Eq. (21.57) and its maximum value is

$$(PE)_{\max} = \frac{1}{2}kC^2 \tag{21.67}$$

A differential element of the spring of length ds has mass equal to $m_s ds/L$ and maximum velocity $\dot{u}_{r\max} = \omega u_{\max} = \omega s \ C/L$. Consequently the total kinetic energy in the system at its maximum value is

$$T_{\max} = \int_0^L \frac{1}{2}\frac{m_s}{L}ds\left(\omega\frac{s}{L}C\right)^2 + \frac{1}{2}m\omega^2C^2 \tag{21.68}$$

After integrating Eq. (21.68) and equating it with Eq. (21.67), we obtain

$$\frac{1}{2}kC^2 = \frac{1}{2}\omega^2C^2\left(m + \frac{m_s}{3}\right) \tag{21.69}$$

Solving for the natural frequency yields

$$\omega = \sqrt{\frac{k}{m + \frac{m_s}{3}}} \tag{21.70}$$

or in cycles per second (cps),

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m + \frac{m_s}{3}}} \tag{21.71}$$

The application of Rayleigh's Method shows that a better value for the natural frequency may be obtained by adding one-third of the mass of the spring to that of the main vibrating mass.

Rayleigh's Method may also be used to determine the natural frequency of a continuous system provided that the deformed shape of the structure is described as a generalized coordinate. The deformed shape of continuous structures and also of discrete structures of multiple degrees of freedom could in general be assumed arbitrarily. However, in practical applications, the success of the method depends on how close the assumed deformed shape will come to match the actual shape of the structure during vibration. Once the deformed shape has been specified, the maximum kinetic energy and the maximum potential energy may be determined by application of pertinent equations such as Eqs. (21.8 and 21.13). However, if the deformed shape has been defined as the shape resulting from statically applied forces, it would be simpler to calculate the work done by the external forces, instead of directly determining the potential energy. Consequently, in this case, the maximum kinetic energy is equated to the work of the forces applied statically. The following examples illustrate the application of Rayleigh's Method to systems with distributed properties.

Illustrative Example 21.5

Determine the natural frequency of vibration of a cantilever beam with a concentrated mass at its end when the distributed mass of the beam is taken into account. The beam has a total mass m_b and length L. The flexural rigidity of the beam is EI and the concentrated mass at its end is m, as shown in Fig. 21.13.

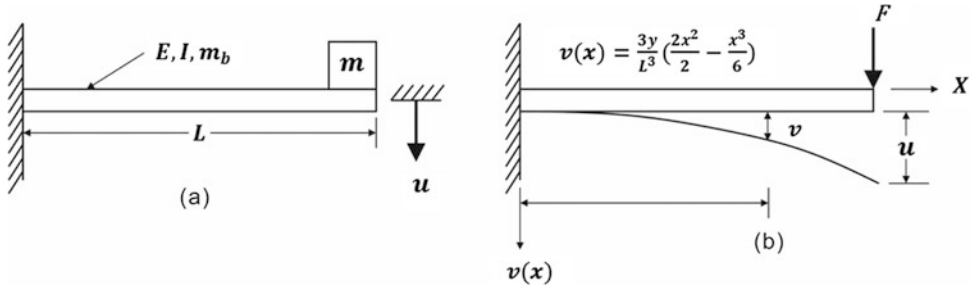

Fig. 21.13 (**a**) Cantilever beam of uniform mass with a mass concentrated at its tip. (**b**) Assumed deflection curve

Solution:

It will be assumed that the shape of deflection curve of the beam is that of the beam acted upon by a concentrated force F applied at the free end as shown in Fig. 21.13b. For this static load the deflection at a distance x from the support is

$$v(x) = \frac{3u}{L}\left(\frac{Lx^2}{2} - \frac{x^3}{6}\right) \tag{21.72}$$

where u = deflection at the free end of the beam. Upon substitution into Eq. (21.72) of $v(x) = C \sin(\omega t + \alpha)$, which is the harmonic deflection of the free end, we obtain

$$v(x) = \frac{3x^2L - x^3}{2L} C \sin(\omega t + \alpha) \tag{21.73}$$

The potential energy (PE) is equated to the work done by the force F as it gradually increases from zero to the final value F. This work is equal to $\frac{1}{2}Fv$, and its maximum value $(PE)_{\max}$ which is equal to the maximum potential energy is then

$$(PE)_{\max} = \frac{1}{2}FC = \frac{3EI}{2L^3}C^2 \tag{21.74}$$

since the force F is related to the maximum deflection by the formula from elementary strength of materials,

$$(PE)_{\max} = u = C = \frac{FL^3}{3EI} \tag{21.75}$$

The kinetic energy due to the distributed mass of the beam is given by

$$T = \int_0^L \frac{1}{2}\left(\frac{m_b}{L}\right)\dot{u}^2{}_r dx \tag{21.76}$$

and using Eq. (21.73) the maximum value for total kinetic energy will then be

$$T_{\max} = \frac{m_b}{2L}\int_0^L \left(\frac{3x^2L - x^3}{2L^3}\omega C\right)^2 dx + \frac{m}{2}\omega^2 C^2 \tag{21.77}$$

After integrating Eq. (21.77) and equating it with Eq. (21.74), we obtain

$$\frac{3EI}{2L^3}C^2 = \frac{1}{2}\omega^2 C^2\left(m + \frac{33}{140}m_b\right) \tag{21.78}$$

and the natural frequency becomes

$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{3EI}{L^3\left(m + \frac{33}{140}m_b\right)}} \tag{21.79}$$

It is seen, then, that by concentrating a mass equal to (33/140) m_b at the end of the beam, a more accurate value for the natural frequency of the cantilever beam is obtained compared to the result obtained by simply neglecting its distributed mass. In practice the fraction 33/140 is rounded to 1/4, thus approximating the natural frequency of a cantilever beam by

$$f = \frac{1}{2\pi}\sqrt{\frac{3EI}{L^3\left(m + \frac{33}{140}m_b\right)}} \tag{21.80}$$

The approximation given by either Eq. (21.79) or Eq. (21.80) is a good one even for the case in which $m = 0$. For this case the error given by these formulas is about 1.5% compared to the exact solution which will be presented in Chap. 21.

Illustrative Example 21.6

Consider in Fig. 21.14 the case of a simple beam carrying several concentrated masses. Neglect the mass of the beam and determine an expression for the natural frequency by application of Rayleigh's Method.

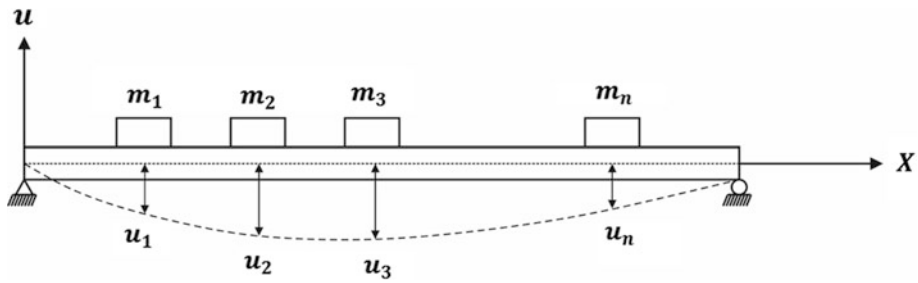

Fig. 21.14 Simple beam carrying concentrated masses

Solution:

In the application of Rayleigh's Method, it is necessary to choose a suitable curve to represent the deformed shape that the beam will have during vibration. A choice of a shape that gives consistently good results is the curve produced by forces proportional to the magnitude of the masses acting on the structure. For the simple beam, these forces could be assumed to be the weights $W_1 = m_1\, g$, $W_2 = m_2\, g$, …, $W_N = m_N g$ due to gravitational action on the concentrated masses. The static

deflections under these weights may then be designated by $U_1, U_2, \ldots, U_N$. The potential energy is then equal to work done during the loading of the beam, thus,

$$(PE)_{\max} = \frac{1}{2}W_1u_1 + \frac{1}{2}W_2u_2 + \quad \ldots \quad + \frac{1}{2}W_Nu_N \tag{21.81}$$

For harmonic motion in free vibration, the maximum velocities under the weights would be ωu_1, $\omega u_2, \ldots, \omega u_N$, and therefore the maximum kinetic energy would be

$$T_{\max} = \frac{1}{2}\frac{W_1}{g}(\omega u_1)^2 + \frac{1}{2}\frac{W_2}{g}(\omega u_2)^2 + \ldots + \frac{W_N}{g}(\omega u_N)^2 \tag{21.82}$$

When the maximum potential energy, Eq. (21.81), is equated with the maximum kinetic energy, Eq. (21.82), the natural frequency is found to be

$$\omega = \sqrt{\frac{g(W_1u_1 + W_2u_2 + \ldots + W_Nu_N)}{W_1u_1^2 + W_2u_2^2 + \ldots + .W_Nu_N^2}}$$

or

$$\omega = \sqrt{\frac{g\sum_{i=1}^{N} W_iu_i}{\sum_{i=1}^{N} W_iu_i^2}} \tag{21.83}$$

Where u_i is the deflection at coordinate i and W_i the weight at this coordinate.

This method is directly applicable to any beam, but in applying the method, it must be remembered that these are not gravity forces at all but substituted forces for the inertial forces. For example, in the case of a simple beam with overhang (Fig. 21.15) the force at the free end should be proportional to m_3 ($F_3 = m_3\, g$) but directed upward in order to obtain the proper shape for the deformed beam.

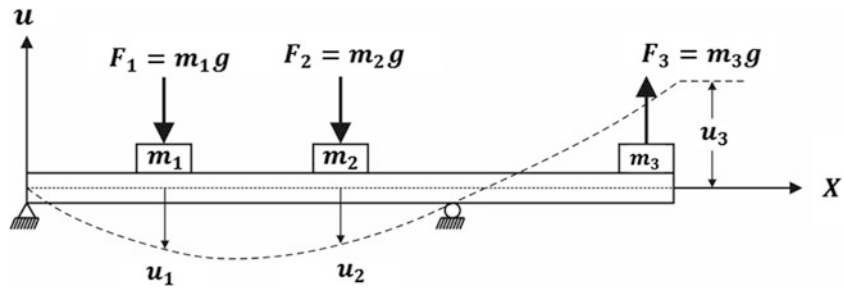

Fig. 21.15 Overhanging massless beam carrying concentrated masses

In the application of Rayleigh's Method, the forces producing the deflected shape do not necessarily have to be produced by gravitational forces. The only requirement is that these forces produce the expected deflection shape for the fundamental mode. For example, if the deflected shape for the beam shown in Fig. 21.14 is produced by forces designated by $f_1, f_2 \ldots f_N$ instead of the gravitation forces $W_1, W_2, \ldots, W_N$, we will obtain, as in Eq. (21.81), the maximum potential energy

$$(PE)_{\max} = \frac{1}{2} f_1 u_1 + \frac{1}{2} f_2 u_2 + \ldots + \frac{1}{2} f_N u_N \tag{21.84}$$

which equated to the maximum kinetic energy, Eq. (21.82), will result in the following formula for the fundamental frequency:

$$\omega = \sqrt{\frac{g \sum_{i=1}^{N} f_i u_i}{\sum_{i=1}^{N} W_i u_i^2}} \tag{21.85}$$

Consequently, the fundamental period could be calculated as

$$T = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{\sum_{i=1}^{N} W_i u_i^2}{g \sum_{i=1}^{N} f_i u_i}} \tag{21.86}$$

21.8 Improved Rayleigh's Method

The concept of applying inertial forces as static loads in determining the deformed shape for Rayleigh's Method may be used in developing an improved scheme for the method. In the application of the improved Rayleigh's Method, one would start from an assumed deformation curve followed by the calculation of the maximum values for the kinetic energy and for the potential energy of the system. An approximate value for natural frequency is calculated by equating maximum kinetic energy with the maximum potential energy. Then an improved value for the natural frequency may be obtained by loading the structure with the inertial loads associated with the assumed deflection. This load results in a new deformed shape which is used in calculating the maximum potential energy. The method is better explained with the aid of numerical examples.

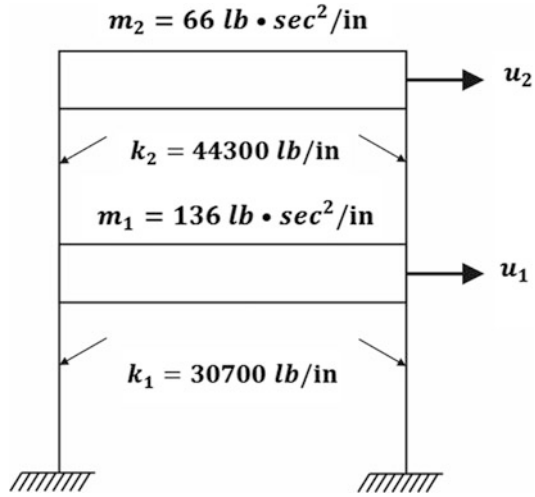

Fig. 21.16 Two-story frame for Example 21.7

Illustrative Example 21.7

By Rayleigh's Method, determine the natural frequency (lower or fundamental frequency) of the two-story frame shown in Fig. 21.16. Assume that the horizontal members are very rigid compared to the columns of the frame. This assumption reduces the system to only two degrees of freedom, indicated by coordinates u_1 and u_2 in the figure. The mass of the structure, which is lumped at the floor levels, has values $m_1 = 136$ lb·sec^2/in and $m_2 = 66$ lb·sec^2/in. The total stiffness of the first story is $k_1 = 30{,}700$ lb/in and of the second story $k_2 = 44{,}300$ lb/in, as indicated in Fig. 21.16.

Solution:

This structure may be modeled by the two mass systems shown in Fig. 21.17. In applying Rayleigh's Method, let us assume a deformed shape for which $u_1 = 1$ and $u_2 = 2$. The maximum potential energy is then

$$\begin{aligned}(PE)_{\max} &= \frac{1}{2}K_1 u_1^2 + \frac{1}{2}K_2(u_2 - u_1)^2 \\ &= \frac{1}{2}(30,700)(1)^2 + \frac{1}{2}(44,300)(1)^2 \\ &= 37,500 \text{ lb.in}\end{aligned} \tag{a}$$

and the maximum kinetic energy

$$\begin{aligned}T_{\max} &= \frac{1}{2}m_1(\omega u_1)^2 + \frac{1}{2}m_2(\omega u_2)^2 \\ &= \frac{1}{2}(136)\omega^2 + \frac{1}{2}(66)(2\omega)^2 \\ &= 200\omega^2\end{aligned} \tag{b}$$

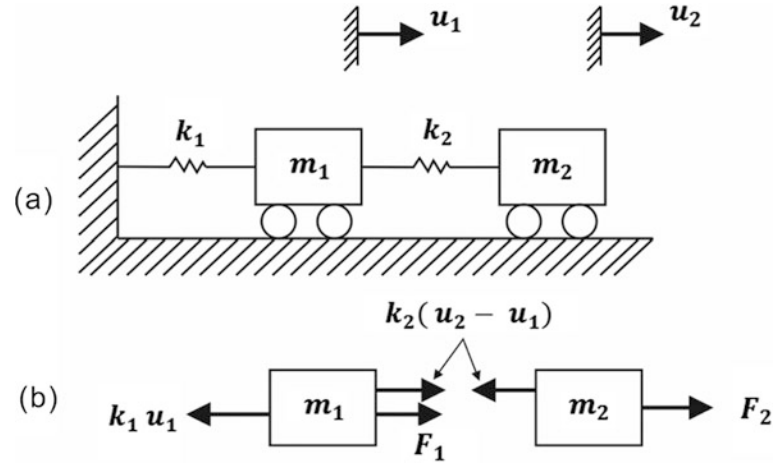

Fig. 21.17 Mathematical model for structure of Example 21.7

Equating maximum potential energy with maximum kinetic energy and solving for the natural frequency gives

$$\omega = 13.69 \text{ rad/sec}$$

or

$$f = \frac{\omega}{2\pi} = 2.18 \text{ cps}$$

The natural frequency calculated as $f = 2.18$ cps is only an approximation to the exact value, since the deformed shape was assumed for the purpose of applying Rayleigh's Method. To improve this calculated value of the natural frequency, let us load the mathematical model in Fig. 21.17a with the inertial load calculated as

$$F_1 = m_1\omega^2 u_1 = (136)(13.69)^2(1) = 25,489$$
$$F_2 = m_2\omega^2 u_2 = (66)(13.69)^2(2) = 24,739$$

The equilibrium equations obtained by equating to zero the sum of the forces in the free body diagram shown in Fig. 21.17b gives

$$30,700 \ u_1 - 44,300(u_2 - u_1) = 25,489$$

$$44,300(u_2 - u_1) = 24,739$$

and solving

$$u_1 = 1.64$$

$$u_2 = 2.19$$

or in the ratio

$$u_1 = 1.00$$

$$u_2 = 1.34 \tag{c}$$

Introducing these improved values for the displacements u_1 and u_2 into Eqs. (a) and (b) to recalculate the maximum potential energy and maximum kinetic energy results in

$$(PE)_{\max} = 25 \tag{d}$$

$$T_{\max} = 160.03\omega^2 \tag{e}$$

and upon equating $(PE)_{\max}$ and $T_{\max}$, we obtain

$$\omega = 12.57 \text{ rad/sec}$$

or

$$f = 2.00 \text{ cps}$$

This last calculated value for the natural frequency $f = 2.00$ cps could be further improved by applying a new inertial load in the system based on the last value of the natural 'frequency and repeating anew cycle of calculations. Table 21.3 shows results obtained for four cycles.

The exact natural frequency and deformed shape, which are calculated for this system in Chap. 7, Example 7.1, as a two-degrees-of-freedom system, checks with the values obtained in the last cycle of the calculations s shown in Table 21.3.

Table 21.3 Improved Rayleigh's Method Applied to Example 21.7

Cycle	Deformed Shape	Inertial Load		Natural Frequency
		F_1	F_2	
1	1:2.00			2.18 cps
2	1:1.34	25,489	24,739	2.00
3	1:1.32	21,489	18,725	1.88
4	1:1.27	19,091	12,230	1.88

21.9 Shear Walls

Horizontal forces in buildings, such as those produced by earthquake motion or wind action, are often resisted by structural elements called shear walls. These structural elements are generally designed as reinforced concrete walls fixed at the foundation. A single cantilever shear wall can be expected to behave as an ordinary flexural member if its length-to-depth ratio (L/D) is greater than about 2. For short shear walls ($L/D < 2$), the shear strength assumes preeminence and both flexural and shear deformations should be considered in the analysis.

When the floor system of a multistory building is rigid, the structure's weights or masses at each floor may be treated as concentrated loads, as shown in Fig. 21.18 for a three-story building. The response of the structure is then a function of these masses and of the stiffness of the shear wall. In practice a mathematical model is developed in which the mass as well as the stiffness of the structure are combined at each floor level. The fundamental frequency (lowest natural frequency) for such a structure can then be obtained using Rayleigh's Method, as shown in the following illustrative example.

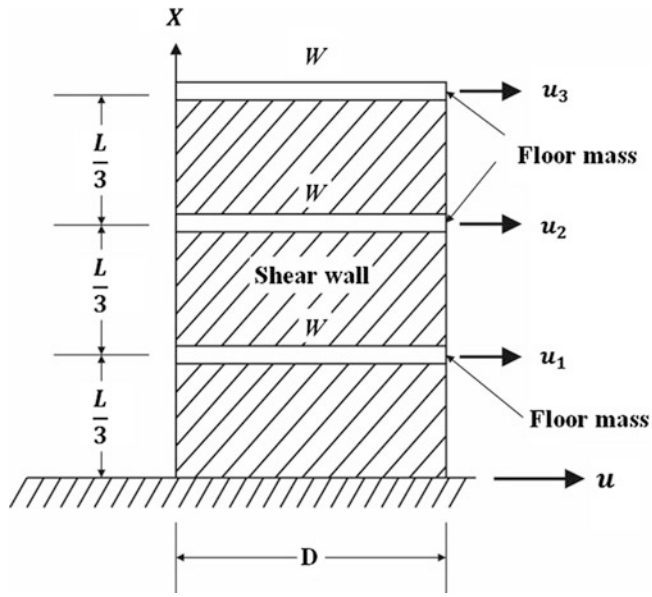

Fig. 21.18 Mathematical model for shear wall and rigid floors

Illustrative Example 21.8

Determine, using Rayleigh's Method, the natural period of the three-story building shown in Fig. 21.18. All the floors have equal weight W. Assume the mass of the wall negligible compared to the floor masses and consider only flexural deformations ($L/D > 2$).

Solution:

The natural frequency can be calculated using Eq. (21.83), which is repeated here for convenience:

$$\omega = \sqrt{\frac{g\sum_{i=1}^{N} W_i u_i}{\sum_{i=1}^{N} W_i u_i^2}}$$

The deformed shape equation is assumed as the deflection curve produced on a cantilever beam supporting three concentrated weights W, as shown in Fig. 21.19. The static deflections u_1, u_2, and u_3 calculated by using basic knowledge of strength of materials are

$$u_1 = \frac{15}{162}\frac{WL^3}{EI} = 0.0926\frac{WL^3}{EI}$$
$$u_2 = \frac{15}{162}\frac{WL^3}{EI} = 0.3025\frac{WL^3}{EI}$$
$$u_3 = \frac{92}{162}\frac{WL^3}{EI} = 0.5679\frac{WL^3}{EI}$$

The natural frequency by Eq. (a) is then calculated as

$$\omega = \sqrt{\frac{386(0.0926 + 0.3025 + 0.5679)}{(0.0926^2 + 0.3025^2 + 0.5679^2)} \cdot \frac{EI}{WL^3}} = 29.66\sqrt{\frac{EI}{WL^3}} \text{ rad/ sec}$$

or

$$f = \frac{\omega}{2\pi W} = 4.72\sqrt{\frac{EI}{WL^3}} \text{cps}$$

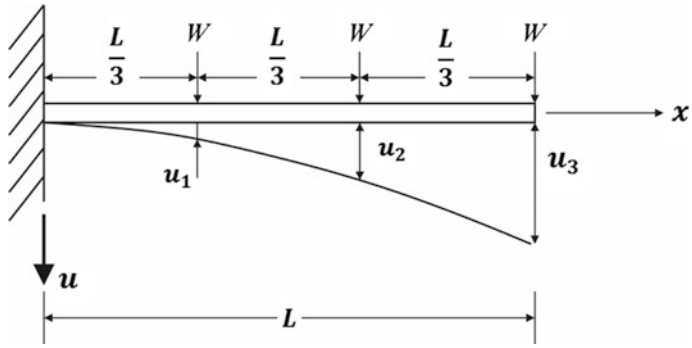

Fig. 21.19 Assumed deflection curve for Example 21.8

Illustrative Example 21.9

For the mathematical model of a three-story building shown in Fig. 21.18, determine the total deflections at the floor levels considering both flexural and shear deformations.

Solution:
The lateral deflection Δu_s, considering only shear deformation for a beam segment of length Δx, is given by

$$\Delta u_s = \frac{V \Delta x}{\alpha A G} \tag{a}$$

where
$V =$ shear force
$A =$ cross-sectional area
$\alpha =$ shear constant ($\alpha = 1.2$ for rectangular sections)
$G =$ shear modulus of elasticity
At the first story, $v = 3\ W$. Therefore, by Eq. (a) the shear deflection at the first story is

$$u_{s1} = \frac{3W(L/3)}{\alpha A G} = \frac{WL}{\alpha A G} \tag{b}$$

At the second floor the shear deflection is equal to the first floor deflection plus the relative deflection between floors, that is

$$u_{s2} = u_{s1} + \frac{2W(L/3)}{\alpha A G} = \frac{5WL}{3\alpha A G} \tag{c}$$

since the shear force of the second story is $V = 2\ W$, and at the third floor

$$u_{s3} = u_{s2} + \frac{W(L/3)}{\alpha A G} = \frac{6WL}{3\alpha A G} \tag{d}$$

The total deflection is then obtained by adding the flexural deflection determined in Example 21.8 to the above shear deflections. Hence,

$$\begin{aligned} u_1 &= \frac{15}{162}\frac{WL^3}{EI} + \frac{WL}{\alpha A G} \\ u_2 &= \frac{49}{162}\frac{WL^3}{EI} + \frac{5WL}{3\alpha A G} \\ u_3 &= \frac{92}{162}\frac{WL^3}{EI} + \frac{6WL}{3\alpha A G} \end{aligned} \tag{e}$$

We can see better the relative importance of the shear contribution to the total deflection by factoring the first terms in Eq. (e). Considering a rectangular wall for which $A = D \times t$, $E/G = 2.5$, $I = tD^3/12$, $\alpha = 1.2$ ($t =$ thickness of the wall), we obtain

$$\begin{aligned} u_1 &= \frac{15}{162}\frac{WL^3}{EI}\left[1 + 1.875\left(\frac{D}{L}\right)^2\right] \\ u_2 &= \frac{49}{162}\frac{WL^3}{EI}\left[1 + 0.957\left(\frac{D}{L}\right)^2\right] \\ u_3 &= \frac{92}{162}\frac{WL^3}{EI}\left[1 + 0.611\left(\frac{D}{L}\right)^2\right] \end{aligned} \tag{f}$$

Table 21.4 Calculation of the Natural Frequency for the Shear Wall Modeled s Shown in Fig. 21.18

D/L[a]	u_1[b](in)	u_2[b](in)	u_3[b](in)	ω[c](rad/sec)	f[c](cps)
0.00	0.09259	0.30247	0.56790	29.66	4.72
0.50	0.13600	0.37483	0.65465	27.67	4.40
1.00	0.26620	0.59193	0.91489	23.30	3.71
1.50	0.48322	0.95376	1.34862	19.05	3.03
2.00	0.78704	1.46032	1.95585	15.71	2.50
2.50	1.17765	2.11161	2.73658	13.21	2.10
3.00	1.65509	2.90764	3.69079	11.33	1.80

[a] $D/L = 0$ is equivalent, to neglect shear deformations
[b] Factor of WL^3/EI
[c] Factor of $\sqrt{EI/WL^3}$

The next illustrative example presents a table showing the relative importance that shear deformation has in calculating the natural frequency for a series of values of the ratio D/L.

Illustrative Example 21.10
For the structure modeled as shown in Fig. 21.18, study the relative importance of shear deformation in calculating the natural frequency. Solution: In this study we will consider, for the wall, a range of values from 0 to 3.0 for the ratio D/L (depth-to-length ratio). The deflections u_1, u_2, u_3 at the floor levels are given by Example 21.9 and the natural frequency by Eq. (21.83). The necessary calculations are conveniently shown in Table 21.4. It may be seen from the last column of Table 21.4, that for this example the natural frequency neglecting shear deformation ($D/L = 0$) is $f = 4.72\sqrt{EI/WL^3}$ cps. For short walls ($D/L > 0.5$) the effect of shear deformation becomes increasingly important.

21.10 Summary

The concept of generalized coordinate presented in this chapter permits the analysis of multiple interconnected rigid or elastic bodies with distributed properties as single-degree-of-freedom systems. The analysis as one-degree-of freedom systems can be made provided that by the specification of a single coordinate (the generalized coordinate) the configuration of the whole system is determined. Such a system may then be modeled as the simple oscillator with its various parameters of mass, stiffness, damping, and load, calculated to be dynamically equivalent to the actual system to be analyzed. The solution of this model provides the response in terms of the generalized coordinate.

The principle of virtual work which is applicable to systems in static or dynamic equilibrium is a powerful method for obtaining the equations of motion as an alternative to the direct application of Newton's law. The principle of virtual work states that for a system in equilibrium the summation of the work done by all its forces during any displacement compatible with the constraints of the system is equal to zero.

Rayleigh's Method for determining the natural frequency of a vibrating system is based on the principle of conservation of energy. In practice, it is applied by equating the maximum potential energy with the maximum kinetic energy of the system. To use Rayleigh's Method for the determining of the natural frequency of a discrete or a continuous system, it is necessary to assume a deformed shape. Often, this shape is selected as the one produced by gravitational loads acting in the direction of the expected displacements. This approach leads to the following formula for calculating the natural frequency:

$$\omega = \sqrt{\frac{g\sum_i W_i u_i}{\sum_i W_i u_i^2}} \qquad (21.83)\ \text{repeated}$$

where u_i is the deflection at coordinate i and W_i concentrated weight at this coordinate. Shear walls are structural walls designed to resist lateral forces in buildings. For short walls ($L/D \leq 2$) shear deformations are important and should be considered in the analysis in addition to the flexural deformations.

21.11 Problems

Problem 21.1

For the system shown in Fig. P21.1 determine the generalized mass M^*, damping C^*, stiffness K^*, and the generalized load $F^*(t)$. Select $U(t)$ as the generalized coordinate.

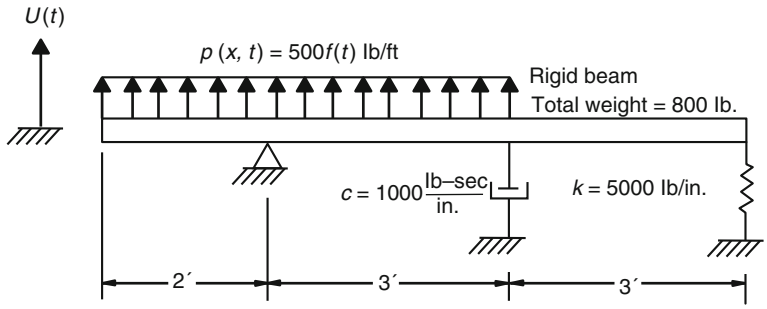

Fig. P21.1

Problem 21.2

Determine the generalized quantities M^*, C^*, K^*, and $F^*(t)$ for the structure shown in Fig. P21.2. Select $U(t)$ as the generalized coordinate.

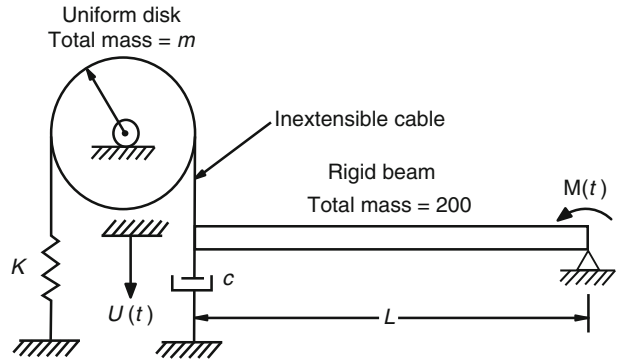

Fig. P21.2

Problem 21.3

Determine the generalized quantities M^*, C^*, K^*, and $F^*(t)$ for the structure shown in Fig. P21.3. Select $\theta(t)$ as the generalized coordinate.

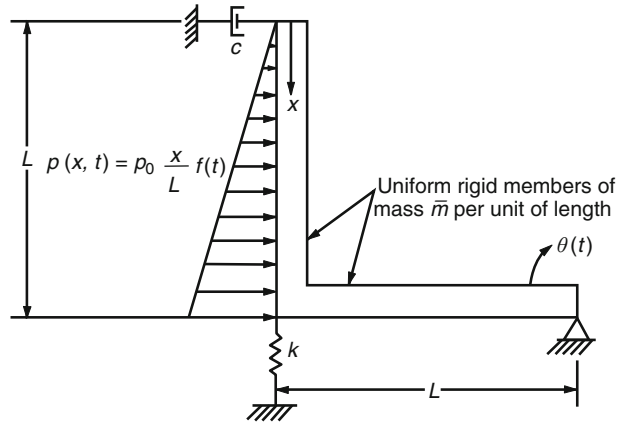

Fig. P21.3

Problem 21.4

For the elastic cantilever beam shown in Fig. P21.4, determine the generalized quantities M^*, K^*, and $F^*(t)$. Neglect damping. Assume that the deflected shape is given by $\phi(x) = 1 - \cos(\pi x/2L)$ and select $U(t)$ as the generalized coordinate as shown in Fig. P21.4. The beam is excited by a concentrated force $F(t) = F_0 f(t)$ at midspan.

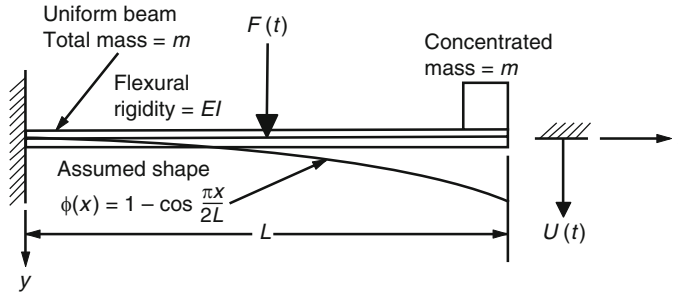

Fig. P21.4

Problem 21.5

Determine the generalized geometric stiffness K^*_G for the system in Fig. P21.4 if an axial tensile force N is applied at the free end of the beam along the x direction. What is the combined generalized stiffness K^*_c?

Problem 21.6

A concrete conical post of diameter d at the base and height L is shown in Fig. P21.6. It is assumed that the wind produces a dynamic pressure $p_0(t)$ per unit of projected area along a vertical plane. Determine the generalized quantities M^*, K^*, and $F^*(t)$ (Take modulus of elasticity $E_c = 3 \times 10^6$ psi; specific weight $\gamma = 150$ lb./ft^3 for concrete.)

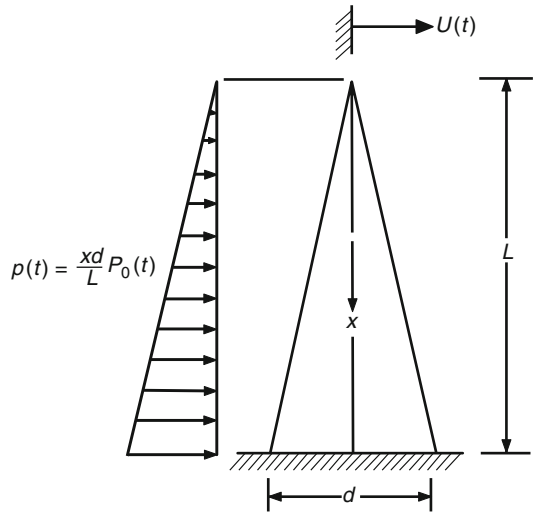

Fig. P21.6

Problem 21.7

A simply supported beam of total uniformly distributed mass m_b, flexural rigidity EI, and length L, carries a concentrated mass m at its center. Assume the deflection curve to be the deflection curve due to a concentrated force at the center of the beam and determine the natural frequency using Rayleigh's Method.

Problem 21.8

Determine the natural frequency of a simply supported beam with overhang which has a total uniformly distributed mass m_b, flexural rigidity EI, and dimensions shown in Fig. P21.8. Assume that during vibration the beam deflected curve is of the shape produced by a concentrated force applied at the free end of the beam.

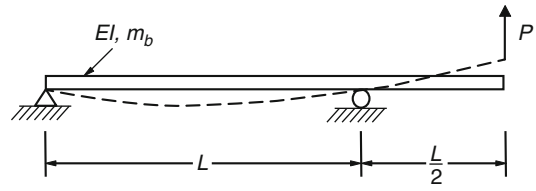

Fig. P21.8

Problem 21.9

Determine the natural frequency of the simply supported beam shown in Fig. P21.9 using Rayleigh's Method. Assume the deflection curve given by $\phi\ (x) = U\sin \pi x/L$. The total mass of the beam is $m_b = 10$ lb. sec^2/in, flexural rigidity $EI = 10^8$ lb.·in^2, and length $L = 100$ in. The beam carries a concentrated mass at the center $m = 5$ lb.·sec^2/in.

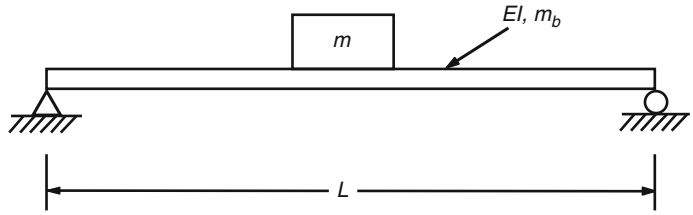

Fig. P21.9

Problem 21.10

A two-story building is modeled as the frame shown in Fig. P21.10. Use Rayleigh's Method to determine the natural frequency of vibration for the case in which only flexural deformation needs to be considered. Neglect the mass of the columns and assume rigid floors. (Hint: Use Eq. (21.83)).

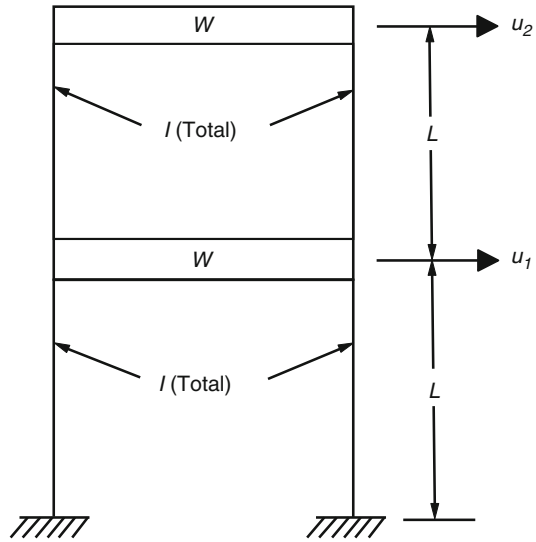

Fig. P21.10

Problem 21.11

Solve Problem 21.10 for the case in which the columns are short and only shear deformation needs to be considered. (The lateral force V for a fixed column of length L, cross-sectional area A, is approximately given by $V = AG\Delta/L$, where G is the shear modulus of elasticity and Δ the lateral deflection.)

Problem 21.12

Calculate the natural frequency of the shear wall carrying concentrated masses at the floor levels of a three-story building as shown in Fig. P21.12. Assume that the deflection shape of the shear wall is that resulting from a concentrated lateral force applied at its tip. Take flexural rigidity, $EI = 3.0 \times 10^{11}$ lb.·in^2; length $L = 36$ ft.; concentrated masses, $m = 100$ lb.·sec^2/in and mass per unit of length along the wall, $\bar{m} = 10\ \text{lb} \cdot \text{sec}^2/\text{in}^2$.

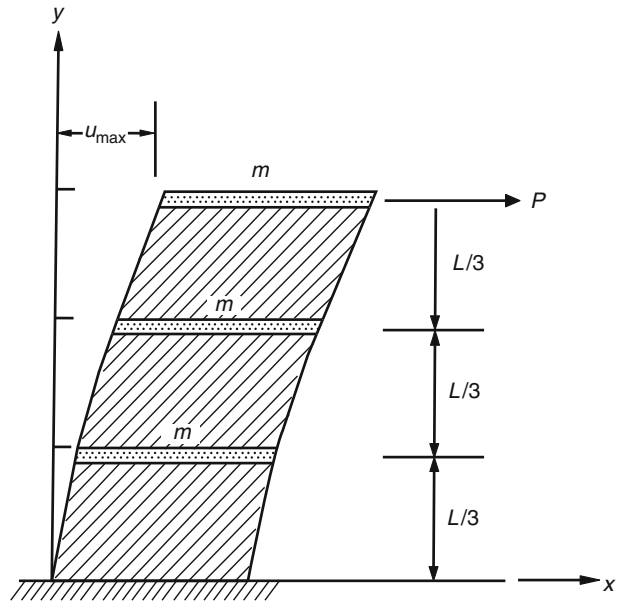

Fig. P21.12

Problem 21.13

Solve Problem 21.12 on the assumption that the deflection shape of the shear wall is that resulting from a lateral uniform load applied along its length.

Part VI

Random Vibration

Random Vibration

22

The previous chapters of this book have dealt with the dynamic analysis of structures subjected to excitations which were known as a function of time. Such an analysis is said to be deterministic. When an excitation function applied to a structure has an irregular shape that is described indirectly by statistical means, we speak of a random vibration. Such a function is usually described as a continuous or discrete function of the exciting frequencies, in a manner similar to the description of a function by Fourier series. In structural dynamics, the random excitations most often encountered are either motion transmitted through the foundation or acoustic pressure. Both of these types of loading are usually generated by explosions occurring in the vicinity of the structure. Common sources of these explosions are construction work and mining. Other types of loading, such as earthquake excitation, may also be considered a random function of time. In these cases the structural response is obtained in probabilistic terms using random vibration theory.

A record of random vibration is a time function such as shown in Fig. 22.1. The main characteristic of such a random function is that its instantaneous value cannot be predicted in a deterministic sense. The description and analysis or random processes are established in a probabilistic sense for which it is necessary to use tools provided by the theory of statistics.

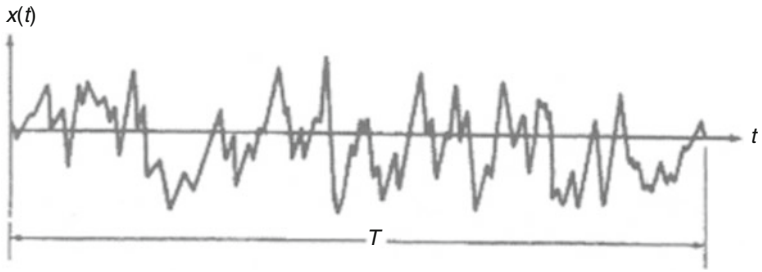

Fig. 22.1 Record of a random function of time

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M. Paz, Y. H. Kim, *Structural Dynamics*, https://doi.org/10.1007/978-3-319-94743-3_22

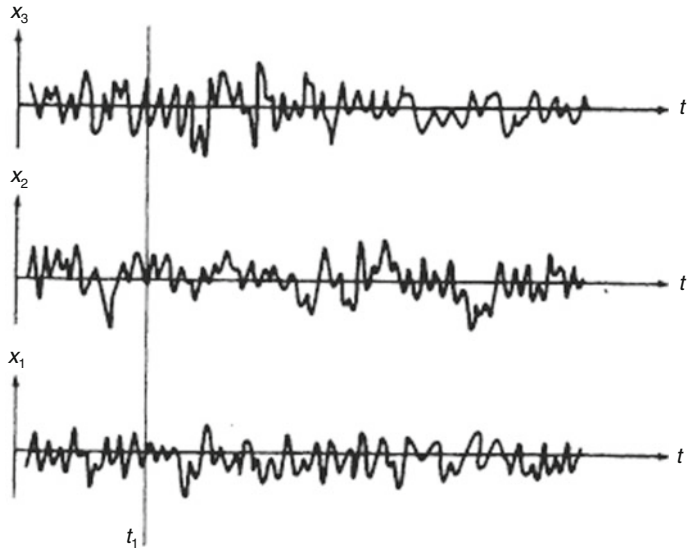

Fig. 22.2 An ensemble of random functions of time

22.1 Statistical Description of Random Functions

In any statistical method a large number of responses is needed to describe a random function. For example, to establish the statistics of the foundation excitation due to explosions in the vicinity of a structure, many records of the type shown in Fig. 22.2 may be needed. Each record is called a sample, and the total collection of samples an ensemble. To describe an ensemble statistically, we can compute at any time t_i the average value of the instantaneous displacements x_i. If such averages do not differ as we select different values of t_i, then the random process is said to be stationary. In addition, if the average obtained with respect to time for any member of the ensemble is equal to the average across the ensemble at an arbitrary time t_i, the random process is called ergodic. Thus in a stationary, ergodic process, a single record may be used to obtain the statistical description of a random function. We shall assume that all random process considered are stationary and ergodic. The random function of time shown in Fig.22.1 has been recorded during an interval of time T. Several averages are useful in describing such a random function. The most common are the mean value $\bar{x}$ which is defined as

$$\bar{x} = \frac{1}{T}\int_0^T x(t)dt \tag{22.1}$$

and the mean-square value $\overline{x^2}$ defined as

$$\overline{x^2} = \frac{1}{T}\int_0^T x^2(t)dt \tag{22.2}$$

Both the mean and the mean-square values provide measurements for the average value of the random function $x(t)$. The measure of how widely the function $x(t)$ differs from the average is given by its variance, σ_x^2, defined as

$$\sigma_x^2 = \frac{1}{T}\int_0^T [x(t) - \bar{x}]^2 dt \tag{22.3}$$

When the expression under the integral is expanded and then integrated, we find that

$$\sigma_x^2 = \overline{x^2} - \left(\bar{x}\right)^2 \tag{22.4}$$

which means that the variance can be calculated as the mean-square minus the square of the mean. Quite often the mean value is zero, in which case variance is equal to the mean square value. The root mean-square RMS_x of the random function $x(t)$ is defined as

$$RMS_s = \sqrt{\overline{x^2}} \tag{22.5}$$

The standard deviation σ_x of the random function $x(t)$ is the square root of the variance; hence from Eq. (22.4)

$$\sigma_x = \sqrt{\overline{x^2} - \left(\bar{x}\right)^2} \tag{22.6}$$

Illustrative Example 22.1

Determine the mean value $\bar{F}$, the mean-square value $\overline{F^2}$, the variance $\sigma^2{}_F$ and the root mean square values RMS_F of the forcing function $F(t)$ shown in Fig. 22.3.

Solution; Since the force $F(t)$ is periodic with period T, we can take the duration of the force equal to T; hence by Eq. (22.1) we have

$$\bar{F} = \frac{1}{T}\int_0^t F(t)dt = \frac{F_{\max}}{2}$$

and noting that

$$F(t) = \frac{2F_{\max}}{T}t \quad \text{for} \quad 0 < t < \frac{T}{2}$$

we obtain by Eq. (22.2)

$$\overline{F^2} = \frac{2}{T}\int_0^T \left(\frac{2F_{\max}}{T}\right)^2 t^2 dt = \frac{F_{\max}^2}{3}$$

The variance may now be calculated from Eq. (22.4) as

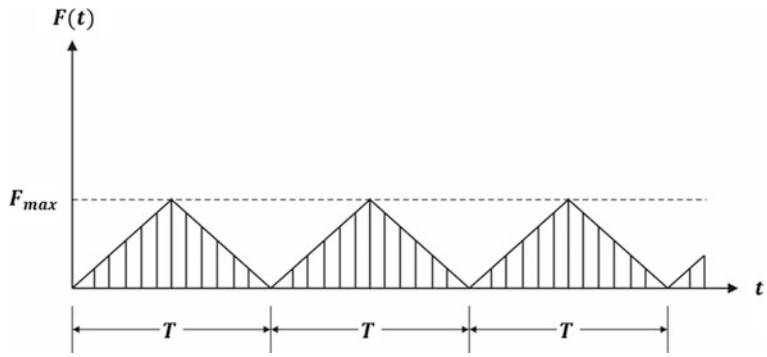

Fig. 22.3 Forcing function for Illustrative Example 22.1

$$\sigma_F^2 = \overline{F^2} - \left(\bar{F}\right)^2$$
$$\sigma_F^2 = \frac{F_{\max}^2}{3} - \frac{F_{\max}^2}{4} = \frac{F_{\max}^2}{12}$$

Finally the root mean square value of $F(t)$ is

$$RMS_F = \sqrt{\overline{F^2}} = \frac{F_{\max}}{\sqrt{3}}$$

22.2 Probability Density Function

Figure 22.4 shows a portion of a record of a random function $x(t)$. If we wish to determine the probability of x having a value in the range (x_1, x_2), we may draw horizontal lines through the values x_1 and x_2, and then measure the corresponding time intervals Δt_1. The ratio given by

$$P(x_1 \leq x \leq x_2) = \frac{\Delta t_1 + \Delta t_2 + \ldots + \Delta t_n}{T} \tag{22.7}$$

and calculated for the entire record length T, is the probability of x having a value between x_1 and x_2 at any selected time t_i during the random process.

Similarly, the probability of $x(t)$ being smaller than a value x can be expressed as

$$P(x) = P[x(t) < x] = \lim_{T-x} \frac{1}{T} \sum_i \Delta t_i \tag{22.8}$$

where the time intervals Δt_i are now those for which the function $x(t)$ has a value smaller than the specified x.

The function $P(x)$ in Eq. (22.8) is known as the cumulative distribution function of the random function $x(t)$. This function is plotted in Fig. 22.5a as a function of x. The cumulative distribution function is a monotonically increasing function for which

$$P(-\infty) = 0, \quad 0 \leq P(x) \leq 1, \quad P(\infty) = 1$$

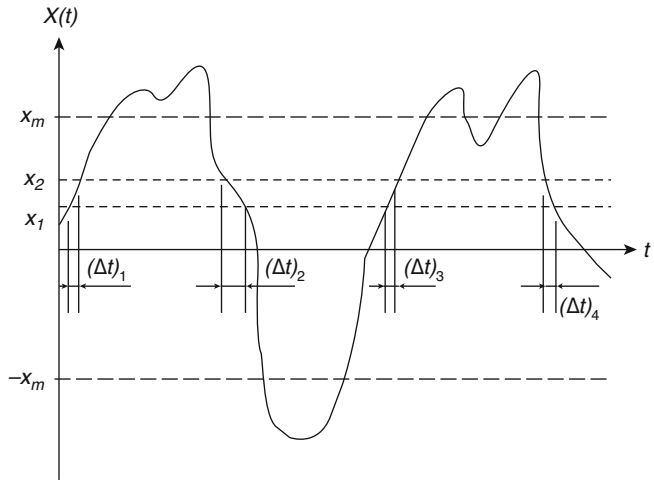

Fig. 22.4 Portion of random record showing determination of probabilities

Now, the probability that the value of the random variable is smaller than the value $x + \Delta x$ is denoted by $P(x + \Delta x)$ and that $x(t)$ takes a value between x and $x + \Delta x$ is $P(x + \Delta x) - P(x)$. This allows us to define the probability density function as

$$p(x) = \lim_{\Delta x \to 0} \frac{P(x - \Delta x) - P(x)}{\Delta x} = \frac{dP(x)}{dx} \tag{22.9}$$

Thus, the probability density function $p(x)$ is represented geometrically by the slope of the cumulative probability function $P(x)$. The functions $P(x)$ and $p(x)$ are shown in Figs. 22.5a, b, respectively. From Eq. (22.9) we conclude that the probability that a random variable $x(t)$ has a value between x and $x + dx$ is given by $p(x)\, dx$, where $p(x)$ is the probability density junction. Having prescribed $p(x)$, for example, as the function plotted in Fig. 22.5b, the probability of x being in the range (x_1, x_2) at any selected time is given by

$$P(x_1 \leq x \leq x_2) = \int_{x1}^{x_2} p(x)dx \tag{22.10}$$

and is equal to the shaded area shown between x_1 and x_2 in Fig. 22.5b. Similarly, the probability of x being greater than x_m, that is, $P\ (|x| > x_m)$ can be represented as the two shaded "tail" areas in Fig. 22.5b. Since every real x lies in the interval $(-\infty,\infty)$, the area under the entire probability density function is equal to 1, that is,

$$\int_{-\infty}^{\infty} p(x)dx = 1 \tag{22.11}$$

Thus as x tends to infinity in either direction, $p(x)$ must asymptotically diminish to zero.

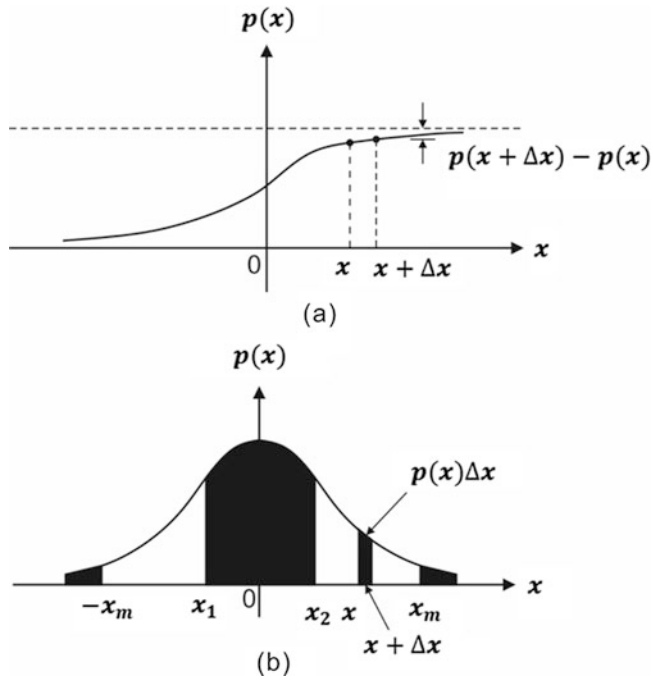

Fig. 22.5 (**a**) Cumulative probability function $P(x)$ and (**b**) Probability density function $p(x)$ of the random variable $x = x(t)$

22.3 The Normal Distribution

The most commonly used probability density function is the normal distribution, also referred to as the Gaussian distribution, expressed by

$$p(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(x-\bar{x}\right)^2/\sigma^2} \tag{22.12}$$

Figure 22.6 shows the shape of this function. It may be observed that the normal distribution function is symmetric about the mean value $\bar{x}$. In Fig. 22.7 the standard normal distribution is plotted non-dimensionally in terms of $(x-\bar{x})/\sigma$. Values of $P(-\infty, x_2)$ [$x_1 = -\infty$ in Eq. (22.10)] are tabulated in many sources including mathematical handbook.[1] The probability of x being between $x - \lambda\sigma$ and $x + \lambda\sigma$, where λ is any positive number, is given by the equation.

$$P\left[\bar{x} - \lambda\sigma < x < \bar{x} + \lambda\sigma\right] = \frac{1}{\sqrt{2\pi}\sigma} \int_{\bar{x}-\lambda\sigma}^{\bar{x}+\lambda\sigma} e^{-\frac{1}{2}\left(x-\bar{x}\right)^{2/\sigma^2}} dx \tag{22.13}$$

Equation (22.13) represents the probability that x lies within λ standard deviations from $\bar{x}$. The probability of x lying more than λ standard deviations from $\bar{x}$ is the probability of $|x-\bar{x}|$ exceeding $\lambda\sigma$, which is 1.0 minus the value given by Eq. (22.13). The following table presents numerical values for the normal distribution associated with $\lambda = 1$, 2, and 3:

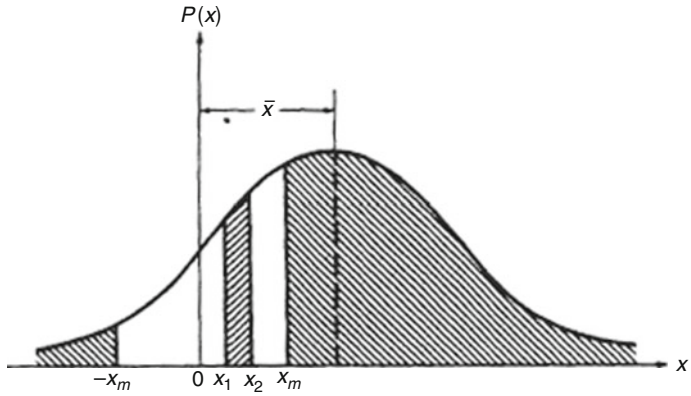

Fig. 22.6 Normal probability density function

λ	$P[\bar{x} - \lambda\,\sigma < x < \bar{x} + \lambda\,\sigma]$	$P[x - \bar{x}] > \lambda\,\sigma]$
1	68.3%	31.7%
2	95.4%	4.6%
3	99.7%	0.3%

[1] *Standard mathematical Tables,* The Chemical Rubber Co. (CRC) 20th Ed. 1972, pp. 566–575.

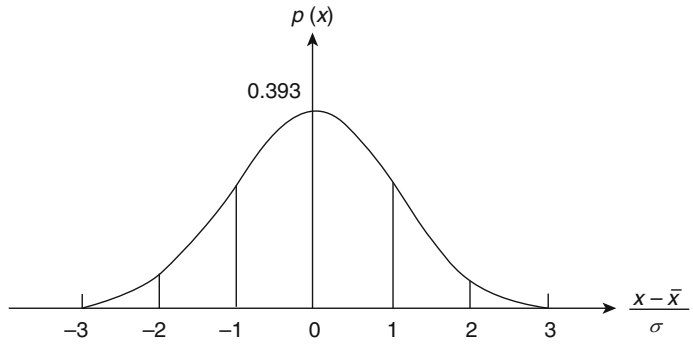

Fig. 22.7 Standard normal probability density function

22.4 The Rayleigh Distribution

Variables that are positive, such as the absolute value A of the peaks of vibration of a random function $x(t)$, often tend to follow the Rayleigh distribution, which is defined by the equation

$$p(A) = \frac{A}{\sigma^2} e^{-A^2/2\sigma^2}, \quad A > 0 \tag{22.14}$$

where σ is a parameter that may be interpreted as the standard deviation of a function $x(t)$.

The probability density $p(A)$ is zero for $A < 0$ and has the shape shown in Fig. 22.8 for positive values A.

The mean and mean square values for the Rayleigh distribution function are given by Eqs. (22.1) and (22.2)as

$$\bar{A} = \int_0^{\chi} Ap(A)dA = \int_0^{\chi} \frac{A^2}{\sigma^2} e^{-A^2/2\sigma^2} dA = \sqrt{\frac{\pi}{2}}\sigma$$

$$\overline{A^2} = \int_0^{\chi} A^2 p(A)dA = \int_0^{\chi} \frac{A^3}{\sigma^2} e^{-A^2/2\sigma^2} dA = 2\sigma^2$$

Therefore, the Root Mean Square value of the random variable A (RMS_A) is

$$RMS_A = \sigma\sqrt{2} \tag{22.15}$$

The variance associated with the Rayleigh distribution function is, by Eq. (22.4),

$$\sigma_A^2 = \overline{A^2} - \left(\bar{A}\right)^2 = \frac{4-\pi}{2}\sigma^2 = 0.429\sigma^2 \tag{22.16}$$

The probability of A exceeding a specified value $\lambda\sigma$, $P\ (A > \lambda\sigma)$ is defined as

$$P(A > \lambda\sigma) = \int_{\lambda\sigma}^{\infty} p(A)dA$$

Which after substituting $p(A)$ from Eq. (22.14) result in

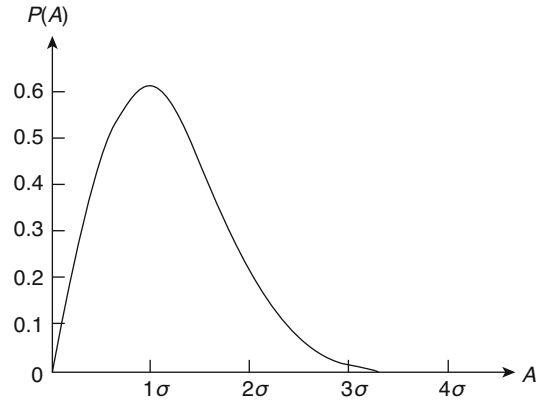

Fig. 22.8 Rayleigh probability density function

$$P(A > \lambda\sigma) = \int_{\lambda\sigma}^{\infty} \frac{A}{\sigma^2} e^{-A^2/2\sigma^2} dA$$

which may be written as

$$P(A > \lambda\sigma) = -\int_{\lambda\sigma}^{\infty} e^{\left(-A^2/2\sigma^2\right)} d\left(-A^2/2\sigma^2\right)$$

Thus,

$$P(A > \lambda\sigma) = e^{-A^2/2} \tag{22.17}$$

Also, the probability of A exceeding a specified value $\lambda\sigma$, $P\ (A > \lambda\sigma)$ is given by

$$P(A > \lambda\sigma_A) = \int_{\lambda\sigma_A}^{\infty} \frac{A}{\sigma^2} e^{-A^2/2\sigma^2} dA$$

Substituting from Eq. (22.16) $\sigma^2 = \sigma^2{}_A/0.429$ and integrating results in

$$P(A > \lambda\sigma_A) = e^{-0.429A^2/2} \tag{22.18}$$

The following table presents values for the probability of A, calculated from Eqs. (22.17) and (22.18), of exceeding $\lambda\sigma$, or $\lambda\sigma_A$ for $\lambda = 1,2,3$, and 4.

λ	$P[A > \lambda\sigma]\%$	$P[A > \lambda\sigma_A]\%$
1	60.65	80.69
2	13.53	42.40
3	1.11	14.51
4	0.03	3.23

22.5 Correlation

Correlation is a measure of the dependence between two random processes. Consider the two records shown in Fig. 22.9. The *correlation* between them is calculated by multiplying the coordinates of these two records at each time t_i and computing the average over all values of t. It is evident that the

correlation so found will be larger when the two records are similar. For dissimilar records with mean zero, some products will be positive and others negative. Hence their average product will approach zero.

We consider now the two records shown in Fig. 22.10 in which $x_1(t)$ is identical to $x(t)$ but shifted to the left in the amount τ, that is, $x_1(t) = x(t + \tau)$. The correlation between $x(t)$ and $x_1(t) = x(t + T)$ is known as the autocorrelation $R(\tau)$ and is given by

$$R(\tau) \lim_{T\to\infty} \frac{1}{T} \int_0^T x(t)x(t+\tau)dt \tag{22.19}$$

When $\tau = 0$, the autocorrelation reduces to the mean square value, that is,

$$R(0) = \lim_{T\to\infty} \frac{1}{T} \int_0^T [x(t)]^2 dt = \overline{x^2} \tag{22.20}$$

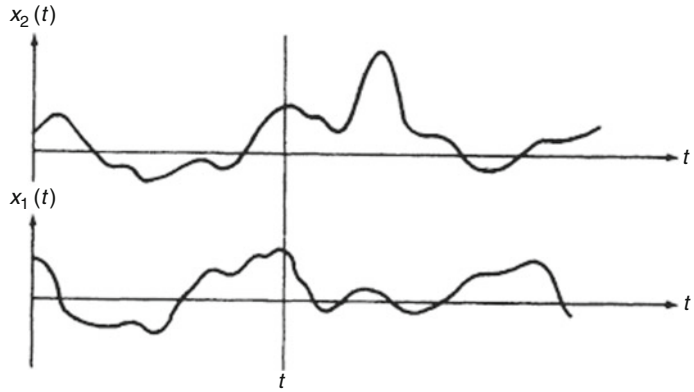

Fig. 22.9 Correlation between x_1 (t) and x_2 (t)

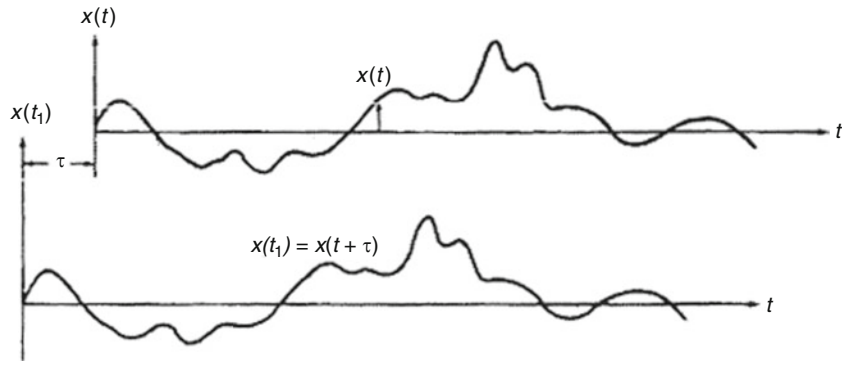

Fig. 22.10 Autocorrelation between x (t) and x $(t + \tau)$

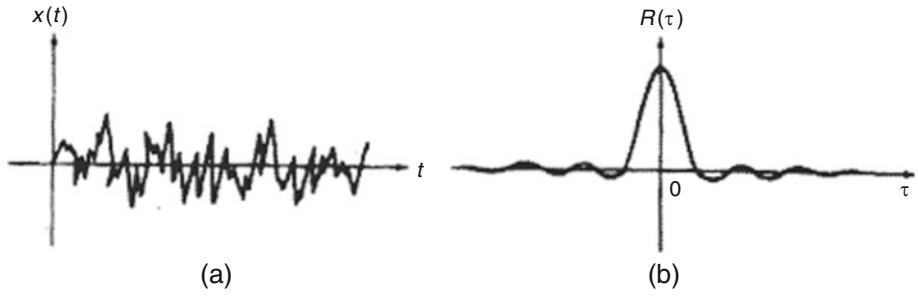

Fig. 22.11 Wide-band random process $x\ (t)$ and its autocorrelation function $R(\tau)$

Since the second record of Fig. 22.10 can be considered to be delayed with respect to the first record, or the first record advanced with respect to the second record, it is evident that $R(\tau) = \mathrm{R}(-\tau)$ is symmetric about the R axis and that $\mathrm{R}(\tau)$ is always less than R(0).

Highly random or wide-band functions such as the one shown in Fig. 22.11a lose their similarity within a short time shift. The autocorrelation of such a function, therefore, is a sharp spike at $\tau = 0$ that drops off rapidly as τ moves away from zero, as shown in Fig. 22.11b. For the narrow-band record containing a dominant frequency as shown in Fig. 22.12a, the autocorrelation has the characteristics indicated in Fig. 22.12b in that it is a symmetric function with a maximum at $\tau = 0$ and frequency ω_0 corresponding to the dominant frequency of $x(t)$.

Illustrative Example 22.2

For the function depicted in Fig. 22.13, determine: (a) the mean, (b) the mean square value, and (c) the autocorrelation function. Also, plot the autocorrelation function.

Solution: because the function $x(t)$ is periodic, averages calculated over a long time approach those calculated over a single period. Considering the period between 0 and T, the function is described analytically by

$$\begin{aligned} x(t) &= x_0 \qquad 0 < x < T/2 \\ &= 0 \qquad T/2 < x < T \end{aligned} \tag{a}$$

(a) Mean value: by Eq. (22.1)

$$\bar{x} = \frac{1}{T}\int_0^{\frac{T}{2}} x_0 dt = \frac{x_0}{2} \tag{b}$$

(b) Mean square value: by Eq. (22.2)

$$\overline{x^2} = \frac{1}{T}\int_0^{\frac{T}{2}} x_0^2 dt = \frac{x_0^2}{2} \tag{c}$$

(c) Autocorrelation function:

To calculate the autocorrelation function, it is necessary to distinguish between the time shift

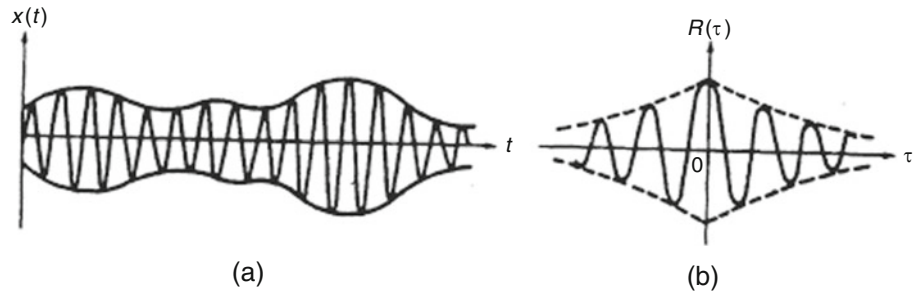

Fig. 22.12 Narrow-band random process $x(t)$ and its autocorrelation function $R(T)$

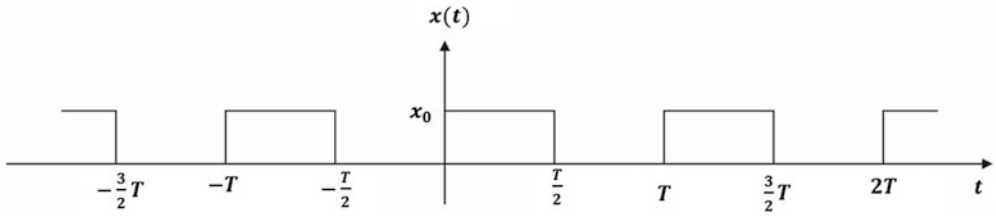

Fig. 22.13 Periodic function $x(t)$ of Illustrative Example 22.2.

$0 < \tau < T/2$ and $T/2 < \tau < T$ as shown, respectively, in Fig. 22.14a, b.

(d) Using Eq. (22.19) and considering Fig. 22.14a, we obtain for $0 < \tau < T/2$

$$R(\tau) = \frac{1}{T}\int_0^T x(t)\cdot x(t+\tau)dt = \int_0^{\frac{T}{2}-\tau} x_0 \cdot x_0 dt$$
$$= \frac{x_0^2}{T}\left(\frac{T}{2}-\tau\right) = x_0^2\left(\frac{1}{2}-\frac{\tau}{T}\right) \quad \text{for} \quad 0 > \tau > \frac{T}{2} \tag{d}$$

in which the limits of integration are given by the overlapping portions of $x(t)$ and $x(t+T)$ as indicated by the shaded area in Fig.22.14a.

When $T/2 < \tau < T$, we obtain from Fig. 22.14b.

$$R(\tau) = \frac{1}{T}\int_{T-\tau}^{\frac{T}{2}} x_0 . x_0 dt = \frac{x_0^2}{T}\left[\frac{T}{2} - T + \tau\right]$$
$$= x_0^2\left(-\frac{1}{2}+\frac{\tau}{T}\right) \quad \text{for} \quad \frac{T}{2} < \tau < T \tag{e}$$

From examination of Fig. 22.14, it can be concluded that the autocorrelation function $R(\tau)$ for the function $x(t)$ must be periodic in t with period T. Therefore, from Eq. (d), (e), and the fact that $R(\tau)$ is periodic, we can plot the autocorrelation function as shown in Fig. 22.15.

22.6 The Fourier Transform

In Chap. 19, we used Fourier series to obtain the frequency components of periodic functions of time.

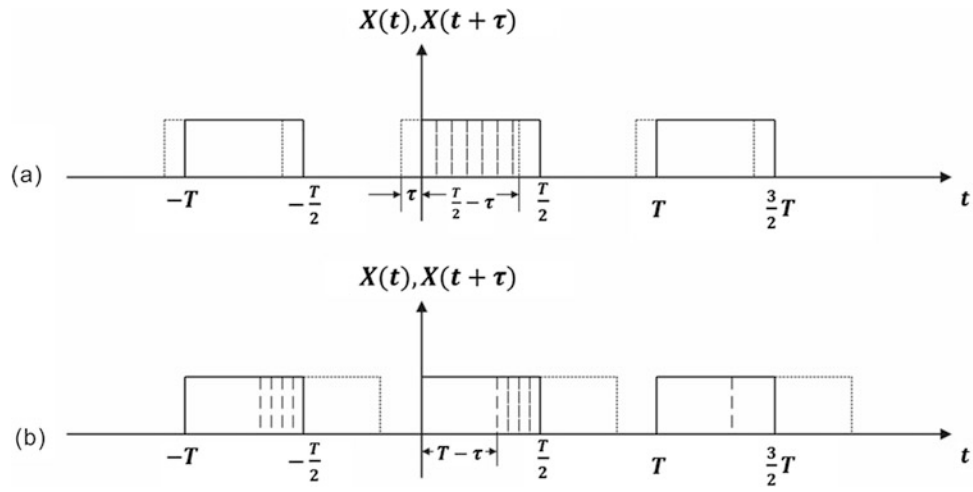

Fig. 22.14 Plot of $x(t)$ and $x(t+\tau)$; (**a**) for $0 < \tau < T/2$. (**b**) for $T/2 < \tau < T$

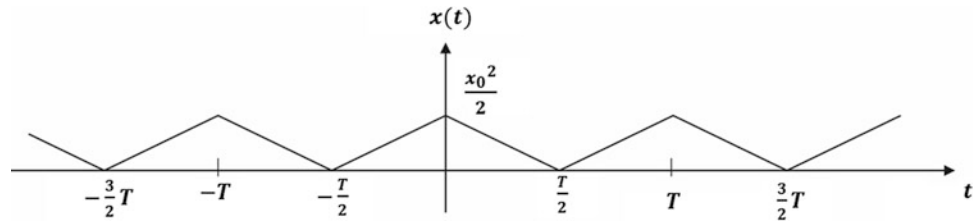

Fig. 22.15 Autocorrelation function $R(\tau)$ for the function of Illustrative Example 22.2

In general, random vibrations are not periodic, and the frequency analysis requires the extension of Fourier series to the Fourier integral for non-periodic functions. Fourier transforms, which result from Fourier integrals, enable a more extensive treatment of the random vibration problem.

We begin by showing that the Fourier integral can be viewed as a limiting case of the Fourier series as the period goes to infinity. Toward this objective we consider the Fourier series in exponential form and substitute the coefficient C_n given by Eq. (19.20) into Eq. (19.19) to obtain:

$$F(t) = \sum_{n=-\chi}^{\chi} \frac{1}{T} \int_{-T/2}^{T/2} F(\tau) \cdot e^{-in\varpi\tau} \cdot e^{in\varpi t} d\tau \tag{22.21}$$

In Eq. (22.21) we have selected the integration period from $-T/2$ to $T/2$ and substituted the symbol τ for t as the dummy variable of integration. The frequency $\omega = n\varpi$ is specified here at discrete, equally spaced values separated by the increment

$$\Delta\omega = (n+1)\varpi - n\varpi = \varpi = \frac{2\pi}{T}$$

We substitute in Eq. (22.21) ω for $n\varpi$ and $\Delta\omega/2\pi$ for $1/T$) and notice that as $T \to \infty$, $\Delta\omega \to d\omega$. Thus, in the limit, Eq. (22.21) becomes

$$\mathrm{F(t)} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty} F(\tau)e^{-i\omega\tau}d\tau\right\}e^{i\omega t}d\omega \tag{22.22}$$

which is the Fourier integral of $F(t)$.

Since the function within the inner braces is a function of only ω, we can write this equation in two parts as

$$C(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(t)e^{-i\omega t}dt \tag{22.23}$$

and

$$F(t) = \int_{-\infty}^{\infty} C(\omega)e^{i\omega t}d\omega \tag{22.24}$$

The validity of these relations, according to classical Fourier transform theory, is subject to the condition that

$$\int_{-\infty}^{\infty} |F(t)|dt < \infty \tag{22.25}$$

The function $C(\omega)$ in Eq. (22.23) is the Fourier transform of $F(t)$ and the function $F(t)$ in Eq. (22.24) is the inverse Fourier transform of $C(\omega)$. The pair of functions $F(t)$ and $C(\omega)$ is referred to as a Fourier transform pair. Equation (22.23) resolves the function $F(t)$ into harmonic components $C(\omega)$, whereas Eq. (22.24) synthesizes $F(t)$ from these harmonic components. In practice it is more convenient to use frequency f in cps rather than the angular frequency ω in rad/sec. Mathematically, since $\omega = 2\pi f$ and $d\omega = 2\pi df$, this also has the advantage of reducing the Fourier transform pair into a more symmetric form, namely,

$$F(t) = \int_{-\infty}^{\infty} C(f)e^{i2\pi ft}df \tag{22.26}$$

and

$$C(f) = \int_{-\infty}^{\infty} F(t)e^{i2\pi ft}dt \tag{22.27}$$

22.7 Spectral Analysis

We have seen in Chap. 19 that the application of Fourier analysis to a periodic function yields the frequency components of the function given by either trigonometric terms [Eq. (19.2)] or exponential terms [Eq. (19.19)]. When the periodic function is known at N discrete, equally spaced times, the frequency components are then given by the terms in Eq. (19.28). Our purpose in this section is to relate the Fourier analysis for a given function $x(t)$ to its mean square value $\overline{x^2}$.

The contributions of the frequency components of $x(t)$ to the value $\overline{x^2}$ are referred to as the spectral function of $x(t)$. Hence spectral analysis consists in expressing $\overline{x^2}$ in terms of either the coefficients of the Fourier series (i.e., a_n and b_n. or equivalently C_n) when $x(t)$ is periodic, or in terms of the Fourier transform [i.e., $C(\omega)$] when $x\ (t)$ is not a periodic function.

We begin by performing the spectral analysis of a periodic function $x(t)$ expressed in Fourier series, Eq. (19.2), which is

$$x(t) = a_0 + \sum_{n-1}^{\infty} \{a_n \cos n\,\varpi t + b_n \sin n\,\varpi t\} \tag{22.28}$$

where the coefficients are given by Eq. (19.3) as

$$\begin{aligned} a_0 &= \frac{1}{T}\int_0^T x(t)dt \\ a_n &= \frac{2}{T}\int_0^T x(t)\cos n\,\varpi dt \\ b_n &= \frac{2}{T}\int_0^T x(t)\sin n\,\varpi dt \end{aligned} \tag{22.29}$$

In Eq. (22.29), T is the period of the function and $\varpi = 2\pi/T$ its frequency. The substitution of $x(t)$ from Eq. (22.28) for one of the factors in the definition of mean square value gives

$$\begin{aligned} \overline{x^2} &= \frac{1}{T}\int_0^T x^2(t)dt \\ &= \frac{1}{T}\int_0^T x(t)\left\{a_0 + \sum_{n=1}^{\infty}(a_n \cos n\varpi t + b_n \sin n\varpi t)\right\}dt \\ &= \frac{a_0}{T}\int_0^T x(t)dt + \sum_{n=1}^{\infty}\left[\frac{a_n}{T}\int_0^T x(t)\cos n\varpi t dt + \frac{b_n}{T}\int_0^T x(t)\sin n\varpi t dt\right] \end{aligned}$$

Finally substituting the integral expressions from Eqs. (22.29) we get the desired result

$$\overline{x^2} = a_0^2 + \sum_{n=1}^{\infty}\left[\frac{a_n^2}{2} + \frac{b_n^2}{2}\right] \tag{22.30}$$

The spectrum of the function $x(t)$ is then given by the terms of the series in Eq. (22.30). Each term of this series is the contribution of the corresponding frequency to the mean square value of $x(t)$.

We now consider a discrete time function $F(t_j)$ expressed as a discrete Fourier transform [Eq. (19.28)], that is, as

$$F(t_j) = \sum_{n=0}^{N-1} C_n e^{2\pi i(nj/N)} \tag{22.31}$$

where C_n is given by Eq. (19.27) as

$$C_n = \frac{1}{N}\sum_{j=0}^{N-1} F(t_j)e^{-2\pi i(nj/N)} \tag{22.32}$$

As indicated in Chap. 19 [Eq. (19.29) or (19.30)], harmonic components of the function $F(t_j)$ higher than the Nyquist frequency, $\omega_{N\,2} = \pi/\Delta t$ rad/sec or $f_{N\,2} = 1/\Delta t$ cps, are not included in the discrete Fourier transform [Eq. (22.31)]. Also as noted in Chap. 19, if there are harmonic components in $F(t_j)$ higher than this limiting value, these higher frequencies introduce distorting contributions to the lower harmonic frequencies. Hence it is imperative that the value of N be selected sufficiently large to include the frequencies that contribute significantly to the original function. To be certain that this condition is satisfied, one may filter the signal of the function electronically to remove all frequencies higher than the Nyquist frequency.

The mean square value of a discrete function $F(t_j)$ ($j = 0, 1, 2, \ldots, N–1$) is obtained from Eq. (22.2) as

$$\overline{F^2} = \frac{1}{T}\sum_{j=0}^{N-1} F^2(t_j)\Delta t \tag{22.33}$$

Substituting $\Delta t/T$ for $1/N$ and using Eqs. (22.31) and (22.32) for one factor $F(t_j)$ in Eq. (22.33), we obtain

$$\begin{aligned}\overline{F^2} &= \frac{1}{N}\sum_{j=0}^{N-1} F(t_j)\sum_{n=0}^{N-1} C_n e^{2\pi i(nj/N)} \\ &= \sum_{n=0}^{N-1} C_n\left[\frac{1}{N}\sum_{j=0}^{N-1} F(t_j)e^{2\pi i(nj/N)}\right] \\ &= \sum_{j=0}^{N-1} C_n C_n^*\end{aligned} \tag{22.34}$$

where C_n^* is the complex conjugate[2] of C_n. Hence

$$\overline{F^2} = \sum_{n=0}^{N-1} |C_n|^2 = |C_0|^2 + |C_1|^2 + |C_2|^2 + \ldots + |C_{N-1}|^2 \tag{22.35}$$

The terms of the summation in Eq. (22.35) are the required spectrum of the discrete function $F(t_j)$; that is, these terms are the frequency contributions to the mean square value $\overline{F^2}$. As we can see in Eq. (22.35), the contribution of each frequency is equal to the square of the modulus of the corresponding complex coefficient C_n which is given by Eq. (22.32).

Illustrative Example 22.3

[2] Since $F(t_j)$ is a real function, its conjugate $F(t_j)^* = F(t_j)$ and also $[e^{-2\pi(nj/N)}]^* = e^{2\pi(nj/N)}$, it follows that $1/N\sum_{j=0}^{N-1} F(t_j)e^{2\pi(nj/N)} = C_n^*$

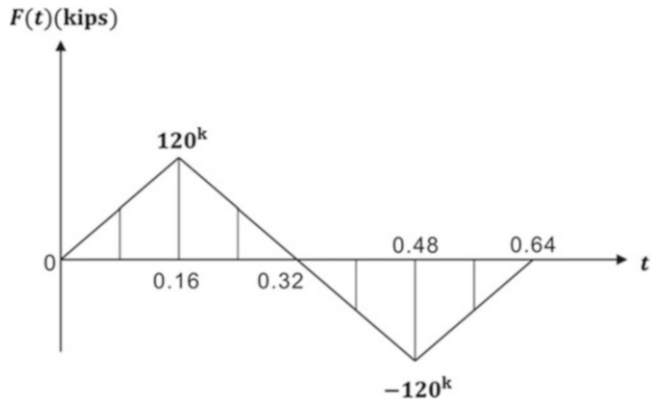

Fig. 22.16 Forcing function for Illustrative Example 22.3

Determine the spectrum of the forcing function $F(t)$ shown in Fig. 22.16. Assume that the function is defined at eight equally spaced time intervals. Use the spectrum to estimate the mean square value $\overline{F^2}$ and compare this result with the mean square calculated directly from the definition of $\overline{F^2}$.

Solution: We use computer Program 4 with $N = 8$ to determine the discrete Fourier coefficients C_n. The values thus obtained are shown in Table 22.1 together with the calculation needed to obtain the spectrum using Eq. (22.35). The summation of the spectral values for $F(t)$ shown in the last column of Table 22.1 is $\overline{F^2} = 0.5400$ E10. We check this value by calculating the mean square value of $F(t)$ directly from the definition Eq. (22.2), namely,

$$\overline{F^2} = \frac{1}{T}\int_0^T F^2(t)dt$$

in which we substitute

$$F(t) = \frac{120,000}{0.16}t$$

$$\overline{F^2} = \frac{4}{0.64}\int_{0.}^{0.16}\left[\frac{120,000}{0.16}\cdot t\right]^2 dt = 0.4800E10$$

Table 22.1 Spectral analysis for the function F(t) in Fig. 22.16

N	Fourier Coefficients C_n Re(Cn)	Im(Cn)	$\lvert C_n\rvert$	Spectrum of $F(t)$ $\lvert C_n\rvert^2$
0	0	0		0
1	0	−51,210	51,210	0.2623 $E10$
2	0	0		0
3	0	8787	8787	0.7721$E8$
4	0	0		0
5	0	8787	8787	0.7721$E8$
6	0	0		0
7	0	51,210	51,210	0.2623$E10$
				$\overline{F^2} = 0.5400\ E10$

Table 22.2 Mean square value for function $F(t)$ in Fig. 22.16

Exponent M	Time intervals $N = 2^M$	Mean square value $\overline{F^2}$
3	8	0.5400 E 10
4	16	0.4900 E 10
5	32	0.4838 E 10
6	64	0.4809 E 10
7	128	0.4802 E 10
		Exact $\overline{F^2} = 0.4800\ E10$

Considering that we have used in this example a relatively small number of intervals or sampled points ($N = 8$), the value of $\overline{F^2} = 0.5400$ E10 obtained from the spectrum of $F(t)$ in Table 22.1 compares fairly well with the exact mean square value $\overline{F^2} = 0.4800$ E10. As discussed before, errors are introduced in the calculations when the number of time intervals N is not large enough to include the higher frequency components of $F(t)$. In order to improve the calculation of the spectrum, it is necessary to use more time intervals in the discrete Fourier Series. Table 22.2 shows results obtained using Program 4 with $M = 3, 4, 5, 6, 7$, corresponding to $N = 2^M = 8, 16, 32, 64, 128$ time intervals for the function $F(t)$ shown in Fig. 22.16. It may be observed that values displayed in the last column of the table are converging to the exact value $\overline{F^2} = 0.4800$ E10 as the number of time intervals, N, is increased.

22.8 Spectral Density Function

If a random process $x(t)$ is normalized (or adjusted) so that the mean value of the process is zero, then, provided that $x(t)$ has no periodic components, the autocorrelation function $R_x(\tau)$ approaches zero as τ increases, that is.

$$\lim_{\tau \to \infty} R_x(\tau) = 0$$

We therefore expect that $R_x(\tau)$ should satisfy the condition in Eq. (22.25). We can then use Eqs. (22.23) and (22.24) to obtain the Fourier transform $S_x(\omega)$ of the autocorrelation function $R_x(\tau)$ and its inverse as

$$S_x(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R_x(\tau) e^{-i\omega\tau} d\tau \tag{22.36a}$$

and

$$R_x(\tau) = \int_{-\infty}^{\infty} s_x(\omega) e^{i\omega\tau} d\omega \tag{22.36b}$$

In Eq. (22.36a), $S_x(\omega)$ is called the spectral density function of $x(t)$. The most important property of $S_x(\omega)$ becomes apparent by letting $\tau = 0$ in Eq. (22.36b). In this case

$$R_x(0) = \int_{-\infty}^{\infty} S_x(\omega) d\omega) \tag{22.37}$$

which by Eq. (22.20) is equal to mean square value of the function $x(t)$, that is,

$$\overline{x^2} = \int_{-\infty}^{\infty} S_x(\omega) d\omega \tag{22.38}$$

The mean square value of a random process is therefore given by the area under the graph of the spectral density function as shown in Fig. 22.17. Consequently, the contribution of an incremental frequency $\Delta\omega$ to the mean square value is

$$\Delta\overline{x^2} = S_x(\omega)\Delta\omega \tag{22.39}$$

The Fourier transform pair for the discrete autocorrelation function $R_j = R(\tau)$ and for the discrete spectral density function $S_n = S(\omega_n)$ of a time function $x = x(t)$, is defined from Eqs. (19.27) and (19.28) as

$$R_j = \sum_{n=0}^{N-1} S_n e^{\pi i(nj/N)} \tag{22.40a}$$

and

$$S_n = \frac{1}{N} \sum_{j=0}^{N-1} R_j e^{-2\pi j(nj/N)} \tag{22.40b}$$

It may be seen then, that the terms in Eq. (22.40b) provide the contributions, $\Delta\overline{x^2}$, to mean square value $\overline{x^2}$, at the frequency as

$$\Delta\overline{x^2} = S_n \tag{22.41}$$

The contribution to the mean square value may also be expressed, in terms of the Fourier transform coefficient C_n and its conjugate C_n^*, using Eq. (22.34) as

$$\Delta\overline{x^2} = C_n C_n^* \tag{22.42}$$

Therefore, it follows from Eqs. (22.41) and (22.42) that.

$$S_n = C_n C_n^* \tag{22.43}$$

The spectral density of a given record can be obtained electronically by an instrument called a frequency analyzer or spectral density analyzer. The output of an accelerometer or other vibration transducer is fed into the instrument, which is essentially a variable frequency narrow-band filter with a spectral meter to display the filtered output. With this instrument the experimenter searches for the predominant frequencies present in a vibration signal. The output of the spectral density analyzer is the contribution to the mean square value $\overline{x^2}$ of the input signal $x(t)$ for a small range $\Delta\omega$ around the set frequency.

When dealing with theory, the natural unit for the frequency is rad /sec. However, in most practical problems the frequency is expressed in cycles per second or Hertz (abbreviated Hz). In the latter case, we rewrite Eq. (22.39) as

$$\Delta\overline{x^2} = S_x(f)\Delta f \tag{22.44}$$

where f is the frequency in Hertz. Since $\Delta\omega = 2\pi\Delta f$, it follows from Eqs. (22.39) and (22.44) that

$$S_x(f) = 2\pi S_x(\omega) \tag{22.45}$$

When the spectral density function for the excitation is known, its mean-square value may be determined from Eq. (22.44) as

$$\overline{x^2} = \int_{-\infty}^{\infty} S_x(f)df \tag{22.46}$$

The spectral density function $S_x(f)$ is expressed in square units of x per Hertz. Since the autocorrelation function $R_x(\tau)$ is real and even, the use of Euler's relationship

$$e^{i\pi\tau} = \cos\,\omega\tau + i\,\sin\,\omega\tau$$

in Eq. (22.36a) yields the cosine transform:

$$S_x(\omega) = \frac{1}{2\pi}\int_{-x}^{x} R_x(\tau)\cos\,\omega\tau d\tau \tag{22.47}$$

It is clear from Eq. (22.47) that $S_x(\omega)$ is also an even function of ω; hence Eq. (22.36b) may be written as

$$R_x(\tau) = \int_{-\infty}^{\infty} S_x(\omega)\cos\,\omega(\tau)d\omega \tag{22.48}$$

Alternatively, Eqs. (22.47) and (22.48) may be written as

$$S_x(\omega) = \frac{1}{\pi}\int_{0}^{\infty} R_x(\tau)\cos\,\omega\tau\,d\tau \tag{22.49}$$

and

$$R_x(\tau) = 2\int_{0}^{\infty} S_x(\omega)\cos\,\omega\tau\,d\omega \tag{22.50}$$

These are the celebrated Wiener-Kinchin equations, which describe how the spectral density function can be determined from the autocorrelation function and vice versa.

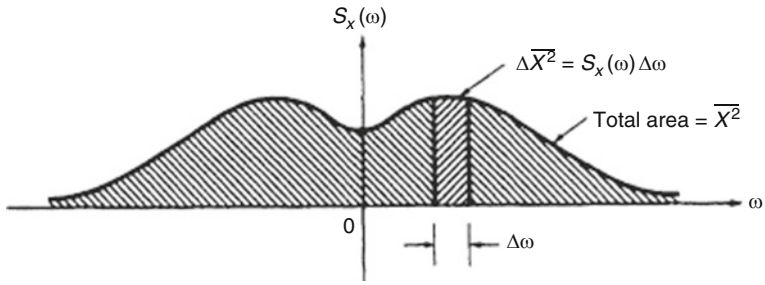

Fig. 22.17 Spectral density function showing area equal to mean-square value

22.9 Narrow-Band and Wide-Band Random Processes

A process whose spectral density function has nonzero values only in a narrow frequency range as shown in Fig. 22.18 is called a narrow band process. In contrast, a wide-band process is one whose spectral density function is nonzero over a broad range of frequencies. The time history of such a process is then made up of the superposition of the whole band of frequencies as shown in Fig. 22.19a. In the limit, when the frequency band extends from $\omega_1 = 0$ to $\omega_2 = \infty$, this spectrum is called white noise. From Eq. (22.38) the mean square value of a white noise process must be infinite; therefore, the white noise process is only a theoretical concept. In practice a process is called white noise when the bandwidth of its frequencies extends well beyond all the frequencies of interest.

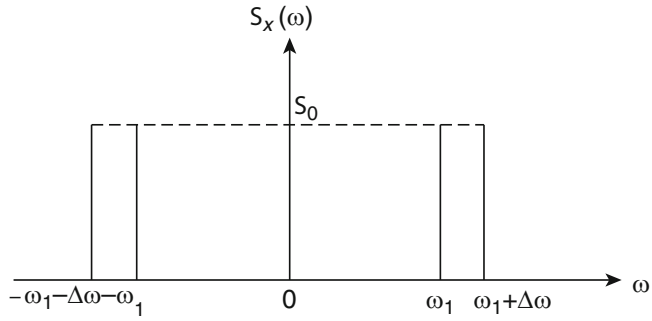

Fig. 22.18 Spectral density function for a narrow-band random process

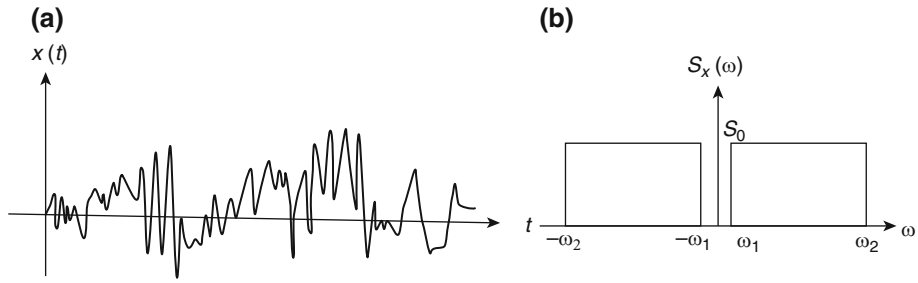

Fig. 22.19 Wide-band process. (**a**) Time history. (**b**) Spectral density function

Illustrative Example 22.4

Determine the mean square value and the autocorrelation function for the narrow-band random process $x(t)$ whose spectral density function is shown in Fig. 22.18.

Solution: From Eq. (22.38)

$$\overline{x^2} = \int_{-\infty}^{\infty} S_x(\omega)d\omega = 2S_0\Delta\omega$$

and from Eq. (22.50)

$$
\begin{aligned}
R_x(\tau) &= 2\int_0^{\infty} S_x(\omega)\cos\omega\tau d\omega \\
&= 2\int_{\omega_1}^{\omega_1+\Delta\omega} S_0\cos\omega\tau d\omega \\
&= \frac{2S_0}{\tau}\,|\sin\omega\tau|_{\omega_1}^{\omega_1+\Delta\omega}
\end{aligned}
\tag{a}
$$

$$
\begin{aligned}
&= \frac{2S_0}{\tau}\,[\sin(\omega_1+\Delta\omega)\tau - \sin\omega_1\tau] \\
R_x(\tau) &= \frac{4S_0}{\tau}\cos\left(\omega_1+\frac{\Delta\omega}{2}\right)\tau\cdot\sin\frac{\Delta\omega}{2}\tau
\end{aligned}
\tag{b}
$$

The autocorrelation function for a narrow-band random process given by Eq. (b) has the form shown in Fig. 22.20, where the predominant frequency of $R_x(\tau)$ is the average value ($\omega_1 + \Delta\omega/2$). The autocorrelation for such a process has a maximum of $2S_0\,\Delta\omega$ when $\tau = 0$ and decreases like a cosine graph as τ moves away from $\tau = 0$.

The autocorrelation function $R_x(\tau)$ for a wide-band random process whose spectral density function extends in the range ω_1 to ω_2 as shown in Fig. 22.19b can be obtained from the result of Illustrative Example 22.4. In this case, letting the lower frequency $\omega_1 = 0$ and the upper frequency $\Delta\omega = \omega_2$ we obtain from Eq. (b) of Illustrative Example 22.4

$$
R_x(\tau) = \frac{4S_0}{\tau}\cos\left(\frac{\omega_2\tau}{2}\right)\sin\left(\frac{\omega_2\tau}{2}\right) = \frac{2S_0}{\tau}\sin\omega_2\tau \tag{22.51}
$$

which has the form shown in Fig. 22.22a.

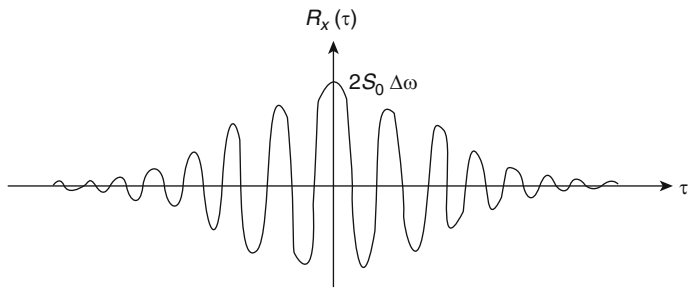

Fig. 22.20 Autocorrelation for a narrow-band random process

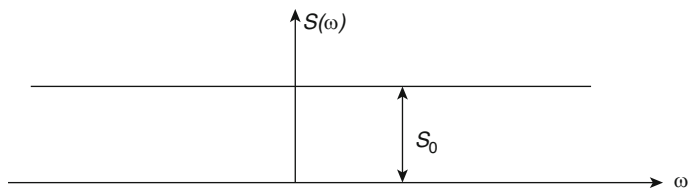

Fig. 22.21 Spectral density function for Illustrative Example 22.5

Illustrative Example 22.5

Determine the autocorrelation function for constant spectral density function $S(\omega) = S_0$ (white noise) for all value of the frequency ω as shown graphically in Fig. 22.21.

The autocorrelation function for white noise may be obtained from Eq. (22.51) by letting $\omega_2 \to \infty$. In this case adjacent cycles come closer together, resulting in a high peak at $\tau = 0$ and zero value elsewhere as shown in Fig. 22.22b.

This high peak will be of infinite height, zero width, but of a finite area. Such behavior may be described mathematically using Dirac's delta function $\delta(\tau)$. The delta function $\delta(\tau)$ is defined as having zero value everywhere except at $\tau = 0$ in such a way that

$$\int_{-\infty}^{\infty} \delta(\tau) f(\tau)\, d\tau = f(0) \tag{22.52}$$

for any function of time $f(\tau)$ defined at $\tau = 0$.

Using the delta function $\delta(\tau)$ we can express the autocorrelation function for white noise as

$$R_x(\tau) = C\delta(\tau) \tag{22.53}$$

where C must be determined from Eq. (22.36a) using the fact that $S_x(\omega)$ must be the constant S_0. Substituting Eq. (22.53) into the Eq. (22.36a) gives

$$S_x(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} C\delta(\tau) e^{-j\omega\tau} d\tau$$

and using Eq. (22.52) yields

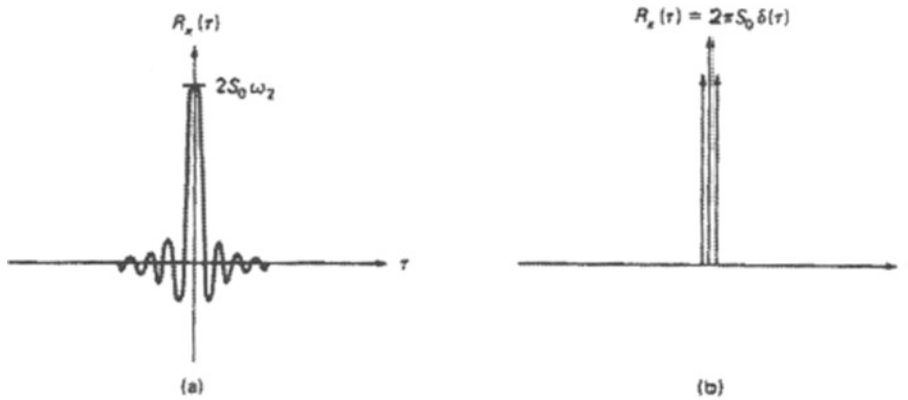

Fig. 22.22 Autocorrelation for a wide-band random process becomes a delta function for white noise

$$S_x(\omega) = \frac{1}{2\pi} C e^{-i\omega 0} = \frac{C}{2\pi} \tag{22.54}$$

Hence

$$S_0 = \frac{C}{2\pi} \tag{22.55}$$

Finally, solving for C and substituting into Eq. (22.53), we obtain the autocorrelation function for white noise shown in Fig. 22.22b, as

$$R(\tau) = 2\pi S_0 \delta(\tau) \tag{22.56}$$

22.10 Response to Random Excitation: Single-Degree-of-Freedom System

To determine the response of a structural system subjected to a random excitation, we need to examine the frequency content of the excitation function. We are mostly interested in estimating the spectral function or the spectral density function of the excitation.

Until recently, the procedure for estimating the spectrum of a discrete time series has been to first determine the autocorrelation function [Eq. (22.19)] and then apply the Fourier transform to this function to obtain the required spectrum [Eq. (22.36a) or Eq. (22.49)]. However, the method of calculation has changed since the development of Fast Fourier Transform (usually abbreviated FFT). As has been indicated in Chap. 19, the FFT is a remarkably efficient method for calculating the Fourier transform of a time series. Rather than estimate the spectrum by first determining the autocorrelation function and then calculating the Fourier transform, it is now more efficient and more accurate to calculate spectra directly from the original time series.

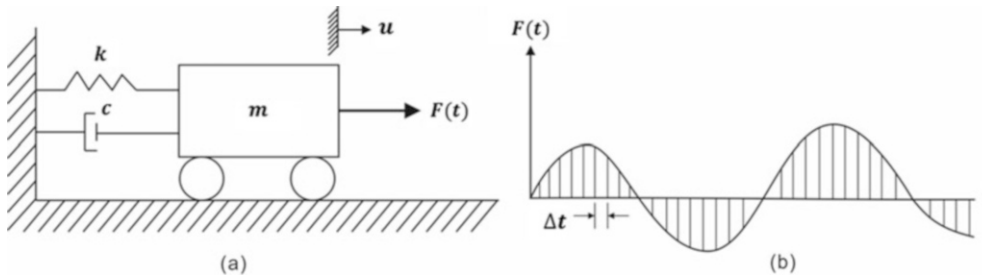

Fig. 22.23 Single-degree-of-freedom system subject to a random force sampled at regular time intervals

Consider the damped, single-degree-of-freedom system shown in Fig.22.23a subjected to a random force $F(t)$, a sample of which is shown in Fig. 22.23b. We will assume that this force is known at N discrete equally spaced time $t_j = j\Delta t$ ($j = 0, 1, 2, \ldots,$ N- 1). Fourier analysis of $F(t)$ results in the frequency component as given by Eqs. (22.31) and (22.32). By superposition the response of the single degree-of-freedom system to the harmonic components of F (t) is given by Eq. (19.35) as

$$u(t_j) = \sum_{n=0}^{N-1} \frac{C_n e^{2\pi(nj/N)}}{k(1 - r_n^2 + 2ir_n\xi_n)} \tag{22.57}$$

in which, as discussed in Sect. 19.6,

$$\begin{aligned} \omega_n &= n\varpi \qquad \text{for } n \le N/2 \\ \omega_n &= -(N-n)\varpi \quad \text{for } n > N/2 \end{aligned} \tag{22.58}$$

$$r_n = \frac{\omega_n}{\omega}, \quad \varpi = \frac{2\pi}{T}, \quad \omega = \sqrt{\frac{k}{m}} \tag{22.59}$$

In these formulas, T is the time duration of the excitation, N the number of equal intervals of the excitation, and ξ_n the damping ratio corresponding to the frequency ω_n.

Equation (22.57) may conveniently be written as

$$u(t_j) = \sum_{n=0}^{N-1} H_{ij} C_n e^{2\pi(nj/N)} \tag{22.60}$$

where the frequency response fonction, H_n is given by

$$H_n = \frac{1}{k} \cdot \frac{1}{1 - r_n^2 + 2ir_n\xi_n} \tag{22.61}$$

The mean square value $\overline{u^2}$ of the response can then be obtained from Eq. (22.35) as

$$\overline{u^2} = \sum_{n=0}^{N-1} |H_n|^2 |C_n|^2 \tag{22.62}$$

Alternatively, Eq. (22.62) may be expressed in terms of the spectral density function. The frequency contributions $\Delta\overline{F^2}$ to the mean square value $\overline{F^2}$ are given by Eq. (22.35) as

$$\Delta\overline{F^2} = |C_n|^2 \tag{22.63}$$

which by Eq. (22.39), may be expressed as

$$\Delta\overline{F^2} = S_F(\omega_n)\Delta\omega \tag{22.64}$$

Now, using Eqs. (22.63) and (22.64), we may write Eq. (22.62) as

$$\overline{u^2} = \sum_{n=0}^{N-1} |H_n|^2 S_F(\omega_n)\Delta\omega \tag{22.65}$$

or as

$$\overline{u^2} = \sum_{n=0}^{N-1} S_u(\omega_n)\Delta\omega \tag{22.66}$$

where

$$S_u(\omega_n) = |H_n|^2 S_F(\omega_n) \tag{22.67}$$

When the frequency is expressed in cps, we may write Eq. (22.66) as

$$\overline{u^2} = \sum_{n=0}^{N-1} S_u(f_n)\Delta f \tag{22.68}$$

where

$$S_u(f_n) = |H_n|^2 S_F(f_n) \tag{22.69}$$

Equation (22.69) states the important result that, when the frequency response function, H_n is known, the spectral density $S_u(f_n)$ for the response can be calculated from the spectral density, $S_F(f_n)$, of the excitation.

Illustrative Example 22.6

Determine the mean square value of the response for the single-degree-of-freedom system in which $k = 100{,}000$ lb/in, $m = 100$ lb · sec^2/in, $c = 632$ lb · sec/in subjected to the $F(t)$ shown in Fig. 22.16. Choose $N = 8$ for the number of intervals.

Solution: The mean square value $\overline{u^2}$ of the response is given by Eq. (22.62) as

$$\overline{u^2} = \sum_{n=0}^{N-1} |H_n|^2 |C_n|^2$$

From Eq. (22.61)

$$|H_n|^2 = \frac{1}{k^2} \cdot \frac{1}{\left(1 - r_n^2\right)^2 + \left(2r_n\xi_n\right)^2}$$

where ω_n/ω is is the frequency ratio and $\xi_n = c_n/c_{cr}$ is the damping ratio.

The natural frequency is calculated as $\omega = \sqrt{k/m} = 31.62$ rad/ sec and the frequency components are given by Eq. (22.58) as

$$\omega_n = n\omega_1, \qquad n \le N/2$$
$$\omega_n = -(N-n)\omega_n, \quad n > N/2$$

in which $\omega_1 = 2\pi/T = 2\pi/0.64 = 9.8175$ rad/sec, since the duration of the applied force is $T = 0.64$ sec.

Values of $|C_n|$, for the function $F(t)$ shown in Fig. 22.16, have been determined in Illustrative Example 22.3 and are shown in Table 22.1. The necessary computations to determine $\overline{u^2}$ are conveniently shown in Table 22.3. From this table the mean square value of the response is $\overline{u^2} = 0.9352$.

The MATLAB program can be used to obtain the response of 0.9352.

```
clear all
close all
clc

%%%%-GIVEN VALUES-
m=100;                  %Mass (lb.sec^2/in.)
k =100000;              %Stiffness (lb/in.)

omega = sqrt(k/m);      %Natural frequency
c=632;                  %Damping coefficient. (lb.sec/in.)
xi =c/(2*m*omega);      %Damping ratio

T = 0.64;               %Time period, T(sec)
omega_bar = 2*pi/T;     %Excitation frequency (rad/sec)
M= 3;                   %Select M, M=3
N = 2^M;                %The number of time increments N

t=0:0.08:0.64;          %Time ranging from 0 to 0.64 sec with deltat = 0.08 sec
Dt = t(2)-t(1);         %Deltat = 0.08
tt= length(t);          %Total number of calculation

for i= 1:tt-1
%%%Define the function of N harmonic force
if t(i)<=0.16
       F(i) = 120000*t(i)/0.16;
elseif t(i) <=0.48
       F(i) =-750000*(t(i)-0.16)+120000;
else
       F(i)=min(0, 750000*(t(i)-0.64)) ;
end

%%%Define the discrete Fourier transform of the series
Cn=fft(F/N);                                          %Eq.19.27

%%%Calculate frequency ratio, r_n
ifi<=N/2
omega_n(i) = (i-1)*omega_bar;                         %Eq.22.58
else
omega_n(i) = -(N-(i-1))*omega_bar;                    %Eq.22.58
end

rn(i)=omega_n(i)/omega;                               %Eq.22.59

    H(i)=sqrt(1/k^2/((1-rn(i)^2)^2+(2*rn(i)*xi)^2));  %Eq.22.61

    Cn2=abs(Cn);

uu = H.^2.*Cn2.^2;                                    %Eq.22.62 (ith)

end
uu_sum = sum(uu)                                      %Eq.18.62
```

Analogous to Eq. (22.65), when the excitation is a random acceleration applied to the support of the structure, the mean-square acceleration response $\overline{a_p^2}$ at a point P of the structure is given by

$$\overline{a_p^2} = \sum S(\omega_n)|H_n|^2 \Delta\omega \tag{22.70}$$

where H_n is now the frequency response in terms of the acceleration at point P resulting from a unit harmonic acceleration at the support of the structure.

The response of a single-degree-of-freedom structure subjected to a single point random excitation can be determined by a simple numerical calculation provided that the spectral function or the spectral density function of the excitation and the frequency response of the structure are known.

The frequency response H_n may be obtained experimentally by applying a sinusoidal excitation of varying frequency at the foundation and measuring the response at the desired point in the structure. The necessary calculations are explained in the following numerical example.

Illustrative Example 22.7

Determine the response at a point P of the structure modeled as a shear building shown schematically in Fig. 22.24a when subjected to a random acceleration at its foundation. The spectral density function of the excitation is known and shown in Fig. 22.24b. The frequency response a_p/a_0 of the structure at point P, obtained experimentally, when the foundation is excited by a sinusoidal acceleration of amplitude a_0 and varying frequency f_n, is shown in Fig. 22.24c.

Table 22.3 Calculation of $\overline{u^2}$ for Illustrative Example 22.6

n	ω_n (rad/sec)	$r = \omega_n / \omega$	$\lvert H_n \rvert$ (in/lb)	$/C_n/$ (lb)	$\Delta\overline{u^2} = \lvert H_n \rvert^2 \lvert C_n \rvert^2$ (in^2)
0	0	0	1.000 E-5	0	0
1	9.8175	0.3105	1.104 E-5	0.5121 E 5	0.3197
2	19.6350	0.6209	1.595 E-5	0	0
3	29.4524	0.9314	4.376 E-5	0.8787 E 4	0.1479
4	39.2699	1.2418	1.677 E-5	0	0
5	−29.4524	−0.9314	4.376 E-5	0.8787 E4	0.1479
6	−19.6350	−0.6209	1.595 E-5	0	0
7	−9.8175	−0.3105	1.104 E-5	0.5121 E 5	0.3197
					$\overline{u^2} = 0.9352$

Solution: The mean square value $\overline{a_p^2}$ of the response at the point P is calculated from Eq. (22.70) as

$$\overline{a_p^2} = \sum S(f_n)|H_n|^2 \Delta f \tag{a}$$

where

$$| H_n | = a_p/a_0$$

Table 22.4 summarizes the computational procedure. By Eq. (a) we obtain from the sum in the last column of this table the mean square value of the response $\bar{a}^2 = 1.8100\text{g}^2$ and by Eq. (22.3) (assuming mean value $\bar{a} = 0$)

$$\sigma = \sqrt{1.8100}\, g = 1.345\, g$$

The probability of exceeding specified accelerations (see Table in Sect. 22.3) can now be found assuming the normal distribution for $/a_p/ > \sigma = 1.345\ g$ and for $/a_p/ > 3\sigma = 4.041\ g$, respectively, as

$$P\left[|a_p| > 1.345g\right] = 31.7\%$$

and

$$P[|a_P| > 4.041g] = 0.3\%$$

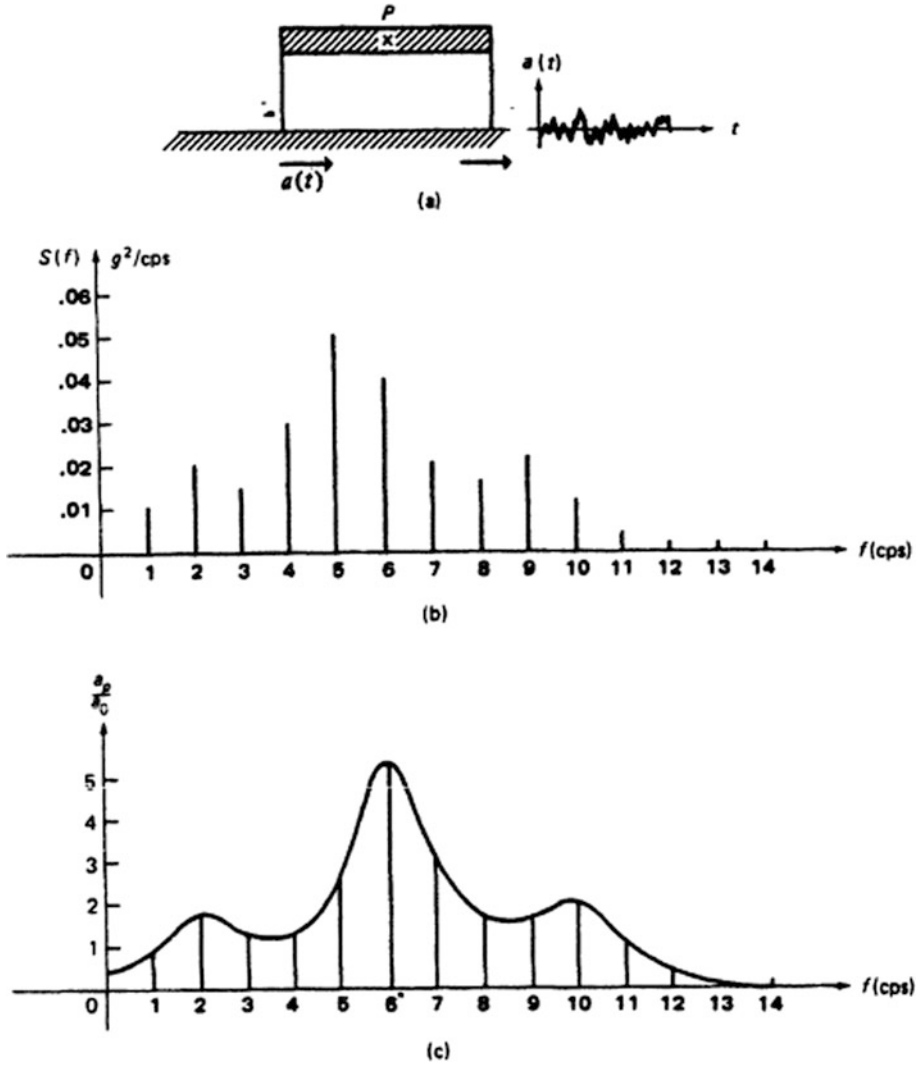

Fig. 22.24 (**a**) Structure subjected to random acceleration at the base. (**b**) Spectral density function of the excitation. (**c**) Relative frequency response at point *P*

Similarly, the probability that the peak acceleration A_p at point P will exceed a specified value (1 σ or 3 σ) is found using the Rayleigh distribution (see Table in Sect. 22.4) as

$$P\left[A_p > 1.345g\right] = 60.65\%$$
$$P\left[A_p > 4.041g\right] = 1.11\%$$

Table 22.4 Calculations of the response for Illustrative Example 22.7

f_n (cps)	Δf (cps)	$S(f_n)$(g^2/cps)	$\lvert H_n\rvert$ (in/lb)	$\lvert H_n\rvert^2\, S(f_n)\Delta f$ (g^2 units)
0	1.0	0	0.5	0
1.0	1.0	0.010.0	1.0	0.100
2.0	1.0	0.020	1.8	0.0648
3.0	1.0	0.015	1.3	0.0253
4.0	1.0	0.030	1.4	0.0588
5.0	1.0	0.050	2.2	0.2420
6.0	1.0	0.040	5.2	1.0816
7.0	1.0	0.020	3.0	0.1800
8.0	1.0	0.015	1.8	0.0486
9.0	1.0	0.020	1.7	0.0578
10.0	1.0	0.010	1.9	0.0361
11.0	1.0	0.005	1.0	0.0050
12.0	1.0	0	0.4	0
13.0	1.0	0	0	0
14.0	1.0	0	0	0
				Sum = 1.8100

22.11 Response to Random Excitation: Multiple-Degree-of-Freedom System

The extension for determining the response to random vibration from a single-degree-of-freedom system to a multiple-degree-of-freedom system can readily be accomplished using the modal superposition method. This method, as we have seen in previous chapters, transforms a system of coupled differential equations into a set of independent or uncoupled differential equations of only one dependent variable in each equation. Thus, each equation of this set is equivalent to the differential equation for a single-degree-of-freedom system and consequently can be solved by the method presented in the preceding section for a single-degree-of-freedom system.

We present first, the relationship between the complex frequency response and the unit impulse response followed by the development of random vibration for a system of two degrees of freedom and the generalization to multiple-degree-of-freedom systems.

22.11.1 Relationship Between Complex Frequency Response and Unit Impulse Response

Consider a linear dynamic system of a single-degree-of-freedom represented by the damped simple oscillator shown in Fig. 22.25.

The random force $F(t)$ acting on this system may be expressed in terms of its frequency content by means of the inverse Fourier transform given by Eq. (22.24) as

$$F(t) = \int_{-\infty}^{\infty} C(\omega) e^{i\omega t}\, d\omega \quad \text{(repeated)} \tag{22.24}$$

where the coefficient $C(\omega)$ is given by Fourier transform, Eq. (22.23) as

$$C(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(t)e^{-i\omega t}\,d\omega \quad \text{(repeated)} \tag{22.23}$$

The differential equation of motion for the system shown in Fig. 22.25 is

$$m\ddot{u} + c\dot{u} + ky = F(t) \tag{22.71}$$

which for an excitation of a unit exponential function, $F(t) = e^{i\ \omega t}$, as presented in Chap. 3, has a steady-state solution, $u_1(t)$ of the form

$$u_1(t) = H(\omega)e^{i\omega t} \tag{22.72}$$

in which

$$H(\omega) = \frac{1}{k - m\omega^2 + ic\omega}$$

Consequently, the response $u(t)$ due to the force expressed by the inverse of Fourier transform in Eq. (22.24) is given by

$$u(t) = \int_{-\infty}^{\infty} C(\omega)H(\omega)e^{i\omega t}\,dt \tag{22.73}$$

Consider now the special case where the total excitation consists of a single impulse of magnitude 1 applied at time $t = 0$.

In this case, $C(\omega)$ in Eq. (22.23) becomes

$$C(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(t)e^{-i\omega t}\,dt = \frac{1}{2\pi} \tag{22.74}$$

because $e^{-i\omega t} = 1$ at $t = 0$ and

$$\int_{-\infty}^{\infty} F(t)dt = 1 \tag{22.75}$$

for a unit impulse.

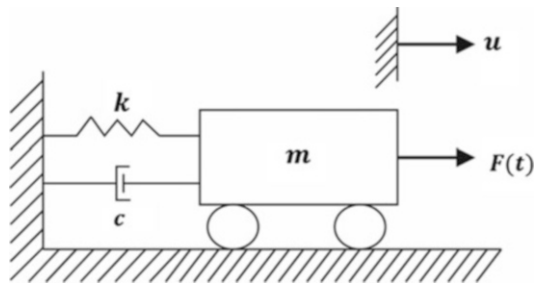

Fig. 22.25 Model of a damped single-degree-of-freedom system

The expression given by Eq. (22.75) is known as the Dirac's δ function. Thus, from Eqs. (22.73) and (22.74), the response, $h(t)$, to unit impulse is given by

$$h(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H(\omega) e^{i\omega t}\, dt \tag{22.76}$$

which is recognized as the inverse Fourier transform of

$$H(\omega) = \int_{-\infty}^{\infty} h(t) e^{-i\omega t}\, d\omega \tag{22.77}$$

We observe that the unit impulse response $h(t)$ of Eq. (22.76) and the complex frequency response $H(\omega)$ of Eq. (22.77) form a Fourier transform pair.

22.11.2 Response to Random Excitation: Two-Degree-of-Freedom System

The concepts and methodology for determining the response of multiple-degree-of-freedom systems can be adequately presented through the treatment of a two-degree-of-freedom system. We begin by presenting a two-degree-of-freedom system subjected to a single random excitation. This case is then extended by subjecting this two-degree-of-freedom system to a second excitation, in addition to the first random excitation. Finally, we present the general case of a multiple-degree-of-freedom system subjected to multiple random excitations.

We consider in Fig. 22.26a the structural model of a two-degree-of-freedom system subjected to a random stationary force $F_2(t)$ applied at coordinate u_2. The equations obtained by establishing the dynamic equilibrium in the free-body-diagram in Fig. 22.26b are:

$$\begin{aligned} m_1\ddot{u}_1 + (k_1 + k_2)u_1 - k_2u_1 &= 0 \\ m_2\ddot{u}_2 - k_1u_1 + k_2u_2 &= F_2(t) \end{aligned} \tag{22.78}$$

The natural frequencies and modal shapes are then found, as presented in Chap. 11, by setting $F_2(t) = 0$ in Eq. (22.78) and substituting a trial solution $u_1 = a_1 \sin \omega t$ and $u_2 = a_2 \sin \omega t$. Hence, we obtain

$$\begin{aligned} (k_1 + k_2 - m_1\omega^2)a_1 - k_2a_2 &= 0 \\ -k_2a_1 + (k_2m_2\omega_2)a_2 &= 0 \end{aligned} \tag{22.79}$$

For a nontrivial solution of Eq. (22.79), we set equal to zero the determinant of the coefficients:

$$\begin{vmatrix} k_1 + k_2 - m_1\omega^2 & -k_2 \\ -k_2 & k_2 - m_1\omega^2 \end{vmatrix} \tag{22.80}$$

The expansion of the determinant in Eq. (22.80) results in the following quadratic equation in ω^2:

$$m_1m_2\omega^4 - [(k_1 + k_2)m_2 + k_2m_1]\omega^2 + k_1k_2 = 0 \tag{22.81}$$

Equation (22.81) provides the roots ω_1^2 and ω_2^2 which are the square values of the natural frequencies for this two-degree-of-freedom system. The substitution of ω_1^2 into one of the equations in Eq. (22.79) results in the first mode a_{11}, a_{21}, and the subsequent substitution of ω_2^2, in the second mode a_{12}, a_{22}. These modal shapes are conveniently normalized by dividing the components a_{11} and a_{21} of the first mode by $\sqrt{m_1a_{11}^2 + m_2a_{21}^2}$ and the components, a_{12} and a_{22}, of the second mode by j

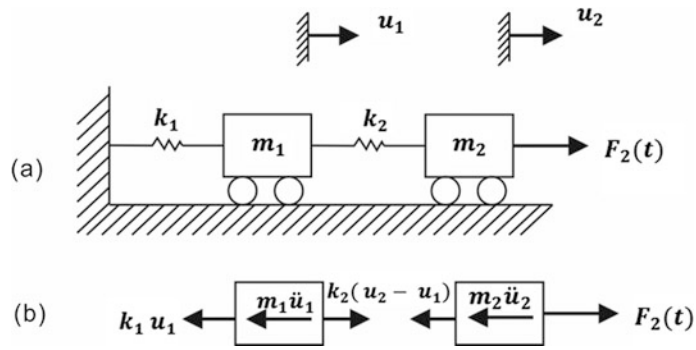

Fig. 22.26 Two-degree-of-freedom system. (**a**) Mathematical model. (**b**) Free-body-diagrams

$\sqrt{m_1 a_{12}^2 + m_2 a_{22}^2}$ to obtain the modal matrix given by Eq. (22.82) in which the normalized modes are arranged in the columns of this matrix.

$$[\phi] = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} \tag{22.82}$$

The uncoupled modal equation, as presented in Chap. 6, is obtained by introducing in Eq. (22.78) the linear transformation

$$\begin{aligned} u_1(t) &= \phi_{11} z_1(t) + \phi_{12} z_2(t) \\ u_2(t) &= \phi_{21} z_1(t) + \phi_{22} z_2(t) \end{aligned} \tag{22.83}$$

resulting in

$$\begin{aligned} \ddot{z}_1 + \omega_1^2 z_1 &= \phi_{21} F_2(t) \\ \ddot{z}_2 + \omega_2^2 z_2 &= \phi_{22} F_2(t) \end{aligned} \tag{22.84}$$

in which z_1 and z_2 are the modal displacement functions.

Damping in the system may readily be included by adding in Eqs. (22.84) damping terms expressed in function of the modal damping ratios ξ_1 and ξ_2. Hence,

$$\begin{aligned} \ddot{z}_1 + 2\xi_2 \omega_1 \dot{z}_1 + \omega_1^2 z_1 &= \phi_{21} F_2(t) \\ \ddot{z}_2 + 2\xi_2 \omega_2 \dot{z}_2 + \omega_2^2 z_2 &= \phi_{22} F_2(t) \end{aligned} \tag{22.85}$$

The solution of Eqs. (22.85) may be expressed by Duhamel's integral which for a damped system is given from Eq. (4.24) as

$$z_1(t) = \int_{-\infty}^{\infty} \phi_{21} F_2(t - \lambda_1) h_1(\lambda_1) d\lambda_1 \tag{22.86}$$

and

$$z_2(t) = \int_{-\infty}^{\infty} \phi_{22} F_2(t - \lambda_2) h_2(\lambda_2) d\lambda_2$$

in which the unit impulse functions $h_1(t)$ and $h_2(t)$ are given for $t \geq 0$ by

$$h_1(t) = \frac{e^{-\xi_1 \omega_1 t}}{\omega_1 \sqrt{1 - \xi_1^2}} \sin\left[\omega_1 \sqrt{1 - \xi_1^2}\, t\right] \tag{22.87}$$

and

$$h_2(t) = \frac{e^{-\xi_2 \omega_2 t}}{\omega_2 \sqrt{1 - \xi_2^2}} \sin\left[\omega_2 \sqrt{1 - \xi_2^2}\, t\right] \tag{22.88}$$

and for $t < 0$ by

$$h_1(t) = h_2(t) = 0$$

The response is then obtained in terms of the displacements $u_1(t)$ and $u_2(t)$ by substituting the solution of the modal equations, $z_1(t)$ and $z_2(t)$, into Eq. (22.83).

We proceed now to the determination of the mean square value of the response, $u_1(t)$ in order to estimate its standard deviation and its confidence interval, due to random excitation. The mean square value of the response may be calculated from the autocorrelation function R_{u1} using Eqs. (22.19) and (22.20) as

$$R_{u_1}(\tau) = \lim_{T \to \infty} \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} u_1(t) u_1(t+\tau)\, dt$$

or using a more convenient notation as

$$R_{u_1}(\tau) = E[u_1(t) u_1(t + \tau)] \tag{22.89}$$

in which the letter E denotes expected or mean value of the function in the bracket. The substitution into (22.89) of $u_1(t)$ and $u_1(t + \tau)$ from Eq. (22.83) results in

$$R_{u_1}(\tau) = E[\phi_{11} z_1(t) + \phi_{12} z_2(t)][\phi_{11} z_1(t + \tau) + \phi_{12} z_2(t + \tau)]$$

or

$$R_{u_1}(\tau) = E\left[\phi_{11}^2 z_1(t) z_1(t + \tau) + \phi_{11}\phi_{12} z_1(t) z_2(t + \tau) + \phi_{12}\phi_{11} z_2(t) z_1(t + \tau) + \phi_{12}^2 z_2(t) z_2(t + \tau)\right] \tag{22.90}$$

The substitution of Eq. (22.86) into the expected value for the product of $z_1(t)\, z_1(t + \tau)$ in Eq. (22.90) results in

$$E[z_1(t) z_1(t + \tau)] = E\left[\int_{-\infty}^{\infty} \phi_{21} F_2(t - \lambda_1) h_1(\lambda_1) d\lambda_1 \int_{-\infty}^{\infty} \phi_{21} F_2(t + \tau - \lambda_2) h_1(\lambda_2) d\lambda_2\right]$$

Interchanging the symbols for expected value and integration, we obtain

$$E[z_1(t)z_1(t+\tau)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E[F_2(t-\lambda_1)F_2(t+\tau-\lambda_2)]h_1(\lambda_1)\,h_1(\lambda_2)\phi_{21}^2 d\lambda_1 d\lambda_2$$

or

$$E[z_1(t)z_1(t+\tau)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_2(\tau-\lambda_2+\lambda_1)h_1(\lambda_1)\,h_1\left(\lambda_2\phi_{21}^2 d\lambda_1\lambda_2\right.$$

since the expected value, $E\,[F_2(t-\lambda_1)F_2(t+\tau-\lambda_2)]$ is equal to the autocorrelation of the function $F_2\,(t-\lambda_2+\lambda_1)$.

Analogously, for the other terms in Eq. (22.88), we obtain

$$E[z_1(t)z_2(t+\tau)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_2(\tau-\lambda_2+\lambda_1)h_1(\lambda_1)h_2(\lambda_2)\phi_{21}\phi_{22}d\lambda_1 d\lambda_2$$

$$E[z_2(t)z_1(t+\tau)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_2(\tau-\lambda_2+\lambda_1)h_1(\lambda_2)h_2(\lambda_1)\phi_{22}\phi_{21}d\lambda_1 d\lambda_2$$

$$E[z_2(t)z_2(t+\tau)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_2(\tau-\lambda_2+\lambda_1)h_2(\lambda_1)h_2(\lambda_2)\phi_{22}^2 d\lambda_1 d\lambda_2$$

which substituted into Eq. (22.90) yields

$$\begin{aligned} R_{u_1}(\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_2(\tau-\lambda_2+\lambda_1)\big[&\phi_{11}^2\phi_{21}^2 h_1(\lambda_1)h_2(\lambda_2) + \phi_{11}\phi_{12}\phi_{21}\phi_{22}h_1(\lambda_1)h_2(\lambda_2) \\ &+ \phi_{12}\phi_{11}\phi_{22}\phi_{21}h_1(\lambda_2)h_2(\lambda_1) + \phi_{12}^2\phi_{22}^2 h_2(\lambda_1)h_2(\lambda_2)\big]d\kappa_1\, d\lambda_2 \end{aligned} \tag{22.91}$$

The spectral density function $S_{u1}(\omega)$ for the response is then given by Eq. (22.36a) as

$$S_{u_1}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R_{u_1}(\omega)e^{-i\omega t}\,d\tau \tag{22.92}$$

The multiplication of Eq. (22.91) by

$$\frac{1}{2\pi}e^{-i\omega t}d\tau = \frac{1}{2\pi}e^{i\omega(\tau-\lambda_2+\lambda_1)}e^{i\omega(\lambda_2-\lambda_1)}\,d\tau$$

followed by integration yields

$$\begin{aligned} S_{u_1}(\omega) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{2\pi}R_2(\tau-\lambda_2+\lambda_1)e^{i\omega(\tau-\lambda_2+\lambda_1)}\big[&\phi_{11}^2\phi_{21}^2 h_1(\lambda_1)h_2(\lambda_1) \\ &+ \phi_{11}\phi_{12}\phi_{21}\phi_{22}h_1(\lambda_1)h_2(\lambda_2)\big] + \phi_{12}\phi_{11}\phi_{22}\phi_{21}h_1(\lambda_2)h_2(\lambda_1) \\ &+ \phi_{12}^2\phi_{22}^2 h_2(\lambda_1)h_2(\lambda_2)\big]e^{i\omega\left((\lambda_2-\lambda_1)\right.}d\tau d\lambda_1 d\lambda_2 \end{aligned}$$

After using Eq. (22.77), $S_{u1}(\omega)$ may be expressed as

$$S_{u_1}(\omega) = \sum_{j=1}^{2}\sum_{k=1}^{2}\phi_{1j}\phi_{1k}\phi_{2j}\phi_{2k}S_2(\omega)H_j(\omega)H_k^*(\omega) \tag{22.93}$$

(22.93)

in which $H_k^*(\omega)$ is the complex conjugate of $H_k(\omega)$ and

$$S_2(\omega) = \frac{1}{2\pi}\int R_2(\tau - \lambda_2 + \lambda_1)e^{i\omega(\tau-\lambda_2+\lambda_1)}\,d\tau$$

Equation (22.93) provides the spectral density function, $S_{u1}(\omega)$, for the response at coordinate 1 of a two-degree-of-freedom system excited by a random force at coordinate 2 expressed by the spectral density function $S_{u2}(\omega)$. Now we use Eq. (22.43) to obtain the discrete spectral function, $S_{u1}(\omega_n)$, evaluated at frequency ω_n as

$$S_{u_1}(\omega) = \sum_{j=1}^{2}\sum_{k=1}^{2}\phi_{1j}\phi_{1k}\phi_{2j}\phi_{2k}C_2(\omega_n)C_2^*(\omega_n)H_j(\omega_n)H_k^*(\omega_n) \tag{22.94}$$

where $C_2(\omega_n)$ is the discrete Fourier transform coefficient of the random force at coordinate 2 and $C_2{}^*(\omega_n)$ its complex conjugate.

If the single excitation is the random force $F_1(t)$ at coordinate 1 instead of $F_2(t)$ at coordinate 2, the expression for the spectral density function for the response at coordinate 1 will be

$$S_{u_1}(\omega_n) = \sum_{j=1}^{2}\sum_{k=1}^{2}\phi_{1j}^2\phi_{1k}^2C_1(\omega_n)C_1^*(\omega_n)H_j(\omega_n)H_k^*(\omega_n) \tag{22.95}$$

Then, the generalization to a system of N degrees of freedom subjected to multiple random excitations leads to the following equation:

$$S_{u_1}(\omega_n) = \sum_{j=1}^{2}\sum_{k=1}^{2}\phi_{ij}\phi_{ik}\sum_{L=1}^{2}\sum_{m=1}^{2}\phi_{Lj}\phi_{mk}C_L(\omega_n)C_m^*(\omega_m)H_j(\omega_n)H_k^*(\omega_n) \tag{22.96}$$

where

$S_{ui}(\omega_n)$ is the discrete spectral density function of the response at coordinate i
$C_L(\omega_n)$ is the Discrete Fourier Transform coefficient for the force at coordinate L
$C_m^*(\omega_n)$ is the complex conjugate of the coefficient $C_m(\omega_n)$

22.11.3 Response to Random Excitation: *N* Degree of Freedom System

A program written in MATLAB has been developed to evaluate using Eq. (22.96) the spectral density function of the response $S_{ui}(\omega)$ at coordinate i of a multidegree-of-freedom system subjected to multiple random stationary excitations.

Illustrative Example 22.8

Determine the response for a two-degree-of-freedom structure modeled by system shown in Fig. 22.26 for which the natural frequencies have bee determined as $\omega_1 = 10.10$ rad/sec and $\omega_2 = 12.32$ rad/sec with corresponding modal shapes $\{\Phi\} = \{0.707, 0.707\}_1$ and $\{\Phi\}_2 = \{0.707, -0.707\}_2$, respectively. Samples of the applied random forces, $F_1(t)$ and $F_2(t)$ are depicted in Fig. 22.27.

Use MATLAB to determine the spectral functions of the response at the two coordinates of this system.

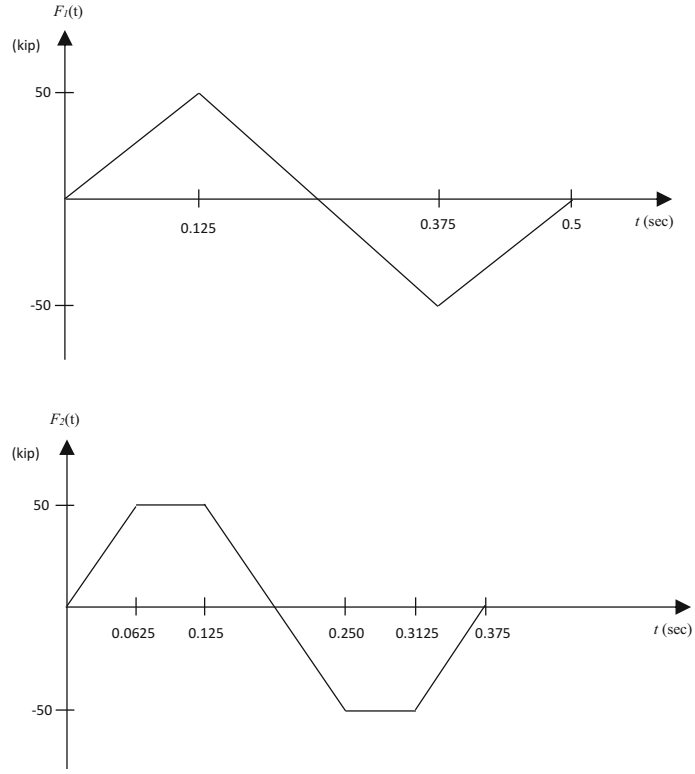

Fig. 22.27

Solution:

```
clear all
close all
clc

w1 = 10;                        %Natural frequency @ 1 mode
w2 = 12.32;                     %Natural frequency @ 2 mode
xi = 0;

omega = [w1 w2];
phi1 = [0.707; 0.707];          %Mode shape @ mode 1
phi2 = [0.707; -0.707];         %Mode shape @ mode 2

T = 1.0;                        %Time period, T(sec)
omega_bar = 2*pi/T;             %Excitation frequency (rad/sec)
M= 3;                           %Select M, M=3
N = 2^M;                        %The number of time increments N

Dt = T/N;                       %Deltat
t=0:Dt:T;                       %Time ranging from 0 to 0.64 sec with deltat

tt= length(t);                  %Total number of calculation


for i= 1:tt-1
%%%Define the function of force F1
if t(i)<=0.125
       F1(i) = 100000*t(i)/0.125;
elseif t(i) <=0.3125
       F1(i) =-800000*(t(i)-0.125)+100000;
else
       F1(i)=min(0, 800000*(t(i)-0.50)) ;
end
```

```
%%%Define the function of force F2
if t(i)<0.0625
        F2(i) = 100000*t(i)/0.125;
elseif t(i)<=0.125 && t(i)>= 0.0625
        F2(i) = 50000;
elseif t(i)<=0.250 && t(i) > 0.125
        F2(i) = -800000*(t(i)-0.250)-50000;
elseif t(i)>0.250 && t(i) <0.3125
        F2(i) = -50000;
elseif t(i)>=0.3125 && t(i) <0.375
        F2(i) = 800000*(t(i)-0.3125)-50000;
else
        F2(i) = 0;
end

%%%Define the discrete Fourier transform of the series
   Cn1 = fft(F1'/N)/1000;
   Cn2 = fft(F2'/N)/1000;

%%%Calculate frequency ratio, r_n
ifi<=N/2
omega_n(i) = (i-1)*omega_bar;       %Eq.22.58
else
omega_n(i) = -(N-(i-1))*omega_bar;  %Eq.22.58
end
for j=1:2
rn(i,j)=omega_n(i)/omega(j);        %Eq.22.59
      H(i,j)=1/(1-rn(i,j)^2);
end

     H1 = 1/omega(:,1)^2*H(:,1);            %Eqs.22.93 and 22.94 for H1
     H2 = 1/omega(:,2)^2*H(:,2);            %Eqs.22.93 and 22.94 for H2

    Su1 =
(Cn1.*conj(Cn1)).*(phi1(1,1)^2*phi1(1,1)^2*H1.*conj(H1)+phi2(1,1)^2*phi1(1,1)^2*H1.*conj(
H2)+...
         phi1(1,1)^2*phi2(1,1)^2*
H2.*conj(H1)+phi2(1,1)^2*phi2(1,1)^2*H2.*conj(H2))            %Eq.22.93

    Su2 =
(Cn2.*conj(Cn2)).*(phi1(2,1)^2*phi1(2,1)^2*H1.*conj(H1)+phi2(2,1)^2*phi1(2,1)^2*H1.*conj(
H2)+...
         phi1(2,1)^2*phi2(2,1)^2*
H2.*conj(H1)+phi2(2,1)^2*phi2(2,1)^2*H2.*conj(H2))            %Eq.22.94

end

%%%Discrete spectral density function of the response at coordinate i

Su1f = [omega_n',Su1];                      %Eq.22.96 (i=1)
Su2f = [omega_n',Su2];                      %Eq.22.96 (i=2)
```

22.12 Summary

The objective of this chapter was to introduce the fundamentals of the theory of random vibrations for application in structural dynamics. In structural dynamics the most common source of random vibration is due to explosions occurring in the vicinity of the structure. The response of a structure to earthquakes may also be predicted using random vibration theory.

A random process is described by a function of time whose value at any time is known only as a set of sample records known as an ensemble. Such a function can only be described in probabilistic terms

using the tools of statistics. The most important statistics of a random process $x(t)$ are its mean value $\bar{x}$, its mean square value $\overline{x^2}$, and its variance σ_x^2 given, respectively, by Eqs. (22.1), (22.2), and (22.3).

The most commonly used probability distribution for a random process is the normal distribution. However, when the random variable can only assume positive values (e.g., the absolute values of the peaks of vibration), the process tends to follow the Rayleigh distribution.

The autocorrelation $R_x(\tau)$ of a random variable $x(t)$ is defined by Eq. (22.19). The spectral density function $S_x(\omega)$ is defined as the Fourier transform of the autocorrelation function $R(\tau)$ [Eq. (22.36a)]. Although the spectrum of $x(t)$ can be obtained from $R_x(\tau)$, it is now more efficient to determine the spectrum of a random function from its discrete Fourier series [Eq. (22.35)] using the fast Fourier transform (FFT).

If the spectrum of the excitation function and the frequency response of a dynamic system are known, it is a simple matter to calculate the mean-square value of the response using Eq. (22.62). Knowing the mean-square value of the response and using standard probability functions (such as the Normal or the Rayleigh distributions), we can predict the response in probabilistic terms.

In this introductory chapter on random vibration, the presentation has been given in detail for single-degree-of-freedom systems and extended to multidegree-of-freedom systems using the modal superposition method. This method, as we have seen in previous chapters, transforms a system of differential equations into a set of independent or uncoupled differential equations. Each equation of this set is equivalent to the differential equation for a single degree-of-freedom system and consequently can be solved for random vibration excitation by the methods presented in this chapter.

22.13 Problems

Problems 22.1–22.5

Determine the mean and mean square values for the functions shown in Figs. P22.1, P22.2, P22.3, P22.4 and P22.5.

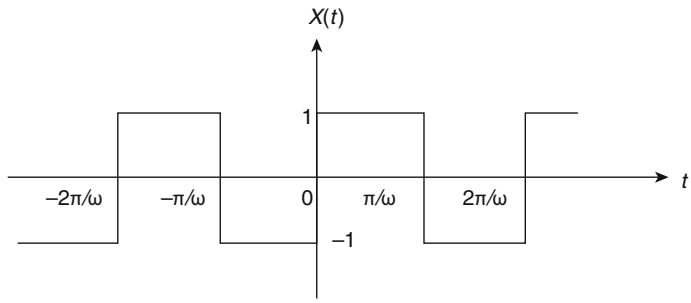

Fig. P22.1

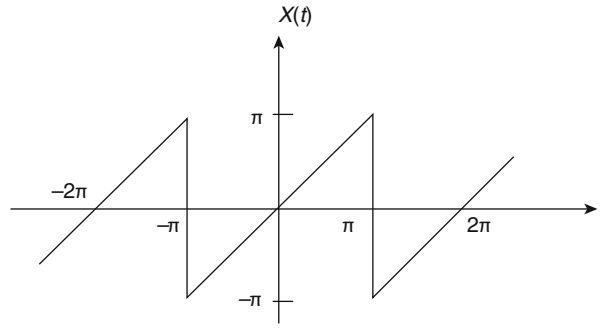

Fig. P22.2

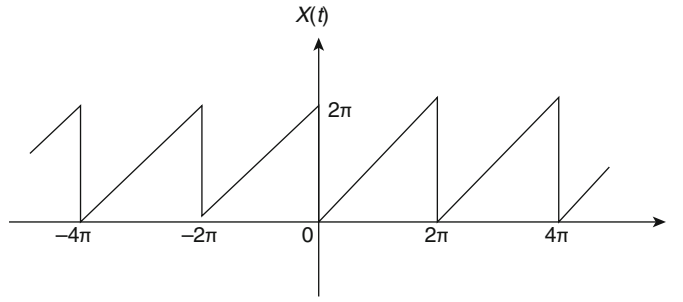

Fig. P22.3

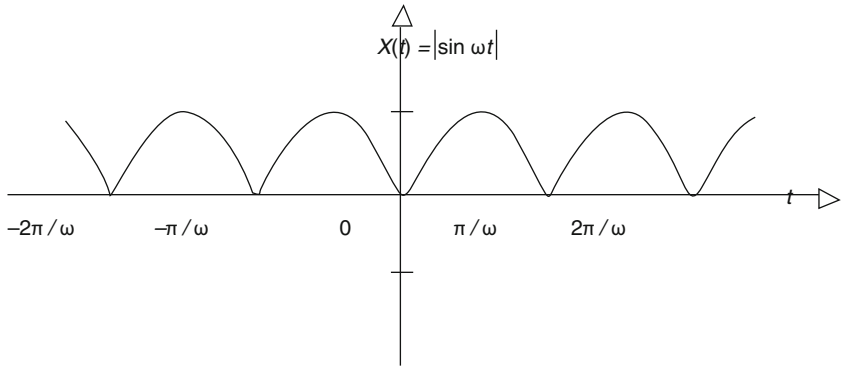

Fig. P22.4

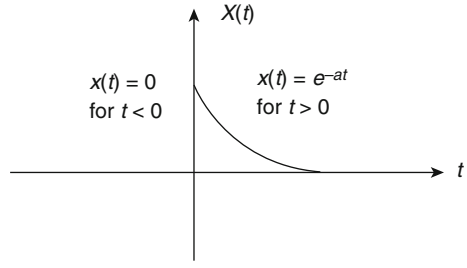

Fig. P22.5

Problems 22.6–22.10
Determine the Fourier series expansions for the periodic functions shown in Figs. P22.1, P22.2, P22.3, P22.4 and P22.5.

Problems 22.11–22.15
Determine and plot the spectral functions for the functions shown in Figs. P22.1, P22.2, P22.3, P22.4 and P22.5.

Problem 22.16
A sine wave with a steady-state component is given by

$$x(t) = A_0 + A_1 \sin \varpi t$$

Determine the mean value j and the mean-square value $\overline{x^2}$.

Problem 22.17
Determine the Fourier coefficients C_n and the spectral function for the periodic function shown in Fig. P22.17.

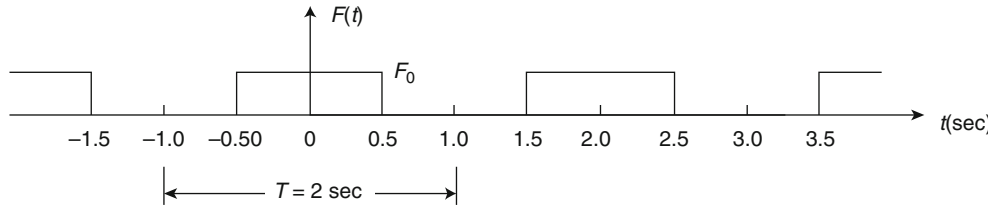

Fig. P22.17

Problem 22.18
A random force has a mean value $F = 2$ Kips and spectral density function shown in Fig. P22.18. Determine its standard deviation σ_F and its root mean square RMS_F.

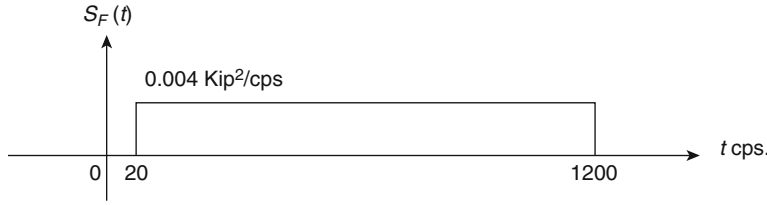

Fig. P22.18

Problem 22.19
Calculate the autocorrelation function for a stationary ergodic random process x (t), for which each sample function is a square wave of.

Each sample function is a square wave of amplitude a and period T.

Problem 22.20

Determine Fourier transform and Fourier integral representation for the function shown in Fig. P22.20.

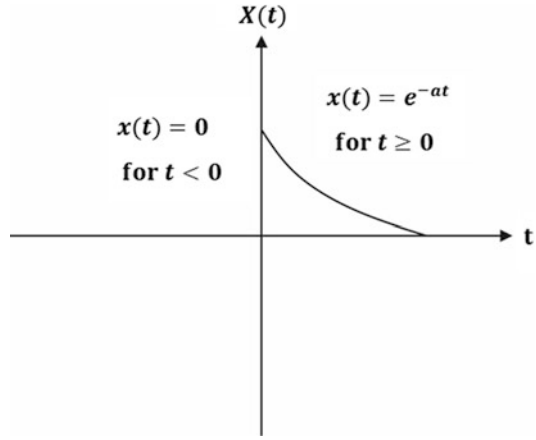

Fig. P22.20

Problem 22.21

A single degree-of freedom system with mass 1.0 lb sec^2/in, stiffness 100 lb/in, and damping $\xi = 0.20$ is excited by the force

$$F(t) = 1000\cos 5t + 1000\cos 10t + 1000\cos 15t\,(\text{lb})$$

1. Determine the spectral function and the mean square value of the response.
2. Also plot the spectra for the input force and the output response displacement.

Part VII

Earthquake Engineering

Dynamic Method

23

In Chap. 5, we introduced the concept of response spectrum as a plot of the maximum response (spectral displacement, spectral velocity, or spectral acceleration) versus the natural frequency or natural period of a single degree-of-freedom system subjected to a specific excitation. In the present chapter, we will use seismic response spectra for earthquake resistant design of buildings modeled as discrete systems with concentrated masses at each level of the building.

Since response spectral charts are prepared for single degree-of-freedom system, it is necessary to perform a transformation of coordinates to obtain the modal equations of motion, and then combine their spectral responses to obtain the maximum response of the structure. In earthquake resistant design of buildings, the maximum responses include displacements, accelerations, shear forces, overturning moments, and torsional moments.

23.1 Modal Seismic Response of Building

The equations of motion of a shear building modeled with lateral displacement coordinates at the N levels and subjected to seismic excitation at the base in the same direction of the lateral displacement, may be written, neglecting damping, from Eq. (8.35) as

$$[M]\{\ddot{u}_r\} + [K]\{u_r\} = -[M]\{1\}\ddot{u}_s(t) \tag{23.1}$$

In Eq. (23.1), $[M]$ and $[K]$ are respectively the mass and the stiffness matrices of the system, $\{u_r\}$ and $\{\ddot{u}_r\}$ are respectively the displacement and acceleration vectors (relative to the base), $\ddot{u}_s(t)$ is the function of the seismic acceleration at the base of the building, and $\{1\}$ a vector with all its elements equal to 1.

23.1.1 Modal Equation and Participation Factor

As presented in Chap. 8, the solution of Eq. (23.1) may be found by solving the corresponding eigenproblem

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M. Paz, Y. H. Kim, *Structural Dynamics*, https://doi.org/10.1007/978-3-319-94743-3_23

$$[[K]] - \omega^2 [M]] \{\Phi\} = \{0\} \tag{23.2}$$

to determine the natural frequencies $\omega_1, \omega_2, \ldots, \omega_N$ (or natural periods $T_1, T_2, \ldots, T_N$) and the modal matrix $[\Phi]$ with its columns containing the normalized modal shapes. Then the linear transformation

$$\{u_r\} = [\Phi]\{z\} \tag{23.3}$$

introduced in Eq. (23.1) yields the modal equations;

$$\ddot{z}_m + \omega_m^2 z_m = -\Gamma_m \ddot{u}_s(t) \qquad (m = 1, 2, \ldots, N) \tag{23.4}$$

in which Γ_m is the participation factor given by Eq. (8.39) as

$$\Gamma_m = \frac{\sum_{i=1}^{N} W_i \phi_{im}}{\sum_{i=1}^{N} W_i \phi_{im}^2} \tag{23.5}$$

For normalized eigenvectors, the participation factor reduces to

$$\Gamma_m = \frac{1}{g} \sum_{i=1}^{N} \phi_{im} W_i \tag{23.6}$$

because for normalized eigenvectors,

$$\sum_{i=1}^{N} \frac{W_i}{g} \phi_{im}^2 = 1$$

where g is the acceleration due to gravity. Damping may be introduced in the modal Eq. (23.4) by simply adding the damping term to this equation, namely,

$$\ddot{z}_m + 2\xi_m \omega_m \dot{z}_m + \omega_m^2 z_m = -\Gamma_m \ddot{u}_s(t) \tag{23.7}$$

where ξ_m is the modal damping ratio. For convenience Eq. (23.7), can be written with omission of the participation factor as

$$\ddot{q}_m + 2\xi_m \omega_m \dot{q}_m + \omega_m^2 q_m = \ddot{u}_s(t) \tag{23.8}$$

with the substitution

$$z_m = -\Gamma_m q_m \tag{23.9}$$

23.1.2 Modal Shear Force

The value of the maximum response in Eq. (23.8) for the modal spectral acceleration, $S_{am} = \left(\ddot{q}_m\right)_{\max}$, is found from an appropriate response spectral chart.

From Eqs. (23.3) and (23.9), the maximum acceleration a_{xm} of the mth mode at the level x of the building is given by

$$a_{xm} = \Gamma_m \phi_{xm} S_{am} \tag{23.10}$$

in which S_{am} and a_{xm} are usually expressed in units of the gravitational acceleration g.

As stated in Chap. 5, the modal values of the spectral acceleration S_{am}, the spectral velocity S_{vm}, and the spectral displacement S_{dm} are related by an apparent harmonic relationship:

$$S_{am} = \omega_m S_{\mathrm{vm}} = \omega_m^2 S_{dm}$$

or in terms of the modal period $T_m = 2\pi/\omega_m$ by

$$S_{am} = \frac{2\pi}{T_m} S_{vm} = \left(\frac{2\pi}{T_m}\right)^2 S_{dm}$$

On the basis of these relations, the modal spectral acceleration S_{am} in Eq. (23.10) may be replaced by the spectral displacement S_{dm} times ω_m^2 or by the spectral velocity S_{vm} times ω_m.

The modal lateral force F_{xm} at the level x of the building is then given by Newton's Law as

$$F_{xm} = a_{xm} W_x$$

or by Eq. (23.10) as

$$F_{xm} = \Gamma_m \phi_{xm} S_{am} W_x \tag{23.11}$$

in which S_{am} is the modal spectral acceleration in g units (Note: ASCE 7-16 requires to scale S_am multiplied by the importance factor, I_e (Table 23.1) and divided by response modifications coefficient, R (Table 23.2)) and W_x is the weight attributed to the level x of the building.

Table 23.1 Modal seismic force F_{xm} (Kip)

Level	Model 1	Model 2	Model 3	Model 4	Design F_x Values
4	46	−29	11	−6	56
3	47	5	−20	10	52
2	30	36	4	−19	51
1	12	24	14	23	38

Table 23.2 Modal torsional moment M_{txm} (Kip-ft)

Level x	Model 1	Model 2	Model 3	Model 4	Design M_{tx} values
4	221	−139	53	−31	268
3	448	−113	−41	14	464
2	590	62	−20	−77	598
1	649	178	46	31	675

The modal shear force V_{xm} at the level x of the building is equal to the sum of the seismic forces F_{xm} above that level, namely,

$$V_{xm} = \sum_{i=x}^{N} F_{im} \tag{23.12}$$

The total modal shear force V_m at the base of the building is then calculated as

$$V_m = \sum_{i=1}^{N} F_{im} \tag{23.13}$$

or using Eq. (23.11)

$$V_m = \sum_{i=1}^{N} \Gamma_m \phi_{im} W_i S_{am} \tag{23.14}$$

23.1.3 Effective Modal Weight

The effective modal weight W_m is defined by the equation

$$V_m = W_m S_{am} \tag{23.15}$$

Then, from Eq. (23.14), the modal weight is

$$W_m = \Gamma_m \sum_{i=1}^{N} \phi_{im} W_i \tag{23.16}$$

Combining Eqs. (23.5) and (23.16) results in the following important expression for the effective modal weight:

$$W_m = \frac{\left[\sum_{i=1}^{N} \phi_{im} W_i\right]^2}{\sum_{i=1}^{N} \phi_{im}^2 W_i} \tag{23.17}$$

It can be proven (Clough and Penzien 1975, pp. 559–560) analytically that the sum of the effective modal weights for all the modes of the building is equal to the total design weight of the building, that is,

$$\sum_{m=1}^{N} W_m = \sum_{i=1}^{N} W_i \tag{23.18}$$

Equation (23.18) is most convenient in assessing the number of significant modes of vibration to consider in the design. Specifically, the ASCE 7-16 (12.9.1.1) requires that, in applying the dynamic method of analysis, a sufficient number of modes are needed to estimate a combined modal mass participation of 100% of the structure's mass. Alternatively, this requirement can be satisfied by including a sufficient number of modes such that their total effective modal weight is at least 90% of the total design weight of the building. Thus, this requirement can be satisfied by simply adding a sufficient number of effective modal weights [Eq. (23.17)] until their total weight is 90% or more of the seismic design weight of the building.

23.1.4 Modal Lateral Forces

By combining Eq. (23.11) with Eqs. (23.15) and (23.16), we may express the modal lateral force F_{xm} as

$$F_{xm} = C_{xm} V_m \tag{23.19}$$

where the modal seismic coefficient C_{xm} at level x is given by

$$C_{xm} = \frac{\phi_{xm} W_x}{\sum_{i=1}^{N} \phi_{im} W_i} \tag{23.20}$$

23.1.5 Modal Displacements

The modal displacement δ_{xm} at the level x of the building may be expressed, in view of Eqs. (23.3) and (23.9), as

$$\delta_{xm} = \Gamma_m \phi_{xm} S_{dm} \tag{23.21}$$

where Γ_m is the participation factor for the mth mode, ϕ_{xm} is the component of the modal shape at level x of the building, and S_{dm} is the spectral displacement for that mode.

Alternatively, the modal displacement δ_{xm} may be calculated from Newton's Law of Motion in the form

$$F_{xm} = \frac{W_x}{g} \omega_m^2 \delta_{xm} \tag{23.22}$$

because the magnitude of the modal acceleration corresponding to the modal displacement δ_{xm} is $\omega_m^2 \delta_{xm}$. Hence, from Eq. (23.22)

$$\delta_{xm} = \frac{g}{\omega_m^2} \cdot \frac{F_{xm}}{W_x} \tag{23.23}$$

or substituting $\omega_m = 2\pi/T_m$

$$\delta_{xm} = \frac{g}{4\pi^2} \cdot \frac{T_m^2 F_{xm}}{W_x} \tag{23.24}$$

where T_m is the mth natural period. In accordance with ASCE 7-16, the value for displacement and drift quantities shall be multiplied by the quantity of C_d/I_e. The importance factor, Ie and the deflection amplification factor, C_d can be determined by Table 23.1 and 23.2, respectively.

23.1.6 Modal Drift

The modal drift $\mathbf{\Delta}_{xm}$ for the xth story of the building, defined as the relative displacement of two consecutive levels, is given by

$$\Delta_{xm} = \delta_{xm} - \delta_{(x-1)m} = <\Delta a \tag{23.25}$$

with $\delta_{0m} = 0$.

Allowable story drift is presented in Table 23.3 (ASCE 7-16: Table 12.12-1).

Table 23.3 Calculation of ratio of secondary to primary moment

Level x	Story weight W_x(Kip)	Above weight P_x (Kip)	Story drift Δ_x (in.)	Story shear V_x (Kip)	Story height H_x (in.)	$Ms/Mp = P_x\Delta_x/V_xH_x$
4	645.1	645.1	0.362	56	144	0.005
3	789.1	1434.2	0.625	97	144	0.012
2	789.1	2223.3	0.605	125	144	0.014
1	789.1	3012.4	0.441	141	144	0.012

23.1.7 Modal Overturning Moment

The modal overturning moment M_{xm} at the level x of the building which is calculated as the sum of the moments of the seismic forces F_{xm} above that level is given by

$$M_{xm} = \sum_{x+1}^{N} F_{im}(h_i - h_x) \tag{23.26}$$

where h_i and h_x are, respectively, the height of levels i and x.

The modal overturning moment M_m at the base of the building then is given by

$$M_m = \sum_{i=1}^{N} F_{im} h_i \tag{23.27}$$

23.1.8 Modal Torsional Moment

The modal torsional moment M_{tmx} at level x, which is due to eccentricity e_x between the center of the above mass and the center of stiffness at that level (measured normal to the direction considered), is calculated as

$$M_{tmx} = e_x V_{xm} \tag{23.28}$$

where V_{xm} is the modal shear force at level x.

As mentioned in Chap. 23, the ASCE 7-16 requires that an accidental torsional moment be added to the torsional moment existent at each level. The recommended way to add the accidental torsion is to offset the center of mass at each level by 5% of the dimension of the building normal to the direction under consideration.

23.2 Total Design Values

The design values for the base shear, story shear, lateral deflection, story drift, overturning moment and torsional moment are obtained by combining corresponding modal responses. Such combination

has been performed by application of the technique SRSS. This technique, as mentioned in Chap. 8, estimates the maximum modal response by calculating the square root of the sum of the squared values of the modal contributions. However, as discussed in Sect. 8.6 of Chap. 8, the SRSS technique may result in relatively large errors when some of the natural frequencies are closely spaced. This situation generally occurs in the analysis of three-dimensional structures. At the present, the more refined technique described in that section, CQC (complete quadratic contribution), and the complete quadratic combination method as modified by ASCE 4 (CQC-4), or an approved equivalent approach (ASCE 7-18 Sect. 12.9.1.3), is becoming the technique of choice for implementation in computer programs. For preliminary or hand calculation, however, the simpler technique SRSS is commonly used. The following formulas may be used to estimate maximum design values by application of the SRSS technique:

1. Design Base Shear:

$$V = \sqrt{\sum_{m=1}^{N} V_m^2} \tag{23.29}$$

where the modal base shear V_m is given by Eq. (23.15).

2. Design Lateral Seismic Force at Level x:

$$F_x = \sqrt{\sum_{m=1}^{N} F_{xm}^2} \tag{23.30}$$

where the seismic modal force F_{xm} is given by Eq. (23.19).

3. Design Shear Force at Story x:

$$V_x = \sqrt{\sum_{m=1}^{N} V_{xm}^2} \tag{23.31}$$

where the modal shear force V_{xm} at level x is given by Eq. (23.12).

4. Design Lateral Deflection at Level x:

$$\delta_x = \sqrt{\sum_{m=1}^{N} \delta_{xm}^2} \tag{23.32}$$

where the modal displacement δ_{xm} at level x is given by Eq. (23.23) or Eq. (23.24).

5. Design Drift for Story x:

$$\Delta_x = \sqrt{\sum_{m=1}^{N} \Delta_{xm}^2} \tag{23.33}$$

where the modal drift Δ_{xm} at story x is given by Eq. (23.25).

6. Design Overturning Moment at Level x:

$$M_x = \sqrt{\sum_{m=1}^{N} M_{xm}^2} \tag{23.34}$$

where the modal overturning moment M_{xm} at level x is given by Eq. (23.26).

7. Design Torsional Moment at Level x:

$$M_{tx} = \sqrt{\sum_{m=1}^{N} M_{txm}^2} \tag{23.35}$$

where the modal torsional moment M_{tum} at level x is given by Eq. (23.28).

23.3 Scaling of Results

When the base shear force calculated by the dynamic method is less than that determined by the equivalent lateral force method, then for irregular buildings, the base shear calculated by dynamic analysis shall be scaled up to match 100% of the base shear determined by the static lateral force procedure. All corresponding response parameters, including member forces and moments, shall be adjusted proportionally.

The code also stipulates that the base shear for a given direction, determined using dynamic analysis, need not exceed the value obtained by the equivalent lateral force method. In this case, all corresponding response parameters are adjusted proportionately.

Illustrative Example 23.1

Model as a plane orthogonal frame the four-story reinforced building.

Solution:

The stiffness method and static condensation method to determine the reduce stiffness matrix that correponds to the lateral coordinate at the various levels of the building. This problem requires numbering the joints of the frame consecutively excluding the fixed joint at the foundation (labeled zero) as shown in Fig. 23.1. In this model each vertical or horizontal member represents all nine elements or members along the building. The flexural stiffness values *EI* for this model are calculated as follows:

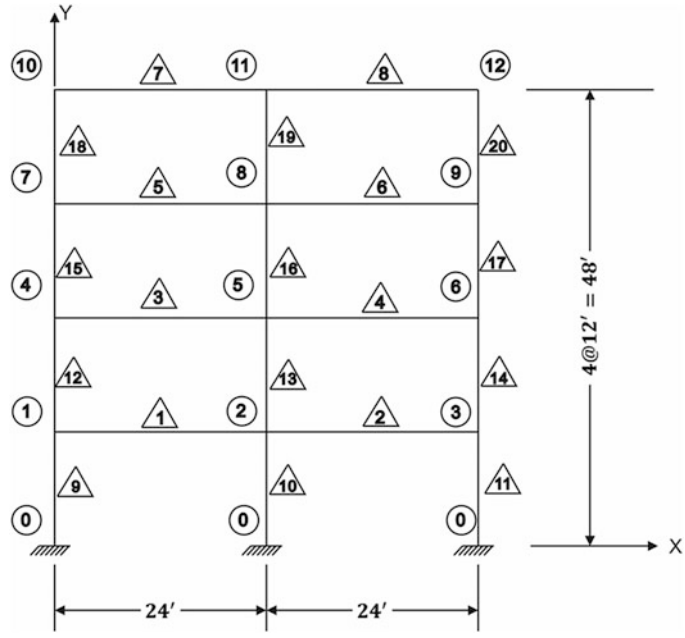

Fig. 23.1 Normalized response spectra shapes. (Reproduced from 1997 Uniform Building Code. © 1997, with permission of the publishers, the International Conference of Building Officials.)

Horizontal Members:

$$EI = 9 \times 3000 \times 12 \times 24^3/12 = 0.37325 \times 10^9 \left(\text{kip-in}^2\right)$$

Exterior Columns:

First and Second stories:

$$EI = 9 \times 3000 \times 8000 = 0.216 \times 10^9 \left(\text{kip} - \text{in}^2\right)$$

Third and Fourth stories:

$$EI = 9 \times 3000 \times 4096 = 0.111 \times 10^9 \left(\text{kip} - \text{in}^2\right)$$

Interior Columns:

First and Second stories:

$$EI = 9 \times 3000 \times 13,824 = 0.373 \times 10^9 \left(\text{kip} - \text{in}^2\right)$$

Third and Fourth stories:

$$EI = 9 \times 3000 \times 8000 = 0.216 \times 10^9 \left(\text{kip} - \text{in}^2\right)$$

Illustrative Example 23.2

Consider again the four-story reinforced concrete building. Model this building as a plane orthogonal frame and perform the seismic analysis: dynamic method.

Solution:
Seismic weights:

$$W_1 = W_2 = W_3 = 789.1 \text{ Kip}, \text{W}_4 = 645.1 \text{ Kip}$$

1. Modeling the Structure: This structure has been modeled as a plane orthogonal frame (using Program in Chaps. 13 and 9) in the solution of Illustrative Example 23.1. The reduced stiffness matrix in reference to the four lateral displacement coordinates of the building, from Example 23.1, is given by

$$[K] = \begin{bmatrix} 5611 & -2916 & 458 & -45 \\ -2916 & 3863 & -1885 & 269 \\ 458 & -1885 & 2893 & -1398 \\ -45 & 269 & -1398 & 1168 \end{bmatrix} (\text{Kip/in})$$

and the mass matrix (W_i/g) by

$$[M] = \begin{bmatrix} 2.036 & 0 & 0 & 0 \\ 0 & 2.036 & 0 & 0 \\ 0 & 0 & 2.036 & 0 \\ 0 & 0 & 0 & 1.671 \end{bmatrix} \left(\text{Kip} \cdot \sec^2/\text{in}\right)$$

2. Natural Periods and Modal Shapes: The natural frequencies and the normalized modal shapes are obtained by solving the eigenproblem

$$\left[[K] - \omega^2[M]\right]\{\phi\} = \{0\} \tag{e}$$

The roots of the corresponding characteristic equation

$$\left|[K] - \omega^2[M]\right| = \{0\} \tag{f}$$

are

$$\begin{aligned} \omega_1^2 &= 77.22, \omega_3^2 = 1939.75 \\ \omega_2^2 &= 678.60, \omega_4^2 = 4052.78 \end{aligned} \tag{g}$$

resulting in the natural frequencies ($f = \omega/2\pi$)

$$\begin{aligned} f_1 &= 1.40\,\text{cps}, \qquad f_3 = 7.01\,\text{cps} \\ f_2 &= 4.14\,\text{cps}, \qquad f_4 = 10.13\,\text{cps} \end{aligned} \tag{h}$$

or natural periods ($T = 1/f$)

$$\begin{aligned} T_1 &= 0.715 \text{ sec}, \qquad T_3 = 0.143 \text{ sec} \\ T_2 &= 0.241 \text{ sec}, \qquad T_4 = 0.099 \text{ sec} \end{aligned} \tag{i}$$

and the corresponding modal shapes arranged in the columns of the modal matrix are

$$[\Phi] = \begin{bmatrix} 0.11277 & -0.31074 & -0.35308 & 0.50518 \\ 0.27075 & -0.46790 & -0.11497 & -0.42860 \\ 0.43123 & -0.06907 & 0.50310 & 0.21286 \\ 0.51540 & 0.45452 & -0.34652 & -0.17766 \end{bmatrix} \quad \text{(j)}$$

3. Spectral Accelerations: The spectral accelerations (multiplied by the importance factor, Ie (Table 23.1) and divided by Response modifications coefficient, R (Table 23.2)) for the natural periods in Eq. (i) obtained from the spectral chart (Note: the procedure of estimations is not presented in this example. More detailed procedure is presented in the illustrative example 23.1)

$$\begin{aligned} S_{a1} &= 0.034\ g, S_{a3} = 0.061\ g \\ S_{a2} &= 0.063\ g, S_{a4} = 0.050\ g \end{aligned} \quad \text{(k)}$$

4. Effective Modal Weights: The effective modal weight is given by Eq. (23.17) as

$$W_m = \frac{\left[\sum_{i=1}^{N} \phi_{im} W_i\right]^2}{\sum_{i=1}^{N} \phi_{im}^2 W_i}$$

Values obtained for W_m ($m = 1, 2, 3, 4$) are shown in Table 23.4. This table also shows the effective modal weight as a percentage of the total seismic weight of the building.

Table 23.4 Modal effective weight and modal base shear

Mode m	Modal effective W_m (Kip)	Weight (%)	Modal base shear V_m (Kip)
1	2465	82	84
2	366	12	23
3	99	3	6
4	82	3	4
Total Weight = 3012 Kips			

5. Modal Base Shear: The modal base shear is given by Eq. (23.15) as

$$V_m = W_m S_{am}$$

Numerical values of V_m also are given in Table 23.4. The total base shear force given by Eq. (23.29) then is calculated from values in the last column of Table 23.4 as

$$V = \sqrt{(84)^2 + (23)^2 + (6)^2 + (4)^2} = 87.4 \quad \text{(Kip)}$$

6. Scaling Modal Effective Weight and Modal Base Shear: The modal values for the effective weight and for the base shear force in Table 23.4 are scaled up by the ratio r equal to 100% of the base shear determined by the equivalent lateral force method and the value for the base shear force calculated by the dynamic method. The base shear determined in Illustrative Example 23.2 using the equivalent lateral force method was equal to 140.75 Kip, while the value calculated for this example using the dynamic method is equal to 87.4 Kip. Therefore, the scaling ratio is

$$r = \frac{1.0 \times 140.75}{87.4} = 1.61$$

Table 23.5 shows the result of scaling (by the factor $r = 1.61$) the values in Table 23.4 for the modal effective weight W_m and the modal base shear V_m.

Table 23.5 Scaled values for modal effective weight and modal base shear

Mode m	Modal effective W_m (Kip)	Weight (%)	Modal base shear V_m (Kip)
1	3969	82	135
2	589	12	37
3	159	3	10
4	132	3	6
$\Sigma = 4849$ Kips			

7. Modal Seismic Force: The numerical values for the seismic coefficients C_{xm} and for the seismic lateral forces F_{xm} calculated from Eqs. (23.20) and (23.19) are shown respectively in Tables 23.1 and 23.6. The design seismic forces calculated by Eq. (23.30) are shown in the last column of Table 23.1.

Table 23.6 Modal seismic coefficient C_{xm}

Level	Model 1	Model 2	Model 3	Model 4
4	0.341	−0.780	1.141	−1.007
3	0.349	0.145	−2.027	1.476
2	0.219	0.983	0.463	−2.972
1	0.091	0.653	1.422	3.503

8. Model Shear Force: Values for the model shear force V_{xm} at level x calculated using Eq. (23.12) and results from Table 23.1 are given in Table 23.7. Design values for the story shear forces V_x calculated by Eq. (23.31) are given in the last column of Table 23.7. It should be noticed that the values shown in Table 23.7 for the modal shear force at level $x = 1$ are precisely the values for the base shear force calculated by Eq. (23.15) and shown in Table 23.5.

Table 23.7 Modal shear force V_{xm} (Kip)

Level x	Model 1	Model 2	Model 3	Model 4	Design V_x values
4	46	−29	11	−6	56
3	93	−24	−9	3	97
2	123	13	−4	−16	125
1	135	37	10	6	141

9. Modal Lateral Displacement: Values for the modal lateral displacement δ_{xm} at level x calculated from Eq. (23.24) are shown in Table 23.8. This table also shows in the last column design values for lateral displacements δ_x calculated by Eq. (23.32). In accordance with ASCE 7-16, the value for displacement and drift quantities shall be multiplied by the quantity of C_d/I_e. For this problem, the importance factor, I_e= 1 and the deflection amplification factor, C_d = 5.5.

Table 23.8 Modal lateral displacement δ_{xm} (in.)

Level	Model 1	Model 2	Model 3	Model 4	Design δ_x values
4	1.965	−0.140	0.019	−0.005	1.970
3	1.644	0.021	−0.027	0.006	1.644
2	1.032	0.144	0.006	−0.013	1.042
1	0.430	0.096	0.019	0.015	0.441

Note: In accordance with ASCE 7-16 Section 12.8.6, the deflection shall be adjusted by the deflection amplification factor and the importance factor. Herein, Cd =5.5 and Ie =1 are used in the calculation of design dx values.

10. Modal Story Drift: Table 23.9 shows the values for modal story drift Δ_{xm} calculated by Eq. (23.25). Design values for story drift obtained from Eq. (23.33) are given in the last column of this table. The maximum story drift permitted by the ASCE 7-16 (Table 23.3) should not exceed 3.6 in. (0.025 H_x), where H_x is the story height. The design values Δ_x calculated in Table 23.9 for this example are well below these limits for story drift. It should be noted that the scaling of drifts are not included in this problem. If the modal base shear (V_t) is less than C_sW, where C_s is determined in accordance with Eq. 23.16, the deflection shall be adjusted by the factor of C_sW/V_t.

Table 23.9 Modal story drift Δ_{xm} (in.)

Story	Model 1	Model 2	Model 3	Model 4	Design Δ_x values
4	0.321	−0.161	0.046	−0.012	0.362
3	0.612	−0.123	−0.033	0.019	0.625
2	0.602	0.048	−0.013	−0.028	0.605
1	0.430	0.096	0.019	0.015	0.441

11. Modal Overturning Moments: Table 23.10 shows the values for modal overturning moment calculated using Eq. (23.26). The last column of this table gives the design values for overturning moments M_x calculated by Eq. (23.34).

Table 23.10 Modal overturning moment M_{xm} (Kip-ft)

Level x	Model 1	Model 2	Model 3	Model 4	Design M_x values
4	–	–	–	–	–
3	553	−347	132	−78	671
2	1673	−629	30	−42	1788
1	3147	−475	−19	−235	3192
Base	4770	−30	97	−158	4774

12. Modal Torsional Moments: Table 23.2 shows the values calculated from Eq. (23.28) for the modal torsional moments assuming only accidental eccentricity e_x of 5% at each level x of the building (for this example $e_x = 0.05 \times 96$ ft $= 4.8$ ft). Design values for torsional moments M_{tx} calculated from Eq. (23.35) are shown in the last column of Table 23.2.

13. The P-Δ Effect: The ASCE 7-16 specifies that the P-Δ effect does not need to be considered when the ratio of the secondary moment $M_s = P_x \Delta_x$ to the overturning or primary moment M_p is less than 0.10 calculated for each level x of the building. If the modal lateral displacement in Table 23.8 is adjusted by the deflection amplification factor and the importance factor, the adjustment should be canceled out in this calculation. Therefore, the quantities shall be multiplied by the quantity of I_e/C_d. The results of the necessary calculations to evaluate this ratio are shown in Table 23.3. The largest moment ratio in Table 23.3 is 0.014, which is well below the code limit of 0.1. Consequently, there is no need to account for the P-Δ effect. In addition, the ratio shall not exceed the maximum value defined in ASCE 7-16 Sect. 12.8.7 (Eq. 12.8-17). This is not necessary in this problem. It is permitted to be conservatively 1.0.

23.4 Summary

The provisions of the ASCE 7-16 for earthquake-resistant design by the dynamic method require the modeling of the building as a discrete system with one coordinate (horizontal displacement) at each level of the building. Possible models include: (1) the shear building in which the horizontal diaphragms are assumed absolutely rigid. (2) the cantilever building in which the horizontal diaphragms are considered absolutely flexible, and (3) the plane orthogonal frame in which the stiffness of horizontal diaphragms is considered as part of the effect of horizontal members of the frame. The implementation of the dynamic method also requires the solution of the corresponding eigenproblem for the modeled structure to determine its natural periods and modal shapes.

The maximum response of the modal equations is then obtained from a spectral chart such as the one provided by the code. Modal seismic forces and modal response in terms of shear forces, overturning moments, torsional moments, lateral displacements, and story drifts are determined at each level of the building. The final design values are calculated using the SRSS technique or the CQC technique to combine the modal contributions, and the complete quadratic combination method as modified by ASCE 4 (CQC-4), or an approved equivalent approach.

The modal method in which the modal responses are combined is valid while the structure remains in linear elastic behavior as expected when subjected to an earthquake of moderate intensity. When subjected to a strong earthquake, the structure will deform in the inelastic range producing plastic deformations and structural damage. However, its ductility will provide a mechanism to absorb energy and structural stability will be maintained, although the structure may continue to undergo large deflections which may create further damage.

23.5 Problems

Problem 23.1
Perform the seismic resistant design of the 10-story building of Problem 23.1 using the dynamic method.

Problem 23.2
Solve Problem 23.2 using the dynamic method of the ASCE 7-16. The total length of the building is 120 ft and its width 24 ft. Consider only the accidental torsional eccentricity.

IBC-2018 and ASCE 7-16

24

The IBC-2018 is based on the design requirements of every structures, and portion including nonstructural components attached to structures and their supports in accordance with ASCE 7-16 Chaps. 11, 12, 13, 15, 17 and 18. In 2000, the UBC is replaced by the IBC published by the international Code Council (ICC). Current IBC-2018 adopts Risk Category instead of Seismic Use Group. In addition, the latest IBC-2018 classifies every building in a Seismic Design Category which determines the analysis procedure to be used, the maximum allowed height and drift limitations.

24.1 Response Spectral Acceleration: S_S, S_1

The reader may recall that the response spectral acceleration is defined as the maximum acceleration of a structural system, modeled as a linear single-degree-of-freedom system of mass m and stiffness k subjected to a specific time- history excitation at its base. For such a system, the maximum response (maximum displacement or maximum acceleration) is only a function of its natural period $\left(T = 2\pi\sqrt{m/k}\right)$ and of damping c expressed as a fraction of the critical damping $\left(\xi = c/c_{cr} \text{ with } c_{cr} = 2\sqrt{km}\right)$. In the IBC-2000 and current IBC-2018, the earthquake time-history excitation assumed at a given geographic location is designated as "Maximum Considered Earthquake" (MCE). The response spectral acceleration of a structural system with a natural period, $T_s = 0.2$ sec (short period) and with a natural period $T_1 = 1$ sec is obtained from response acceleration of two sets of maps prepared for various the regions of the United States. These maps are available from the Federal Emergency Management Agency (FEMA). They provide contour lines for the values of the response spectral acceleration S_S for the short period ($T_s = 0.2$ sec) and for the response spectral acceleration S_I for period $T_1 = 1$ sec. Interpolation or the value of the next higher contour line may be used for sites located between contour lines. Because these two values, S_S and S_1 are obtained from maps, they are usually referred to as mapped response spectral accelerations. These maps provide the response spectral acceleration for an assumed damping in the structure equal to 5% of the critical damping and for a type of soil classified as Site Class B (rock). For a different type of soil other than rock, the mapped spectral response accelerations must be modified by factors that depend on the Site Class as described in the Sect. 24.2.

The Spectral Acceleration Maps are provided in two sets of eight maps each; a set of maps for the short period (T = 0.2 sec) a set for the long period (T = 1.0 sec).

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M. Paz, Y. H. Kim, *Structural Dynamics*, https://doi.org/10.1007/978-3-319-94743-3_24

Eectronic values of mapped acceleration parameters, and other seismic design parameters, are provided at the USGS Web site at http://earthquake.usgs.gov/designmaps, or through the SEI Web site at http://content.seinstitute.org.

24.2 Soil Modified Response Spectral Acceleration: S_{MS}, S_{M1}

To obtain the soil modified response spectral acceleration S_{MS} for the short period and S_{M1} for the period $T_1 = 1$ sec the values of the mapped response spectral accelerations S_S and S_1, for the short period ($T_s = 0.2$ sec) and for the period $T_1 = 1$ sec must be modified by Site Class Coefficients designated, respectively, as F_a and F_v, namely,

$$S_{MS} = F_a S_S \tag{24.1}$$

and

$$S_{M1} = F_v S_1 \tag{24.2}$$

where F_a is the Site Class Coefficient for the short period and F_v is the Site Class Coefficient for period $T_1 = 1$ sec. Numerical value for the coefficients F_a and F_v are given, respectively, in Tables 24.1 and 24.2 as functions of the Site Class and the values for S_S and S_1. In Tables 24.1 and 24.2, straight-light interpolation for intermediate values of mapped spectral response acceleration can be used for both short and long periods. Site Class classification and definition are described in Sect. 24.5.

24.3 Design Response Spectral Acceleration: S_{DS}, S_{D1}

The Design Response Spectral Accelerations (for 5% damping), S_{DS} for the short period and S_{D1} for the period $T = 1$ sec are calculated by

$$S_{DS} = \frac{2}{3} S_{MS} \tag{24.3}$$

and

$$S_{D1} = \frac{2}{3} S_{M1} \tag{24.4}$$

where S_{MS} is the Soil Modified Response Spectral Acceleration for the short period given by Eq. (24.1) and S_{M1} is the Soil Modified Earthquake Response Spectral Acceleration for the period $T = 1$ sec given by Eq. (24.2).

Table 24.1 Site coefficient (F_a) [IBC-2018: Table 1613.2.3 (1)]

Site class	Mapped risk targeted maximum considered earthquake (MCE_R) Spectral response acceleration at short periods					
	$S_S \leq 0.25$	$S_S = 0.50$	$S_S = 0.75$	$S_S = 1.00$	$S_S = 1.25$	$S_S \geq 1.25$
A	0.8	0.8	0.8	0.8	0.8	0.8
B	0.9	0.9	0.9	0.9	0.9	0.9
C	1.3	1.3	1.2	1.2	1.2	1.2
D	1.6	1.4	1.2	1.1	1.0	1.0
E	2.4	1.7	1.3	[a]	[a]	[a]
F	[a]	[a]	[a]	[a]	[a]	[a]

[a]Values shall be determined in accordance with Section 11.4.8 of ASCE 7

Table 24.2 Site coefficient (F_v) [IBC-2018: Table 1613.2.3(2)]

Site class	Mapped risk targeted maximum considered earthquake (MCE_R) Spectral response acceleration at 1-Second period					
	$S_1 \leq 0.1$	$S_1 = 0.2$	$S_1 = 0.3$	$S_1 = 0.4$	$S_1 = 0.5$	$S_1 \geq 0.6$
A	0.8	0.8	0.8	0.8	0.8	0.8
B	0.8	0.8	0.8	0.8	0.8	0.8
C	1.5	1.5	1.5	1.5	1.5	1.4
D	2.4	2.2[a]	2.0[a]	1.9[a]	1.8[a]	1.7[a]
E	4.2	3.3[a]	2.8[a]	2.4[a]	2.2[a]	2.0[a]
F	[b]	[b]	[b]	[b]	[b]	[b]

[a]See requirements for site-specific ground motions in Section 11.4.8 of ASCE 7
[b]Values shall be determined in accordance with Section 11.4.8 of ASCE 7

24.4 Site Class Definition: A, B . . .F

The classification of the Site Class (A, B, C, D, E or F) where the structure is located is based on the average properties in the top 100 ft of the soil profile according to the description given in Table 24.3. The Site Class is to be estimated or measured at the building site by a geotechnical engineer or by a geologist/seismologist.

Table 24.3 Site class definitions [ASCE 7-16: Table 20.3-1]

Site class	Shear wave velocity, $\bar{v}_s$	Standard penetration blow/ft, $\bar{N}$ or $\bar{N}_{ch}$	Undrained shear strength, $\bar{s}_u$
A. Hard rock	>5000 *ft/s*	NA	NA
B. Rock	2500 – 5000 *ft/s*	NA	NA
C.Very dense soil and soft rock	1200 – 2500 *ft/s*	>50	>2000 *psf*
D. Stiff soil	600 – 1200 *ft/s*	15 *to* 50	1000 – 2000 *psf*
E. Soft clay soil	<600 *ft/s*	<15	<1000 *psf*
	Any profile with more than 10 ft of soil having the following characteristics: Plasticity index $PI > 20$ Moisture content $w \geq 40\%$ Undrained shear strength, $\bar{s}_u < 500$ *psf*		
F. Soil requiring site response analysis in accordance with ASCE 7-16 Section 21.1	ASCE 7-16 Section 20.3.1		

24.5 Risk Category and Seismic Importance Factor (I_e)

Each Structure is assigned a Seismic Risk Category based on the use of the building. An Importance Factor (I_e) is then assigned to each Risk Category. Four category designations (I, II, III and IV [defined in IBC-2018 Table 1604.5 and ASCE-16 Table 1.5-1]) and corresponding values of the Importance Factor (I_e) are defined and assigned as described in Table 24.4.

Table 24.4 Importance factor (I_e) by risk category of buildings and other structures for earthquake loads [IBC-2018: Table 1604.5 and ASCE 7-16: Table 1.5-2]

Brief description of occupancy or functions of structure (More detail in IBC 2018: Table 1604.5)	Risk category	Seismic importance factor, I_e
Miscellaneous structures	I	1.0
Standard occupancy	II	1.0
Hazardous structure	III	1.25
Essential structures	IV	1.50

24.6 Seismic Design Category (A, B, C, D, E and F)

The IBC-2018 stipulates that every structure must be designed and constructed to resist the effects of earthquake ground motions as required for the assigned Seismic Design Category. This assignment is based on the Risk Category described in Sect. 24.5 and the design response spectral acceleration coefficients S_{DS} and S_{D1}, determined in accordance with Sect. 24.3. Each building and structure should be assigned to the most severe seismic design category obtained from Table 24.5 or from Table 24.6, irrespective of the actual fundamental period, T, of the structure.

Where $S1$ is less than or equal to 0.04 and S_S is less than or equal to 0.15, the structure is permitted to be assigned to Seismic Design Category A.

Table 24.5 Seismic design category for short period response acceleration [IBC-2018: 1613.2.5(1)]

	Risk category		
VALUE OF S_{DS}	I or II	III	IV
$S_{DS} < 0.167$ g	A	A	A
$0.167 \text{ g} \le S_{DS} < 0.33$ g	B	B	C
$0.33 \text{ g} \le S_{DS} < 0.50$ g	C	C	D
$0.50 \text{ g} \le S_{DS}$	D	D	D

Table 24.6 Seismic design category for T = 1 sec period response acceleration [IBC-2018: 1613.2.5(2)]

	Risk category		
VALUE OF S_{D1}	I or II	III	IV
$S_{D1} < 0.067$ g	A	A	A
$0.067 \text{ g} \le S_{D1} < 0.133$ g	B	B	C
$0.133 \text{ g} \le S_{D1} < 0.20$ g	C	C	D
$0.20 \text{ g} \le S_{D1}$	D	D	D

Note: Seismic Risk Category I, II, and III structures located on sites with mapped maximum considered earthquake spectral response acceleration at 1-sec period, S_1, equal to or greater than 0.75 g, shall be assigned to Seismic Design Category E, and Risk Category IV structures located on such sites shall be assigned to Seismic Design Category F

The ASCE 7-16 provides for an exception in the determination of the Response Spectral Acceleration, which in turn affects the value for the Design Response Spectral Acceleration and the final Seismic Design Category for the structure.

Exception: For regular structures having five or fewer number of stories and having a fundamental period $T \leq 0.5$ sec (see Section 24.8 for estimation of T), the value for the mapped short period needs not exceed $S_S = 1.5$g (ASCE 7-16 Section 12.8.1.3).

The MCE_R response can be determined by multiplying the design response spectrum by 1.5 in accordance with ASCE 7-16 (Section 11.4.6).

24.7 Design Response Spectral Curve: S_a Vs. T

The general procedure to plot the Design Response Spectrum Curve for a given geographical location results in a plot as the one shown in Fig. 24.1. This plot provides the Response Spectral Acceleration S_a as a function of the fundamental period T of the structure. The construction of the Design Response Spectral Curve requires the calculation of the following parameters:

$$T_0 = 0.2\frac{S_{D1}}{S_{DS}} \quad \text{and} \quad T_s = \frac{S_{D1}}{S_{DS}} \tag{24.5}$$

where S_{DS} and S_{D1} are, respectively, the Design Response Spectral Acceleration for the short period [Eq. (24.3)] and the Design Response Spectral Acceleration for the period T = 1 sec [Eq. (24.4)]. T_L is the long-period transition period(s).

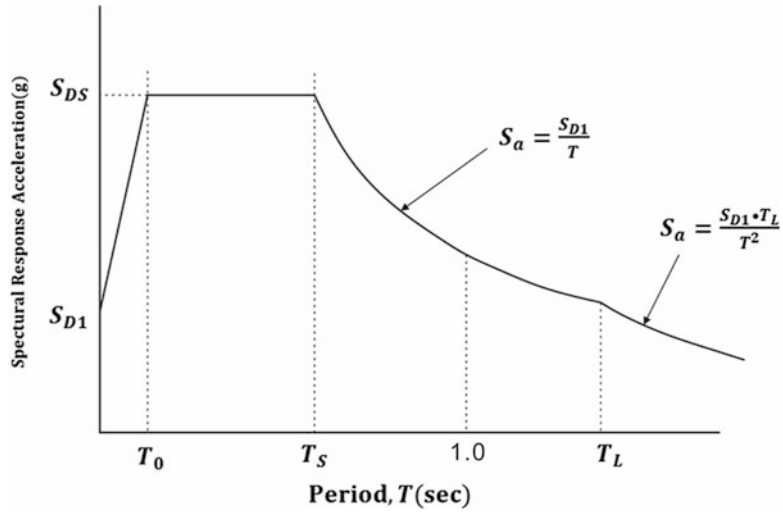

Fig. 24.1 Design response spectrum plot

The construction of the Design Response Spectrum Plot for a given geographical location such as the plot shown in Fig. 24.1, can be obtained by entering the values of T_0, T_s, as defined in Eq. (24.5) and 1.0 in the abscissa and the values of the Design Spectral Acceleration, S_{D1} and S_{Ds}, in the ordinate axis Then, on the left, two straight lines are drawn through established points, (0, S_{D1}), (T_0, S_{Ds}) and (T_s, S_{Ds}). For the period of $0 < T < T_0$, the straight line can be expressed as, $S_a = S_{DS}\left(0.4 + 0.6\frac{T}{T_0}\right)$.

Finally, a hyperbolic curve defined by the function $S_a = S_{D1}/T$ is drawn on the right portion of the plot from $T_S < T < T_L$ as shown in Fig. 24.1. Where $T > T_L, S_a = \frac{S_{D1}T_L}{T^2}$ (Fig. 24.2).

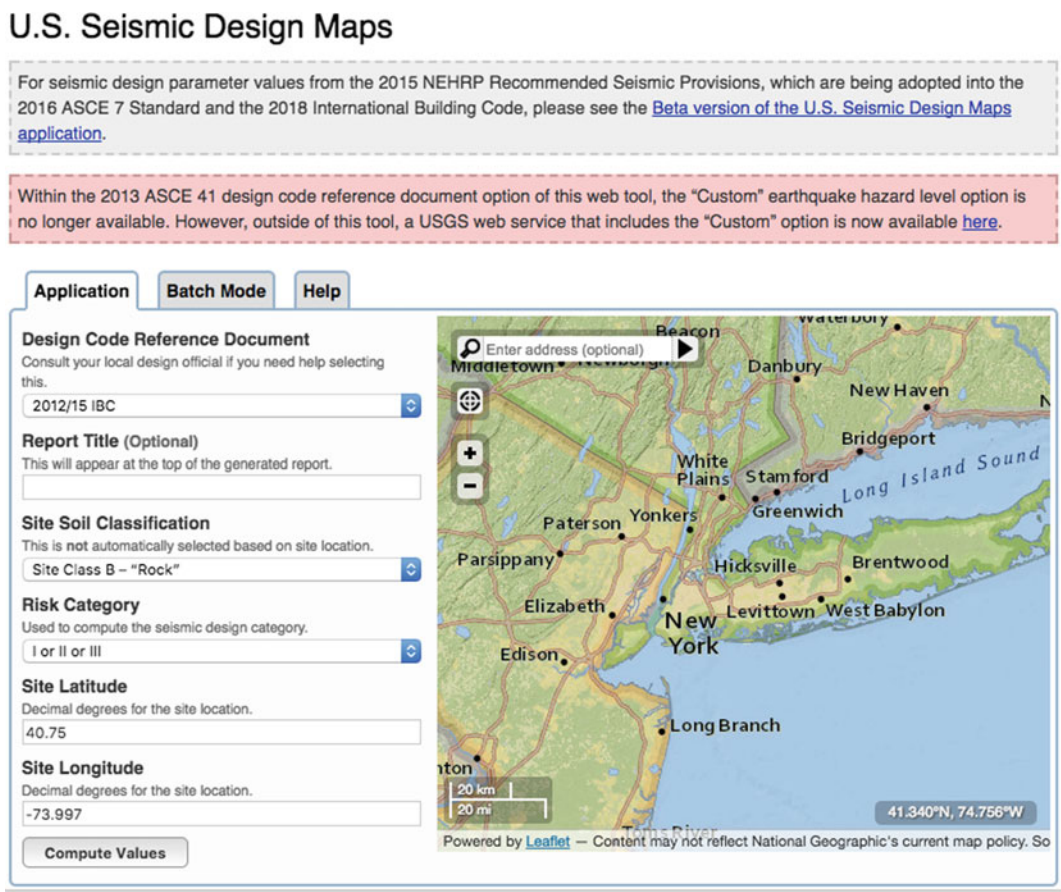

Fig. 24.2 Screen caption for the inputs for Illustrative Example 24.1. (Source: USGS Web site)

Illustrative Example 24.1

Use the USGS Seismic Design tool to determine Seismic Design Category at a location with Zip Code 10001 corresponding to Central Latitude = 40.750 and Central Longitude = −73.997. Assume Risk Category = I or II and Soil Class = D (Stiff Soil). Using the Tables 24.5 and 24.6, $F_a = 1.576$ and $F_v = 2.40$.

Results: The following results are as follows:

For $T_s = 0.2$ sec: $S_S = 0.280$ g, $S_{MS} = F_a Ss = 0.441$ g,
$S_{DS} = \frac{2}{3} S_{MS} = 0.294$ g, Seismic Design Category = B
For $T_1 = 1.0$ sec: $S_1 = 0.072$ g, $S_{M1} = F_V S_1 = 0.172$ g,
$S_{D1} = \frac{2}{3} S_{M1} = 0.115$ g, Seismic Design Category = B
If two categories are different, the most severe category must be used. Therefore, the Seismic Design Category is B.

24.8 Determination of the Fundamental Period

The ASCE 7-16 allows the determination of the fundamental period of the building using: (1) approximate empirical formulas, or (2) calculations by rational analysis using structural properties of the resisting elements in a properly substantiated analysis. In this relation, the Code provides the following information:

1. Natural Period determined using approximate formulas
   The fundamental period of the building may be taken as the approximate value T_a calculated by the following formula:

$$T_a = C_t h_n^x \tag{24.6}$$

where

h_n = total height of the building in feet, C_t and x are determined from Table 12.8-2 (ASCE 7-16). These values are depending on structural types (Table 24.7).

Table 24.7 Values of approximate period parameters C_t and x[ASCE 7-16: Table 12.8-2]

Structure type	C_t	x
Moment-resisting frame system in which the frames resist 100% of the required seismic force and are not enclosed or adjoined by components that are more rigid and will prevent the frames from deflecting where subjected to seismic forces:		
Steel moment-resisting frames	0.028	0.8
Concrete moment-resisting frames	0.016	0.9
Steel eccentrically braced frames in accordance with Table 12.2-1 lines B1 or D1 (ASCE 7-16)	0.03	0.75
Steel buckling-restrained braced frames	0.03	0.75
All other structural systems	0.02	0.75

Alternatively, the fundamental natural period of moment-resisting frames (steel or concrete) not exceeding 12 stories in height and having a minimum story height of 10 ft may be approximately determined as

$$T_a = 0.1N \tag{24.7}$$

where N is the number of stories.

2. Natural Period calculated by rational analysis.
   The fundamental period of the building, T, in the direction under consideration may be calculated using the structural properties of resisting elements in a properly substantiated analysis. However, the ASCE 7-16 requires that the calculated fundamental period T be less or equal to the coefficient C_u times the value of the period obtained by an approximate formula, that is

$$T \leq C_u T_a \tag{24.8}$$

where the upper limit coefficient C_u is given in Table 24.8 as a function of the design response spectral acceleration S_{D1} corresponding to the period $T = 1$ sec.

Table 24.8 Upper limit coefficient C_u [ASCE 7-16: Table 12.8-1]

Design response spectral acceleration S_{D1}	Upper limit coefficient C_u
<0.4	1.4
0.3	1.4
0.2	1.5
0.15	1.6
>0.1	1.7

Note: Upper limit calculated for T does not apply for interstory drift determination

24.9 Minimum Lateral Force Procedure [ASCE 7-16: Section 1.4.3]

The Minimum Lateral Force Procedure is applicable to regular and irregular structures assigned to Seismic Design Category A in accordance with ASCE 7-16 Section 11.7. This procedure requires a complete lateral-force-resistant system designed to resist minimum forces, F_x, simultaneously applied at the various levels of the building as shown in Fig. 24.3. These minimum forces are calculated by

$$F_x = 0.01 \ w_x \tag{24.9}$$

in which w_x is the seismic weight allocated to level x of the building.

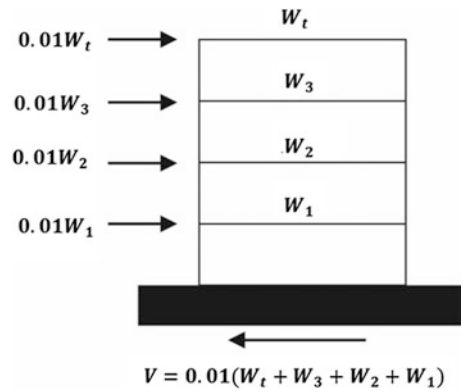

Fig. 24.3 Force distribution for Minimum Lateral Force Procedure

In the Minimum Lateral Force Procedure, the design seismic forces may be applied independently in two orthogonal directions (orthogonal combined effects are permitted to be neglected).

24.10 Simplified Lateral Force Analysis Procedure [ASCE 7-16: Section 12.14.8 and IBC 2018 Section 1613.2.5.2]

The use of the Simplified Analysis Procedure is permitted for one- or two- story bearing wall structures with limited irregularity (e.g, light-frame wood structural bearing wall). The simplified lateral-force procedure is only applicable to structures a permitted by Table ASCE 7-16:

Table 12.6-1. Generally, this is not intended to use for routine practice. Also, the estimated base shear is generally conservative compared to other IBC/ASCE lateral load procedure.

24.10.1 Seismic Base Shear

The seismic base shear, V, in a given direction shall be determined with the following equation:

$$V = \frac{1.2\,S_{DS}}{R} W \tag{24.10}$$

where

S_{DS} = The Design Response Spectral Acceleration for the Short Period. (Eq. 24.3).
R = The Response Modification Factor from Table 24.9.
W = Seismic weight of the structure that includes the dead weight and any permanent weight on the building. It also includes (1) a minimum of 25% of the live load, (2) partition load or a minimum of 10 pounds per square foot where partition load is included in the floor load design, (3) total operating weight of permanent equipment, and (4) twenty percent of flat roof snow load where the design flat roof snow load exceeds 30 pounds per square foot.
Note: The design story drift (Δ) may be taken as 1% of story height.

24.10.2 Response Modification Factor *R*

The Response Modification Factor R used in the calculation of base shear force V and of the lateral forces F_x, respectively, given by Eqs. (24.10) and (24.11) serve to reduce the design loads to account for the ductility in the structural system as well as for increase damping as the structure is subjected to large deformations beyond the elastic range. Table 24.9 contains an abbreviated set of values for R obtained from a much more detailed table provided by ASCE 7-16.

Table 24.9 Response modification factors R and deflection amplification factor C_d [ASCE 7-16: Abbreviated Table 12.2-11]

Basis seismic force resisting system	Response modifications coefficient, R	Deflection amplification factor, C_d
Bearing Wall Systems		
Special reinforced concrete shear walls	5 1/2	5
Ordinary reinforced concrete shear walls	4 1/2	4
Special reinforced masonry shear walls	5	3 1/2
Ordinary reinforced masonry shear walls	2 1/2	1 3/4
Building Frame Systems		
Special steel concentrically braced frames	6	5
Special reinforced concrete shear walls	6	5
Moment Resisting Frame Systems		
Special steel moment frames	8	5 1/2

(*continued*)

Table 24.9 (continued)

Basis seismic force resisting system	Response modifications coefficient, *R*	Deflection amplification factor, C_d
Intermediate steel moment frames	6	−5
Special steel truss moment frames	7	5 1/2
Ordinary steel moment frames	4	3 1/2
Special reinforced concrete moment frames	8	5 1/2
Dual Systems with Intermediate Moment Frames		
Special steel concentrically braced frames[a]	6	5
Ordinary steel concentrically braced frames[a]	5	4 1/2
Special reinforced concrete shear walls	6	5
Ordinary reinforced concrete shear walls	5 1/2	4 1/2
Inverted Pendulum Systems and Cantilevered Column Systems		
Special steel moment frames	2 1/2	2 1/2
Ordinary steel moment frames	1 1/4	2 1/2

[a]Ordinary moment frame is permitted to be used in lieu of intermediate moment frame in Seismic Design Categories B and C

24.10.3 Vertical Distribution of Lateral Forces

Lateral equivalent forces, F_x, applied at each level of the building shall be calculated by the following equation:

$$F_x = \frac{F S_{DS}}{R} w_x \tag{24.11}$$

where

w_x is the portion of the Seismic Weight, W allocated at level x of the building, and F is 1.0 for one-story building, 1.1 for two-story building, and 1.2 for three-story building.

24.11 Equivalent Lateral Force Procedure [ASCE 7-16: Section 12.8]

The earthquake resistant design provisions in the International Building Code (IBC-2018) refers to mainly on the ASCE 7-16. Originally, 1997 NEHRP publication (National Earthquake Hazard Reduction Program) that is distributed by FEMA (Federal Emergency Management Agency) are used to develop IBC-2000. The key provisions of the ASCE 7-16 for the Equivalent Seismic Lateral Force Method are presented and in this section.

The ASCE 7-16 stipulates that the structure should be designed for a total base shear force calculated by the formula:

$$V = C_S W \tag{24.12}$$

in which W is seismic weight of the structure that includes the dead weight and any permanent weight

on the building. It also includes (1) a minimum of 25% of the reduced live load, (2) partition load or a minimum of 10 pounds per square foot where partition load is included in the floor load design, (3) total operating weight of permanent equipment, and (4) twenty percent of flat roof snow load where the design flat roof snow load exceeds 30 pounds per square foot.

The coefficient C_S is the Seismic Response Coefficient given by

$$C_S = \frac{S_{DS}}{R/I_e} \tag{24.13}$$

in which

S_{DS} is the Design Spectral Acceleration for short period defined by Eq. (24.3).
R is the response modification factor from Table 24.9.
I_e is the Occupancy Importance Factor described in Sect. 24.8.

The value of the seismic response coefficient, C_S, calculated by Eq. (24.13) cannot be less than 0.01, nor can it exceed the following:

When $T \leq T_L$,

$$C_S \leq \frac{S_{D1}}{(R/I_e)T} \tag{24.14a}$$

When $T > T_L$,

$$C_S \leq \frac{S_{DS}T_L}{T^2(R/I_e)} \tag{24.14a}$$

but shall not be less than

$$C_S \geq 0.044S_{DS}I_e \geq 0.01 \tag{24.15}$$

and for buildings in categories E and F and buildings for which $S_1 \geq 0.6$ g, the value of CS shall not be less than

$$C_S \geq \frac{0.5S_1}{R/I_e} \tag{24.16}$$

where

R is the Response Modification Factor from Table 24.9.
I_e is the Importance Factor given in Table 24.4.
S_{DI} is the Design Response Spectral Acceleration defined by Eq. (24.4).
S_1 is the mapped earthquake response spectral; acceleration at T = 1-sec period determined in accordance with Sect. 24.1.
T_L is the long-period transition period in ASCE 7-16 Section 11.4.5.

24.11.1 Distribution of Lateral Forces [ASCE 7-16: Section 12.8.3]

The total base shear force V calculated from Eq. (24.12) is distributed over the height of the structure as a lateral force, F_x at each level calculated by

$$F_x = \frac{w_x h_x^k}{\sum_{i=1}^{N} w_i h_i^k} V \tag{24.17}$$

where

V = total base shear
N = total number of stories above the base of the building
h_x = height of level x
w_x = seismic weight assigned to 5 the x level of the building
k = 1.0 for buildings having a period $T \leq 0.5$ sec
k = 2.0 for buildings having a period $T \geq 2.5$ sec
k is determined by linear interpolation for buildings having a period 0.5 < T < 2.5 sec

The distribution of the lateral forces F_x is shown in Fig. 24.4 for a multistory building. The ASCE 7-16 stipulate that the force F_x at level x, be applied over the area of the building according to the mass distribution at that level.

The story shear, V_x, is the sum of the force F_x above that story. The seismic design story shear in any story shall be determined from the following equation:

$$V_x = \sum_{i=x}^{n} F_i \tag{24.18}$$

where F_i = the protion of the seismic base shear (V) induced at level, i.

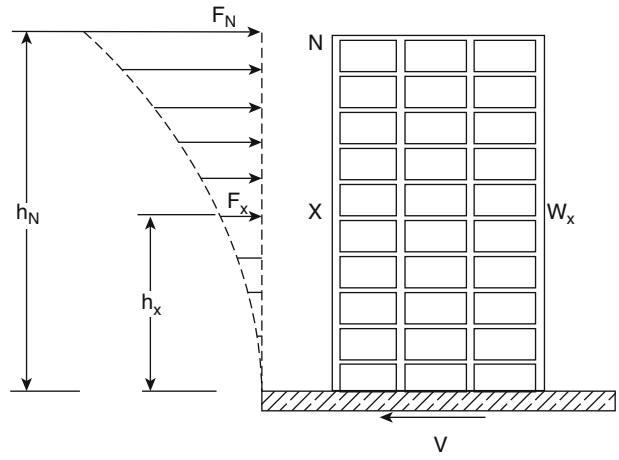

Fig. 24.4 Vertical distribution of the base shear force

24.11.2 Overturning Moments [ASCE 7-16: Section 12.8.5]

The ASCE 7-16 requires that overturning moments be determined at each level of the building. The overturning moment is calculated by static from the equivalent lateral forces F_x [Eq. (24.17)] applied at the levels of the building above the level under consideration. However, the code allows for a reduction in the design overturning moment. Hence, the overturning moment M_x at level x of the building is given by

$$M_x = \sum_{i=x+i}^{N} F_i(h_i - h_x) \tag{24.19}$$

In accordance with ASCE 7-16 Section 2.13.4, overturning effects at the soil–foundation interface are permitted to be taken as at least 75% for foundations of structures that satisfy both of the following conditions. This 25% reduction is permitted when higher mode are unlikely to occur simultaneously with mode 1.

24.11.3 Horizontal Torsional Moment

The ASCE 7-16 states that consideration should be given for the increased shear force resulting from horizontal torsion where diaphragms are not flexible. Diaphragms are considered flexible when the maximum lateral deformation of the diaphragm is more than twice the average story drift of the associated stories. The torsional moment at a given story results from the difference in location of the center of the mass between the applied seismic lateral forces at the levels above that story and the center of stiffness of the resisting elements of the story. The code also requires that when diaphragms are not flexible an accidental eccentricity be added by displacing the center of mass in five percent of the building dimension at that level perpendicular to the direction of the applied forces.

Further provisions in the code account for torsional irregularities in buildings in Seismic Design Categories C through F by increasing the accidental torsion by an amplification factor A, determined as

$$A = \left(\frac{\delta_{\max}}{1.2\delta_{avg}}\right)^2 \leq 3.0 \tag{24.20}$$

Where

$\delta_{\max}$ = The maximum displacement at level x
δ_{avg} = The average displacement at the extreme points of the structure at level x

24.11.4 *P*-Delta Effect (P–Δ) [ASCE 7-16: Section 12.8.7]

The $P - \Delta$ effect was presented and discussed in Sect. 24.10 of Chap. 24. The ASCE 7-16 treats the $P - \Delta$ effects in exactly the same manner as in the UBC-97, except that the calculation of the ratio θ_x, at level x, between the secondary overturning moment, $M_x^{'}$, and the primary moment M_x includes the Deflection Amplification Factor C_d. That is, in the ASCE 7-16 the ratio θ_x is calculated by

$$\theta_x = \frac{M_x'}{M_x} = \frac{P_x \Delta_x I_e}{V_x h_{sx} C_d} \tag{24.20}$$

where:

P_x = total weight at level x and above, no individual load factor need exceed 1.0.
Δ_x = Drift of story x
V_x = shear force of story x
h_{sx} = height of story x
C_d = Deflection Amplitude Factor (Table 24.9)
I_e = Importance Factor determined in accordance to Sect. 24.5

24.11.5 Story Drift [ASCE 7-16: Sect. 12.8.6]

The ASCE 7-16 (Sect. 12.8.6) specifies that the design deflection δ_x, at level x, be determined in accordance with the following equation:

$$\delta_x = \frac{C_d \delta_{xe}}{I_e} \tag{24.21}$$

where

C_d = Deflection Amplification Factor (Table 24.9)
δ_{xe} = Deflection determined by an elastic analysis of the seismic-force-resisting system
I_e = Importance Factor determined in accordance to Sect. 24.5

The design story drift, Δ, is computed as the difference of the lateral deflections of the center of mass at the top and bottom levels of the story under consideration. However, for structures assigned to Seismic Design Categories C, D, E or F (see Sect. 24.6) with plan irregularities, the design story drift Δ shall be computed as the largest difference of the deflection along any of the edges of the structure at the top and bottom levels of the story under consideration.

The ASCE 7-16 specified a maximum allowable story drift Δ_a which depends on the type of building and on the Risk Category. Table 24.10 provides values for Δ_a, the allowable story drift. Therefore, the limitation on the story drift, Δ_x may be expressed as:

$$\Delta_x = (\delta_x - \delta_{x-1}) \leq \Delta_a \tag{24.22}$$

where

δ_x is the lateral displacement at level x calculated by Eq. (24.21).
Δ_a is the allowable story drift from Table 24.10.

Table 24.10 Allowable story drift (Δ_a) [ASCE 7-16: Table 12.12-1][a, b]

Building	Risk category		
	I or II	III	IV
Buildings 4 stories in height (other than masonry)	0.025 h_{sx}[c]	0.020 h_{sx}	0.015 h_{sx}
Masonry cantilever shear wall bldgs.	0.010 h_{sx}	0.010 h_{sx}	0.010 h_{sx}
Other masonry shear wall buildings[d]	0.007 h_{sx}	0.007 h_{sx}	0.007 h_{sx}
All other buildings	0.020 h_{sx}	0.015 h_{sx}	0.010 h_{sx}

[a]h_{sx} = story height below level x
[b]For seismic force-resisting systems comprised solely of moment frames in Seismic Design Categories D, E, and F, the allowable story drift shall comply with the requirements of Section 12.12.1.1
[c]There shall be no drift limit for single-story structures with interior walls, partitions, ceilings, and exterior wall systems that have been designed to accommodate the story drifts. The structure separation requirement of ASCE 7-16 Section 12.12.3 is not waived
[d]Structures in which the basic structural system consists of masonry shear walls designed as vertical elements cantilevered from their base or foundation support which are so constructed that moment transfer between shear walls (coupling) is negligible

24.12 Redundancy [ASCE 7-16: Section 12.3.4]

Redundancy is an important characteristic of a structure to provide multiple paths of resistance. Higher redundancy indicates better reliability. When the redundancy is low, inelastic behavior during major seismic event can cause the collapse of the structure. When the structures are potentially exposed to severe inelastic demand, the structure should be designed with high redundancy to increase the numbers of load paths. Loads can be redirected to be distributed to other lateral-force-resisting elements. The redundancy factor is applied to increase the horizontal forces (ASCE 7-16: Section 12.3.4). The redundancy factor value varies between 1.0 and 1.3.

The value of ρ is 1.0 for the following type of calculation:

- Structures assigned to Seismic Design Category B or C.
- Drift calculation and P-delta effects.
- Design of nonstructural components.
- Design of nonbuilding structures that are not similar to buildings.
- Design of systems and members such as collector elements, splices, and their connections where overstrength factor are used.
- Diaphragm loads and structures with damping systems
- Design of structural walls for out-of-plane forces, including their anchorage.

For structures assigned to Seismic Design Category D, E, or F, ρ is 1.3 unless one of the following two conditions is met, whereby ρ is permitted to be taken as 1.0:

1. Each story resisting more than 35% of the base shear in the direction of interest shall comply with Table 24.11 (ASCE 7-16 Section Table 12.3-3).
2. Structures are regular in plan at all levels, provided that the seismic force-resisting systems consist of at least two bays of seismic force-resisting perimeter framing on each side of the structure in each orthogonal direction at each story resisting more than 35% of the base shear.

Table 24.11 Requirements for each story resisting more than 35% of the base shear [ASCE 7-16: Table 12.3-3]

Lateral force-resisting element	Requirement
Braced frames	Removal of an individual brace, or connection thereto, would not result in more than a 33% reduction in story strength, nor does the resulting system have an extreme torsional irregularity (horizontal structural irregularity type 1b)
Moment frames	Loss of moment resistance at the beam-to-column connections at both ends of a single beam would not result in more than a 33% reduction in story strength, nor does the resulting system have an extreme torsional irregularity (horizontal structural irregularity type 1b).
Shear walls or wall pier with a height-to-length ratio of greater than 1.0	Removal of a shear wall or wall pier with a height-to-length ratio greater than 1.0 within any story, or collector connections thereto, would not result in more than a 33% reduction in story strength, nor does the resulting system have an extreme torsional irregularity (horizontal structural irregularity type 1b).
Cantilever columns	Loss of moment resistance at the base connections of any single cantilever column would not result in more than a 33% reduction in story strength, nor does the resulting system have an extreme torsional irregularity (horizontal structural irregularity type 1b).
Other	No requirements

24.13 Earthquake Load Effect [ASCE 7-16 Section 12.4.2]

The ASCE 7-16 specifies that the Earthquake Load, E, on a structural element be calculated as the sum of the effects due to the lateral seismic forces amplified by the redundancy factor ρ plus the effect of the vertical component of the earthquake ground motion. Namely,

$$E = \rho Q_E + 0.2 S_{DS} D \tag{24.24}$$

where

Q_E = the effect of horizontal seismic forces
ρ = Redundancy Factor to be taken as the largest of the values for ρ_i obtained in accordance to Sect. 24.1
S_{DS} = Design Spectral Response Acceleration for short periods calculated by Eq. (24.4)
D = vertical seismic load on an element

24.14 Building Irregularities [ASCE 7-16 Section 12.3.2.1]

The Equivalent Lateral Force Procedure is based on the assumptions and characteristics of regular structures. Building irregularities are the cause of stress concentrations leading to structural damage and poor performance. If the structure has irregularities, it must comply with additional code requirements and assignment of seismic design categories listed in Table 12.3-1. Figures 24.5 and 24.8 show, respectively, examples of plan irregularities and of vertical irregularities (Fig. 24.6).

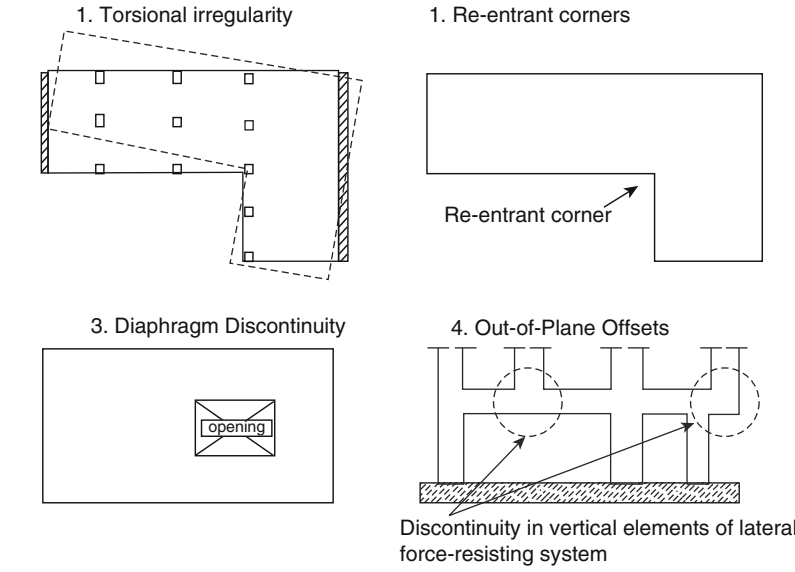

Fig. 24.5 Examples of structural plan irregularities

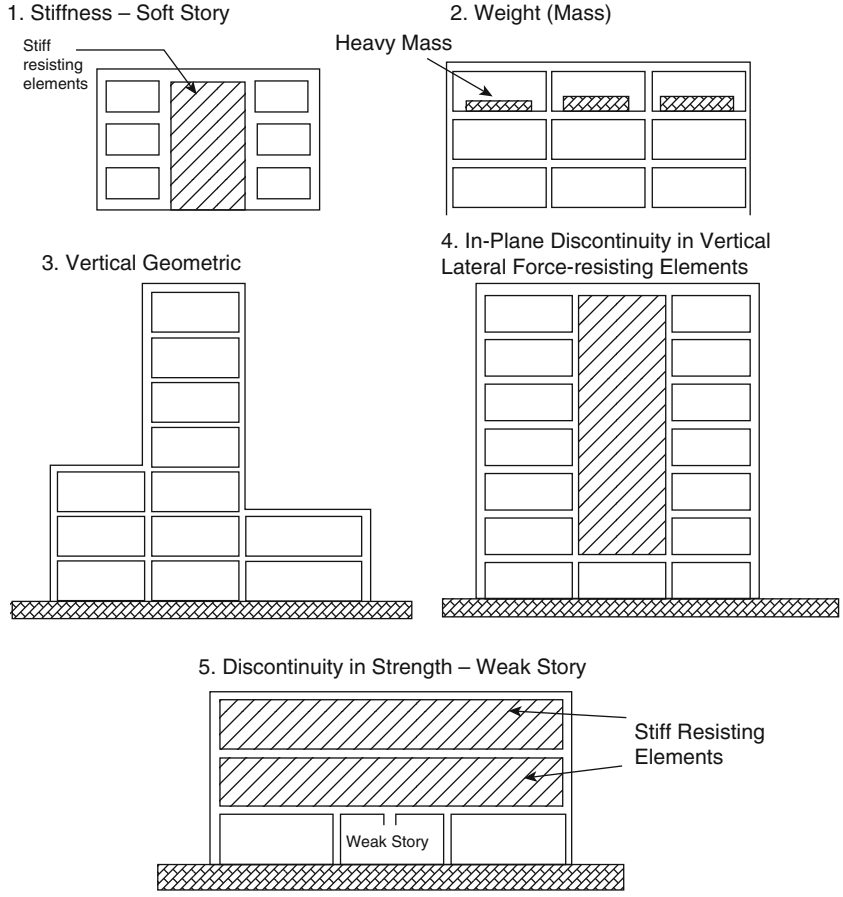

Fig. 24.6 Examples of vertical structural irregularities

Illustrative Example 24.2 Using the Equivalent Lateral Force Method of the IBC-2018 and ASCE 7-16, perform the seismic analysis of four-story concrete building of Illustrative Example 23.1 presented in Chap. 23. The building site is in Seattle, Washington with Zip Code 94704.

Solution:

The following values are obtained from Illustrative Example 24.1:

Seismic weights:

$$w_1 = w_2 = w_3 = 781.1 \text{ kip}$$
$$w_4 = 645.1 \text{ kip}$$

Total weight of the building:

$$W = 781.1 \times 3 \times 645.1 = 3012.4 \text{ kip}$$

Fundamental period:

$T_a = C_t \ h_N^x$	Eq. (24.6) repeated

where

$C_t = 0.016$ (for reinforced concrete moment-resisting frame)
$x = 0.9$
$h_N = 48$ ft (total height of the building)

Then

$T = 0.016 \times 48^{0.9} = 0.52$ sec

Importance Factor (Warehouse)

$I = 1.0$	(Table 24.4)

Mapped Response Spectral Acceleration:

(Use USGS Web site at http://earthquake.usgs.gov/designmaps)

Results:

Short Period: ($T = 0.2$ sec):	$S_S = 1.360$ g
Long Period ($T = 1.0$ sec)::	$S_1 = 0.527$ g

The response spectra are obtained for Illustrative Example 24.2.

Site Class = B for rock	(Table 24.3)
Site coefficient $F_a = 1.0$	(Table 24.1)
Site Coefficient $F_B = 1.0$	(Table 24.2)

Soil Modified Response Spectral Acceleration:

$S_{MS} = F_a S_S = 1.360$	by Eq. (24.1)
$S_{M1} = F_v S_1 = 0.527$	by Eq. (24.2)

Design Response Spectral Acceleration:

$S_{DS} = \frac{2}{3} S_{MS} = 0.907$	by Eq. (24.3)
$S_{D1} = \frac{2}{3} S_{M1} = 0.351$	by Eq. (24.4)

Response Modification Factor: (Table 24.9)

$R = 8$ (Special Reinforced Concrete Moment Frame).

Seismic Design Category = D (Tables 24.5 and 24.6)
Seismic Coefficient

$C_S = \frac{S_{DS}}{R/I_e} = \frac{0.907}{8/1.0} = 0.113$	by Eq. (24.13)

Check maximum value for C_S:
Since $T \leq T_L = 6\text{sec}$

$C_S \leq \frac{S_{D1}}{(R/I_e)T} = \frac{0.351}{(8/1)0.52} = 0.084$	by Eq. (24.14a)

Check minimum value for C_S:

$C_S \geq 0.044 S_{D1} I_e$ $= 0.044 \times 0.351 \times 1.0 = 0.0154 \geq 0.01$	OK	by Eq. (24.15)

Then

$C_S = 0.084$

Base Shear Force:

$V = C_S W$	Eq. (24.12) repeated

$$V = 0.084 \times 3012.4 = 253.04 \text{ kips}$$

Vertical Force Distribution:

$F_x = \frac{w_x h_x^k}{\sum_{i=1}^{N} w_i h_i^k} V$	Eq. (24.17) repeated

$$T = 0.52 \text{ sec} > 0.5 \text{ sec}$$

k = 1.01 (by interpolation)	(Sect. 24.13)

Table 24.12 contains the necessary calculations to determine F_x at the various levels of the building in Illustrative Example 24.2.

Overturning Moments:

$M_x = \sum_{i=x+1}^{N} F_i(h_i - h_x)$	Eq. (24.18) repeated

Calculated values for the overturning moments at the various levels of the building are shown in the last column of Table 24.12.

Story Drift and Lateral Displacement:

For a structure modeled as a shear building the drifts, Δ_x, or relative displacement between consecutive levels is given by

$$\Delta_x = \frac{V_x}{K_x}$$

where

V_x is the story shear force calculated in Table 24.12.

K_x is the stiffness of the story calculated in the solution of Illustrative Example 24.1, now listed in Table 24.13.

Lateral Displacement and Story Drift:

The elastic lateral displacement δ_{ex} may be calculated at each level of the building by adding story drifts for the story at that level and those below as shown calculated in Table 24.13.

The design lateral displacement, δ_x, is then determined by

$\delta_x = \frac{C_d \delta_{xe}}{I}$	Eq. (24.21) repeated

$C_d = 5.5$	(Deflection Amplification Factor)	(Table 24.9)
$I_e = 1.0$	(Importance Factor)	(Table 24.4)

Calculated values for the design lateral displacements, δ_x at the various levels of the building are shown in Table 24.13.

Table 24.12 Calculation of seismic lateral forces for Illustrative Example 24.2

Level	h_x (ft)	W_x (kip)	h_x^k (ft)	$w_x h_x^k$ (kip-ft)	F_x (kip)	V_x (kip)	M_x (kip-ft)
4	48	645.1	49.89	32,187	90.20	90.20	
3	36	781.1	37.31	29,146	81.68	171.88	1082
2	24	781.1	24.77	19,352	54.23	226.11	3145
1	12	781.1	12.30	9609	26.93	253.04	5858
				90,293			10,086

The inelastic story drift is then given by

$\Delta_{Mx} = \delta_x - \delta_{x-1} \leq \Delta_a$	Eq. (24.22) repeated

where the allowable story drift Δ_a is given by

$\Delta_a = 0.025h_x = 0.025 \times 144 = 3.6$ in	(Table 24.10)

Values calculated for the inelastic story drift Δ_{Mx} shown in the last column of Table 24.13 are well below the allowable limit $\Delta_a = 3.6$ in.

Table 24.13 Lateral displacement and story drift for Illustrative Example 24.2

Level x	Story shear V_x (kip)	Story stiffness K_x (kip/in.)	Story drift (Elastic) Δ_x (in.)	Lateral displ. (Elastic) δ_{xe} (in.)	Lateral displ. (Inelastic) δ_x (in.)	Story drift (Inelastic) Δ_{Mx} (in.)	Allowable story drift (Inelastic) Δ_a (in.)
4	90.20	1757.7	0.0513	0.2972	1.63	0.28	3.6
3	171.88	1757.7	0.0978	0.2459	1.35	0.54	3.6
2	226.11	3236.0	0.0699	0.1481	0.81	0.38	3.6
1	253.04	3236.0	0.0782	0.0782	0.43	0.43	3.6

Redundancy Factor:

The Seismic Design Category is assigned to be D, ρ is determined to be 1.3 without satisfying one of the two conditions presented in Sect. 24.12 is met.

$$\underline{\rho = 1.3}$$

24.15 Summary

The International Building Code was prepared by the International Code Council (ICC), whose members are representatives of BOCA (Building Officials and Code Administrators), ICBO (International Conference of Building Officials) and SBCCI (Southern Building Code Congress International). The unified effort of all three agencies resulted in the International Building Code, which contains provisions for earthquake resistant design specified in the latest versions of several building codes. These codes are in current use in different regions of the country. This chapter shows the application of USGS Seismic Design tool to determine Seismic Design Category. This chapter is updated with current IBC-2018 and ASCE 7-16 using Equivalent Lateral Force Procedure. The modal response spectrum analysis is presented in Chap. 23.

Appendices

Appendix I: Answers to Problems in Selected Chapters

Chapter 1

1.1 $T = 2\pi L\sqrt{\frac{W}{g}\cdot\frac{L}{3EI+2kL^3}}$

1.2 $u(t=1) = -0.89$ in
$\dot{u}\,(t=1) = 22.66$ in/sec

1.3 $f = 2.24$ cps

1.4 (a)$f = 2.87$ cps
(b)$f = 2.74$ cps

1.5 $f = \frac{4}{\pi}\sqrt{\frac{3EIg}{L^3W}}$

1.6 $u(t=2) = -0.474$ in
$\dot{u}\,(t=2) = -21.05$ in/sec
$\ddot{u}(t=2) = 4065$ in/sec^2

1.7 $\theta = \theta_0 \cos\sqrt{\frac{g}{L}}t + \frac{\dot{\theta}_0}{\omega}\sin\sqrt{\frac{2}{L}}t$

1.8 $k = 73.64$ lb/in
$EI = 3.39\times 10^5(\text{lb}\cdot\text{in}^2)$

1.9 $u(t) = 0.0995\ \sin 48.03t$ (in)
$\dot{u}\,(t) = 4.78\ \cos 48.03t$ (in/sec)

1.10 $u(t) = 0.5\cos 27.78t$

1.11 $f = \frac{1}{2\pi}\sqrt{\frac{ka^2-mgL}{mL^2}}$

1.13 $f = \frac{1}{2\pi}\sqrt{\frac{3EI}{mL^3}-\frac{3g}{2L}}$

1.14 $f - f_0\sqrt{1-\frac{w}{w_{er}}}$

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M. Paz, Y. H. Kim, *Structural Dynamics*, https://doi.org/10.1007/978-3-319-94743-3

1.15 (a) $f = \dfrac{1}{2\pi}\sqrt{\dfrac{3EIkg}{(3EI + kL^3)W}}$

(b) $f = \dfrac{1}{2\pi}\sqrt{\dfrac{48EIkg}{(48EI + kL^3)W}}$

(c) $f = \dfrac{1}{2\pi}\sqrt{\dfrac{3EILg}{a^2b^2W}}$

(d) $f = \dfrac{1}{2\pi}\sqrt{\dfrac{3EILkg}{(3EIL + a^2b^2k)W}}$

1.16 $f = \dfrac{1}{2\pi}\sqrt{k\left(\dfrac{1}{m_1} + \dfrac{1}{m_2}\right)}$

1.17 $\omega = \sqrt{\frac{k}{m}}$

1.18 $\omega = 35.018$ (rad/sec)

1.19 $\omega = \sqrt{\dfrac{k - Wh}{I_0}}$

unstable for $k \leq Wh$

Chapter 2

2.1 $u(t = 1) = -0.037$ in

$\dot{u}\,(t = 1) = -0.570$ in/sec

2.2 $u(t = 2) = -4.65 \times 10^{-9}$ in

$\dot{u}\,(t = 2) = -4.083 \times 10^{-8}$ in/sec

$\ddot{u}(t = 2) = 4.18 \times 10^{-5}$ in/sec^2

2.3.1 c = 0.73 lb · sec/in

2.3.2 $\xi = 1.5\%$

2.5 (a) for $\xi = 1, u = [u_0(1 + wt) + v_0 t]e^{-wt}$

(b) for $\xi > 1, u = e^{-\xi\omega t}\left[u_0 \cosh\omega_D^{'} t + \dfrac{v_0 + u_0\xi\omega}{\omega_D^{'}} \sinh\omega_D^{'} t\right]$

where $\omega_D^{'} = \omega\sqrt{\xi^2 - 1}$

2.6 (a) $\xi = 0.4167$

(b) $T_D = 0.2765$ sec

(c) $\delta = 2.8801$

(d) $\dfrac{u_1}{u_2} = 17.8161$

2.9 $f_D = 0.5\sqrt{\dfrac{ka^2 - mgL}{mL^2}}$

2.10 (a) $\xi = 0.076$

(b) $f_D = 8.69$ cps

(c) $\delta = 0.48$

(d) $\dfrac{u_1}{u_2} = 1.61$

2.11 (a) $\xi = 0.018$
(b) $\omega_D = 57.76$ rad/sec
(c) $\delta = 0.113$
(d) $\dfrac{u_1}{u_2} = 1.12$

2.12 (a) $K_e = \dfrac{3EIk}{3EI + kL^3}$

$$C_E = 0.2\sqrt{\frac{3EIkw}{g\left(3EI + kL^3\right)}}$$

(b) $K_E = \dfrac{48EIk}{48EI + kL^3}$

$$C_E = 0.2\sqrt{\frac{48EIkm}{48EI + kL^3}}$$

(c) $K_E = \dfrac{3EIL}{a^2b^2}$

$$C_E = 0.2\sqrt{\frac{3EILw}{a^2b^2g}}$$

(d) $K_E = \dfrac{3EIkL}{3EIL + ka^2b^2}$

$$C_E = 0.2\sqrt{\frac{3EIkLw}{\left(3EIL + ka^2b^2\right)g}}$$

2.13 $\xi = 6.7\%$

2.14 $m_1m_2\ddot{u} + (m_1 + m_2)c\dot{u} + (m_1 + m_2)ku = 0$

2.15 $\ddot{u} + 2\xi\omega\dot{u} + \omega^2u = 0$

where: $\omega = \sqrt{\dfrac{k}{M}}$, $M = \dfrac{m_1m_2}{m_1 + m_2}$, $\omega_D = \omega\sqrt{1 - \xi^2}$, $\xi = \dfrac{c}{c_{cr}}$, $c_{cr} = 2\sqrt{kM}$

Chapter 3

3.1 $Y = 0.0037$ in

3.2 $A_T = 51.2$ lb

3.3 $Y = 0.823$ in

3.4 $Y = 0.746$ in

3.5 (a) $A_T = 15,803$ lb
(b) $T_R = 3.16$

3.6 $k = 93$ lb/in

3.7 $u = 0.013$ in

3.8 $T_R = 0.34$

3.9 (a) $Y_1 = 0.064$ in for $f_1 = 800$ RPM
$Y_2 = 0.0446$ in for $f_2 = 1000$ RPM
$Y_3 = 0.0302$ in for $f_3 = 1200$ RPM
(b) $Y(r = 1) = 0.076$ in

3.10 $\xi = 3.3\%$

3.11 $\xi = 6.3\%$
$\omega = 49.2\,(\text{rad/sec})$

3.12 $Y = 0.387\,(\text{cm})$
$\sigma_{\max} = 110.7\,(\text{Mpa})$

3.13 $f = f_r\sqrt{1\frac{m_s}{m}}$

3.14 $\omega_p = \omega\sqrt{1-2\xi^2}$ for $\xi < \dfrac{1}{\sqrt{2}}$

$$U_p = \frac{u_{st}}{2\xi\sqrt{1-\xi^2}}$$

3.15 (a) $f = 18.58$ cps
(b) $\xi = 0.0735$
(c) $F_0 = 4825$ lb
(d) $F_0 = 4840$ lb

3.16 $\xi = \dfrac{U_1\left(1-r_1^2\right)}{2r_1\sqrt{U_r^2r_1^2-U_1^2}}$

$$F_r = \frac{U_rU_1\left(1-r_1^2\right)k}{r_1\sqrt{U_r^2r_1^2-U_1^2}}$$

3.17 (a) $M\ddot{u} + c\dot{u} + ku = \dfrac{m_1F_0}{m_1+m_2}\sin t$

where $M = \dfrac{m_1m_2}{m_1+m_2}$

(b) $u = \dfrac{m_1F_0\sin(t-\theta)}{k(m_1+m_2)\sqrt{(1-r^2)^2+(2r\xi)^2}}$

Chapter 4

4.1 (a) $u(t=0.5) = -0.407$ in
(b) $u_{\max} = 1.37$ in

4.2 (a) $u(t=0.5) = -0.102$ in
(b) $u_{\max} = 1.17$ in

4.3 $DLF = \dfrac{t}{t_d} - \dfrac{\sin\omega t}{\omega t_d}$ for $t \le t_d$

$DLF = 1 + \dfrac{1}{\omega t_d}(\sin\omega t - \sin\omega(t+t_d))$ for $t \ge t_d$

4.4 $V_{\max} = 18,093$ lb for left column
$V_{\max} = 1908$ lb for right column

4.5 $V_{\max} = 15,640$ lb for left column
$V_{\max} = 1649$ lb for right column

4.8 $u_1(t) = -\frac{v}{\omega}\cos\omega t + vt$

4.9 $u(t) = \dfrac{u_{st}}{t_d}\left[t - \dfrac{\sin \omega t}{\omega}\right]$ for $t \leq t_d$

$= \dfrac{u_{st}}{t_d}\left\{t_d - \dfrac{\sin \omega t}{\omega}\right] \cos \omega(1 - t_d) + \dfrac{1}{\omega}\sin \omega(t - t_d)\Big\}$ for $t \geq t_d$

4.10 $u_{\max} = 1.348$ (in)
$\sigma_{\max} = 11,906$ (psi)

4.11 $u_{\max} = 0.72$ (in)
$\sigma_{\max} = 16,282$ (psi)

4.12 $u(t = 0.5) = -1.903$ in

4.13 $u(t = 0.5) = -0.060$ in

4.14 $u(t = 1) = -2.809$ in

4.15 $u(t = 1) = -2.397$ in

4.16 $u_{\max} = 6.03$ in (undamped system)
$u_{\max} = 4.59$ in (with 20% daming)

4.17 $u_{\max} = 1.51$ in

4.18 $u_{\max} = 1.42$ in

4.19 $u_{\max} = 0.58$ in

4.20 $u_{\max} = 1.00$ in

4.21 $u_{\max} = 0.79$ in

4.22 $u_{\max} = 0.66$ in

4.23 $u_{\max} = 0.34$ in

4.24 $\sigma_{1\max} = 3842$ psi
$\sigma_{2\max} = 6831$ psi

4.25 $u_{\max} = 0.71$ in

4.26 (a) $\sigma = \pm 6193$ psi
(b) $F_{\max} = 11,376$ lb

Chapter 5

5.1 $u_{\max} = 0.374$ in

5.2 $\sigma_{\max} = 7.246$ ksi

5.3 $u_{\max} = 0.418$ in

5.4 $\sigma_{\max} = 12.788$ ksi

5.5 $S_D = 1.9$ in
$S_v = 22.4$ in/sec
$S_a = 0.68$ g

5.6 $S_D = 1.28$ in
$S_v = 15.36$ in/sec
$S_a = 0.48$ g

5.7 $S_D = 11.0$ in

5.8 $(F_s)_{\max} = 88.0$ Kip

5.9 $(F_s)_{\max} = 36.0$ Kip
$(F_T)_{\max} = 44.0$ Kip

5.10 $S_D = 8.0$ in
5.11 $\mu = 1.8$
5.12 (a) $S_D = 40$ in
$S_v = 50.3$ in/sec
$S_a = 1.63\,\mathrm{g}$
(b) $S_D = 4.8$ in
$S_v = 60.0$ in/sec
$S_a = 1.96\,\mathrm{g}$
5.13 (a) $S_D = 0.46$ in
$S_v = 5.8$ in/sec
$S_a = 0.19\,\mathrm{g}$
(b) $S_D = 6.0$ in
$S_v = 18$ in/sec
$S_a = 0.6\,\mathrm{g}$
5.14 $S_D = 0.78$ in (at $f = 0.5$ cps)
$S_v = 2.46$ in/sec
$S_a = 7.72$ in/sec^2
5.15 $S_D = 8.03$ in (at $f = 1.00$ cps)
$S_v = 50.45$ in/sec
$S_a = 317.00$ in/sec^2
5.16 $S_D = 3.26$ in (at $f = 1.00$ cps)
$S_v = 20.38$ in/sec
$S_a = 127.40$ in/sec^2

Chapter 6

6.1 $u_{\max} = -10.27$ in
6.2 $u_{\max} = 2.56$ in
6.3 $u_{\max} = 6.47$ in
6.4 $u_{\max} = 1.03$ in
6.5 $u_{\max} = 5.19$ in
6.6 $u_{\max} = 2.38$ in
6.7 $\mu = 1.7$
6.8 $u(t = 0.5) = 0.4477$ in
6.9 $u(t = 0.5) = 0.2654$ in
6.10 $u(t = 0.5) = 0.1423$ in
6.11 $u(t = 0.5) = 0.1340$ in
6.12 $a_0 = 2.78$

Chapter 19

19.1 $F(t) = \frac{120}{\pi}\left[\sin 2\pi t + \frac{1}{3}\sin 6\pi t + \frac{1}{5}\sin 10\pi t \ldots\right]$
19.2 $F(t) = 10^{-6}[357 \sin 2\pi t - 26 \cos 2\pi t + 36 \sin 6\pi t - 532 \cos 6\pi t - 35 \sin 10\pi t - 7 \cos 10\pi t + \ldots]$
19.3 $u(t = 0.5) = 0.3518$ in

19.4 (a) $a_n = \dfrac{720}{\pi(1-n^2)}, n = 2, 4, 6, \ldots$
$a_n = 0, \ n = 1, 3, 5, \ldots$
$b_n = 0, \ n = 1, 2, 3, \ldots$

19.5 $u(t = 0.05) = \ -0.2065$ in

19.6 $u(t = 0.05) = \ -0.2064$ in

19.7 $u(t = 0.05) = 0.1295$ in

19.8 (a) $a_0 = 0.0350$
$a_1 = 0.0069 \quad b_1 = -0.0361$
$a_2 = -0.0724 \quad b_2 = -0.0402$
(b) $u(t = 0.35) = 0.2570$ in

19.9 $u(t = 0.35) = 0.2327$ in

19.10 $u\left(\dot{t} \ = 0.5\right) = 0.027$ in

19.11 $u(t = 0.5) = (0.02842 \ - \ 0.00011\text{i})$ in

(a) $a_0 = \dfrac{P_0}{\pi}$

19.12 $a_n = 0, n = 1, 3, 5, \ldots; b_1 = \dfrac{P_0}{2}$
$a_n = \dfrac{P_0}{\pi} \cdot \dfrac{2}{1-n^2}, n = 2, 4, 6, \ldots; b_n = 0, n > 1$

19.13 $u(t = 0.5) = 0.0731$ in

19.14 $u(t = 0.5) = 0.0543$ in

19.15 $u_1(t) = 0.229(\sin \pi t - 0.75 \sin 4.19t)$
$u_2(t) = 0.229[\sin \pi(t-1) - 0.75 \sin 4.19(t-1)]$
$u(t) = u_1(t)$ for $0 \le t \le 1.0$ sec
$u(t) = u_1(t) + u_2(t)$ for $t \le 1.0$ sec

19.16 $u(t = 1 \text{ sec}) = 0.2419$(in)

19.17 $u(t = 1 \text{ sec}) = 0.0532$(in)

19.18 $u(t = 1 \text{ sec}) = 0.0035$(in)

19.19 $u(t = 1 \text{ sec}) = 0.0034$(in)

19.20 $u(t = 1 \text{ sec}) = 0.0028$(in)

Chapter 21

21.1 $M^* = 4.48 \text{ lb} \cdot \text{ sec}^2/\text{in}$
$C^* = 2250 \text{ lb} \cdot \text{ sec}/\text{in}$
$K^* = 45,000 \text{ lb/in}$
$F^*(t) = 625 \cdot f(t)\text{lb}$

21.2 $M^* = \dfrac{5}{6}m$
$C^* = c$
$K^* = k$
$F^*(t) = \dfrac{M(\text{t})}{L}$

21.3 $M^* = \frac{2}{3}\bar{m}L^2$

$C^* = cL$

$K^* = kL$

$F^* = \frac{P_0 L}{6} f(t)$

21.4 $M^* = \frac{m}{2\pi}(5\pi - 8)$

$K^* = \frac{EI\pi^4}{32L^3}$

$F^*(t) = 0.2929 F_0 f(t)$

21.5 $K_G^* = -\frac{N\pi^4}{8L}$

21.6 $M^* = 0.1237\,\frac{\gamma d}{g}$

$K^* = -\frac{E_c \pi d^4}{128L^3}$

$F^*(t) = -0.1807 P_0(t) Ld$

21.7 $\omega = \sqrt{\frac{48EI}{L^3\left(m + \frac{17}{35}m_b\right)}}\ \text{rad/sec}$

21.8 $\omega = 7.825\sqrt{\frac{EI}{m_b L^3}}\ \text{rad/sec}$

21.9 $f = 3.51$ cps

21.10 $\omega = 2.15\sqrt{\frac{gEI}{WL^3}}\ \text{rad/sec}$

21.11 $\omega = 0.62\sqrt{\frac{gAG}{WL}}\ \text{rad/sec}$

21.12 $f = 0.496$ cps

21.13 $f = 0.492$ cps

Appendix II: Glossary

Accelerometer	An instrument for measuring ground acceleration as a function of time.
Aliasing	The phenomenon in which higher harmonies introduce spurious low frequency components. This occurs when the number of sampled points of a function is insufficient to describe the function. (See Nyquist Frequency.)
Amplitude	Maximum value of a function as it varies with time. If the variation with time can be described by either a sine or cosine function, it is said to vary harmonically.
Angular Frequency/Circular Frequency	The frequency of periodic function in cycles per second (Hertz) multiplied by 2π; expressed in rad/sec.

Autocorrelation of a Random Function $x(t)$	Correlation between the function $x(t)$ and the out-of-phase function $x(t + \tau)$ as defined by Eq. (22.19).
Base Shear Force	The total lateral force on the structure equivalent to the earthquake excitation at the base of the structure.
Basic Design Spectra	Smooth or average plots of maximum response of single degree-of-freedom systems used in seismic design of structures.
Boundary Condition	A constraint applied to the structure independent of time.
Braced Frame	An essentially vertical truss system of the concentric or eccentric type which is provided to resist lateral forces.
Building Frame System	An essentially complete space frame which provides support for gravity loads.
Characteristic Equation	An equation whose roots are the natural frequencies.
Circular Frequency	See Angular Frequency.
Complementary Solution	The solution of a homogeneous differential equation (no external excitation).
Complete Quadratic Combination (CQC)	A method of combining maximum values of modal contributions which is based on random vibration theory and includes cross correlation terms.
Concentric Braced Frame	A brace frame in which the members are subjected primarily to axial forces.
Consistent Mass	Mass influence coefficients determined by assuming that the dynamic displacement functions are equal to the static displacement functions.
Correlation between Random Variables $x_1(t)$ and $x_2(t)$	The time average of the product of the functions $x_1(t)$ and $x_2(t)$.
Coupled Equations	A system of differential equations in which the equations are not independent from each other.
Critical Damping	Minimum amount of viscous damping for which the system will not vibrate.
D' Alembert Principle	This principle states that a dynamic system may be assumed to be in equilibrium provided that the inertial forces are considered as external forces.
Damped Frequency	The frequency at which a viscously damped system oscillates in free vibration.
Damping	The property of the structure to absorb vibrating energy.
Damping Ratio	The ratio of the viscous damping coefficient to the critical damping.
Degrees of Freedom	The number of independent coordinates required to completely define the position of the system at any time.
Deterministic Vibration	A process which can be predicted by an exact mathematical expression.
Dirac's Delta Function	A generalized function having the properties described in Eq. (22.52).
Direct Stiffness Method	The method of assembling the system stiffuess matrix by proper summation of the stiffness coefficients of the elements in the system.

Discrete Fourier Transform	A summation of harmonic terms to express the Fourier transform for a function defmed by a finite number of points.
Dual System	A combination of a special or intermediate moment-resisting space frame and shear walls or braced frames.
Ductility Ratio	The ratio between the maximum displacement for elastoplastic behavior and the displacement corresponding to yield point.
Dynamic Condensation	A method of reducing the dimension of the eigenproblem by establishing the dynamic relationship between primary and secondary coordinates.
Dynamic Magnification Factor	The ratio of the maximum displacement of a single degree of freedom excited by a harmonic force to the deflection that would result if a force of that magnitude were applied statically.
Earthquake	The vibrations of the Earth caused by the passage of seismic waves radiating from some source of elastic energy.
Eigenproblem	The problem of solving a homogeneous system of equation: containing a parameter which should be determined to provide nontrivial solutions.
Elastic Rebound Theory	The theory of earthquake generation proposing that faults remain locked while strain energy slowly accumulates in the surrounding rock, and then suddenly slip, releasing this energy.
Elastoplastic	A system which behaves elastically for a force that does not exceed a maximum value and plastically above this maximum.
Ensemble	A set of samples or records of a random process.
Epicenter	The point on the Earth's surface directly above the focus.
Ergodic Process	A stationary random process for which the time average of any record is equal to the average across the ensemble.
Fast Fourier Transform (FFT)	A very efficient algorithm implemented in a computer program for the calculation of the response in the frequency domain.
Flexibility Coefficient	f_{ij} is the displacement at coordinate i due to a unit force (or unit moment) applied at coordinate j.
Forced Vibration	Vibration in which the response is due to external excitation of the system.
Fourier Analysis	Method of determining the response by superposition of the responses to the harmonic components of the excitation.
Fourier Transform	The Fourier transform $C(\omega)$ of a function $F(t)$ is defined by Eq. (22.23).
Fourier Transform Pair	In reference to the function $F(t)$, the Fourier transform pair is given by Eqs. (22.23) and (22.24).
Free Body Diagram	A sketch of the system, isolated from all other bodies, in which all the forces external to the body are shown.
Free Vibration	The vibration of a system in absence of external excitation.
Frequency Analyzer or Spectral Density Analyzer	An instrument that measures electronically the spectral density function of a signal.
Frequency Ratio	The ratio between the forcing frequency to the natural frequency for a system excited by a harmonic load.
Fundamental Frequency	The lowest natural frequency of a multidegree-of freedom vibrating system.

Gauss-Jordan Reduction or Elimination	A computational technique in which elementary row operations are applied systematically to solve a linear system of equations.
Generalized Coordinates	A set of independent quantities which describe the dynamic system at any time. These quantities are generally functions of the geometric Coordinates.
Geometric Stiffness Coefficient	k_{Gij} is the force at coordinate i due to a unit displacement at coordinate j and resulting from the axial forces in the structure.
Harmonic	A sinusoidal function having a frequency that is an integral multiple of the fundamental frequency.
Harmonic Force	A force expressed by a sine, cosine (or equivalent exponential) function.
Hypocenter or Focus	The Point in the interior of the earth at which rupture is initiated during an earthquake.
Impulsive Load	A load that is applied during a relatively short time interval producing an instantaneous change in velocity.
Initial Conditions	The initial values of specific functions such as displacement, velocity, or acceleration evaluated at time $t = 0$.
Intensity (of Earthquakes)	A measure of ground shaking obtained from the damage done to man-made structures, changes in the Earth's surface, and witness reports.
Intermediate Moment-Resisting Space Frame (IMRSF)	A concrete space frame designed in conformance with Section 1921.8 (k) of UBC-97.
Isolation	The reduction of severity of the response, usually attained by proper use of a resilient support.
Lateral Force Method	A method of analysis in which lateral horizontal forces at various levels of the structure are considered equivalent to seismic excitation at the base of the structure.
Lateral Force-Resisting System	That part of the structural system assigned to resist lateral forces.
Linear Acceleration Method	A step-by-step method for the integration of the differential equations of motion in which the acceleration is assumed to be a linear function during each time step.
Linear System	A system of differential equations in which no term contains products (or exponents) of the dependent variables or their derivatives.
Logarithmic Decrement	The natural logarithm of the ratio of any two successive amplitudes of the same sign obtained in the decay curve in a free vibration test.
Lumped Mass	A method of discretization in which the distributed mass of the elements is lumped at the nodes or joints.
Mathematical Model	The idealization of a system including all the assumptions imposed on the physical problem.
Mean-Square Value	The time average of the square of a random function as defined by Eq. (22.2).
Mean Value	The time average of a random function defined by Eq. (23.l).
Modal Shapes (also Normal Modes)	The relative amplitude of the displacements at the coordinates of a multidegree-of-freedom system vibrating at one of the natural frequencies.

Modal Superposition Method	A method of solution of multidegree-of-freedom systems in which the response is determined from the solution of independent modal (or normal) equations.
Modified Mercalli Intensity (MMI)	A measure of the effect of an earthquake at a particular location.
Moment-Resisting Space Frame	A space frame in which the members and joints are capable of resisting forces primarily by flexure.
Narrow-Band Process	A random process whose spectral density function has nonzero values only in a narrow frequency range.
Natural Frequency	The number of cycles per second at which a single degree of-freedom system vibrates freely or a multidegree-of-freedom system vibrates in one of the normal modes.
Natural Period	The time interval for a vibrating system in free vibration to do one oscillation.
Newmark Beta Method	A numerical method to calculate the response of a structure subjected to external excitation (force or motion).
Node or Joint	A point joining elements of the structure and at which displacements are known or to be determined.
Normal Distribution or Gaussian Distribution	A function whose probability density function is given by Eq. (22.12).
Normal Modes	See modal Shapes.
Nyquist Frequency	The maximum frequency component that can be detected from a function sampled at time spacing Δt [$N_y = 1/2\Delta t$ (Hz)].
Occupancy Factor	A numerical factor in the calculation of the base shear force that depends on the intended use of the structure.
Ordinary Moment-Resisting Space Frame (OMRSF)	A moment-resisting not meeting special detailing requirements for ductile behavior.
P-Delta Effect	The secondary effect on shears, axial forces, and moments of frame members induced by the vertical loads on the laterally displaced building frame.
Periodic Function	A function that repeats itself at a fixed time interval known as the period of the function.
Power Spectral Density	A term used to describe the intensity of random vibration at a given frequency, measured in g^2/H_z.
Principle of Virtual Work	The work done by all the forces acting on a system in static or dynamic equilibrium, which occurs during a virtual displacement compatible with the constraints of the system equal to zero, for a rigid system and for an elastic frame, equal to the work of the internal forces during the virtual displacement.
Probability Density Function	A function $p(x)$ such that the probability of $x(t)$ of being in the range $(x, x + dx)$ is $p(x)\,dx$.
Pseudo-Velocity	The velocity calculated by analogy with the apparent harmonic motion for a system seismically excited.
Random Function	A function (as opposed to a deterministic function) whose value at any time cannot be determined exactly, but can only be predicted in probabilistic terms by statistical methods.

Random Vibration or Random Process	A process which cannot be predicted in a deterministic sense, but only probabilistically using the theory of statistics.
Rayleigh Distribution	A function whose probability density function is given by Eq. (22.14).
Rayleigh's Formula	A formula to estimate the fundamental period of the structure.
Resonance	The condition in which the frequency of the excitation equals the natural frequency of the vibrating system.
Response	The force or motion that results from external excitation on the structure.
Response Spectrum	A plot of maximum response (displacement, velocity, or acceleration) for a single degree-of-freedom system defined by its natural frequency (or period) subjected to a specific excitation.
Richter Magnitude (M)	A measure related to total energy released during an earthquake.
Root Mean Square (RMS)	The square root of the mean-square value of a random function [Eq. (22.5)].
Sample	A record of random process.
Seismic Zone Factor	A numerical factor in the calculation of the base shear force at a given geographic location.
Seismograph	An instrument for recording, as a function of time, the motions of the Earth's surface that are caused by seismic waves.
Shear Wall	A wall designed to resist lateral forces parallel to the plane of the wall (sometimes referred to as a vertical diaphragm or structural wall).
Shock Spectrum	See Response Spectrum.
Simple Harmonic Motion	The motion of a system which may be expressed by a sine or cosine function of time.
Site-Structure Resonance Coefficient	A numerical factor in the calculation of the base shear force that depends on the condition of the soil.
Space Frame	A three-dimensional structural system, without bearing walls, composed of members interconnected so as to function as a complete self-contained unit with or without the aid of horizontal diaphragms or floor-bracing systems.
Special Moment-Resisting Space Frame (SMRSF)	A moment-resisting space frame specially detailed to provide ductile behavior and comply with the requirements given in Chap. 19 or 22 of UBC-97 (International, Conference of Building Officials 1997).
Spectral Analysis or Spectrum	A description of contributions of the frequency components to the mean-square value of a random function.
Spectral Density Function	A function that describes the intensity of random vibration in terms of the mean-square value per unit of frequency.
Square Root Sum of Squares (SRSS)	A method of combining maximum values of modal contributions by taking the square root of the sum of the squared modal contributions.
Standard Deviation	The square root of the variance. It may be calculated by Eq. (22.6).

Static Condensation	A method of reducing the dimensions of the stiffness and of the mass matrices by establishing the static relation between primary and secondary coordinates.
Stationary Process	A random process for which the average across the ensemble has the same value at any selected time.
Steady-State Vibration	The motion of the system that remains after the transient motion existing at the initiation of the motion has vanished.
Stiffness Coefficient	k_{ij} is the force at coordinate i due to a unit displacement at coordinate j.
Story Drift	The relative lateral displacements of consecutive levels of a building.
Spring Constant	The change in load on a linear elastic structure required to produce a unit increment of deflection.
Strong Motion Accelerograph	An instrument to register seismic motions higher than a specified amplitude.
Structure	An assemblage of framing members designed to support gravity loads and resist lateral forces. Structures may be categorized as building structures or non-building structures.
Structural Factor	A numerical factor in the calculation of the base shear force that depends on the type of structural system.
Tectonic Earthquakes	Earthquakes resulting from the sudden release of energy stored by a major deformation of the earth.
Time History Response	The response (motion or force) of the structure evaluated as a function of time.
Transient Vibration	The initial portion of the motion which vanishes due to the presence of damping forces in the system.
Transmissibility	The non-dimensional ratio, in the steady-state condition, of the response motion to the input motion. Or the non-dimension the amplitude of the force transmitted to the foundation to the amplitude of the force exciting the system.
Variance of $x(t)$	The average of the squares of the deviations of $x(t)$ values from the mean value x.
Viscous Damping	Dissipation of energy such that the motion is resisted by a force proportional to the velocity but in the opposite direction.
White Noise	A wide-band random process for which the spectral density function is constant over the whole frequency range.
Wide-band Process	A random process whose spectral function has nonzero value over a large range of frequencies.
Wiener-Kinchin Equations	These are equations that relate the autocorrelation function and the spectral density function [Eqs. (22.49) and (22.50)].
Wilson's θ Method	A modification of the step-by-step linear acceleration method in which the time step is multiplied by a factor necessary to render the method unconditionally stable.
Wood-Anderson Seismograph	An instrument used to register seismic motions.

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M. Paz, Y. H. Kim, *Structural Dynamics*, https://doi.org/10.1007/978-3-319-94743-3

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Index

A
Accelerometer, 72, 123, 548
Aliasing, 459, 467, 469
Angular frequency, 12, 543
Autocorrelation, 539–542, 547–553, 563, 564, 568, 570
Axial force
  effect on flexural stiffness, 499
  force-displacement relation, 267, 349
  stiffness coefficient, 266, 268, 291, 295, 349
Axis
  global/system, 291, 297, 317, 321, 363
  local/element, 297

B
Bandwidth method (half power), 55–57
Base shear force, 580, 583, 584, 595, 596, 598, 605
Beam
  definition, 3
  dynamic stress, 432, 433
  forced vibration, 427–432
  geometric stiffness, 266–269, 499
  mass matrix, 259, 262–264, 281, 286, 287, 306, 337, 342
  stiffness matrix, 251, 256–258, 262, 263, 268, 270, 271, 281, 285, 287, 291, 292, 305, 335, 336
  uniform, 13, 25, 38, 251, 252, 255, 256, 259, 263, 270, 273, 278, 285–287, 291, 293, 295–297, 319, 335, 337, 349, 402–404, 406, 415, 416, 418, 420, 422–424, 426, 427, 429, 431, 435, 437
Bernoulli-Euler, 415
Betti's theorem, 182, 255
Body forces, 373, 377–379, 384
Building Officials and Code Administrators (BOCA), 607

C
Characteristic equation, 30, 31, 177, 184, 188, 302, 582
Circular frequency, 12
Clough, R.W., 480, 576
Complementary solution, 45, 47, 49, 97, 98, 116
Complete quadratic equation, 30, 65, 68, 179, 226, 561
Complex equations, 210
Complex frequency, 460, 559
Complex frequency response, 460, 559, 561
Condensation
  dynamic, 219, 222–227, 233–245, 247, 248
  modified dynamic, 241–244, 248
  static, 219–228, 233, 245–248, 260, 262, 286–288, 293, 313, 321, 332, 580, 599
Consistent geometric stiffness matrix, 268
Consistent mass
  definition, 260–264
  matrix, 262–265, 286, 287, 293, 297, 299, 319–321, 337, 342, 350, 351, 362, 377, 445–447
Constant acceleration method, 132, 142, 145–147, 150, 151, 203, 204
Constants of integration
  continuous system, 415
  multi-degree-of-freedom system, 215, 378, 409
  single-degree-of freedom system, 378
Cooley, J.W., 461, 462
Coordinate transformation, 194, 195, 275, 291, 297–299, 301–304, 312, 317, 321–325, 327, 330, 338, 340, 341, 347, 351, 352, 477–479, 573
Correlation, 538, 539
Coupled differential equations, 193, 195, 559
Critical buckling load, 499, 500
Critical damping, 30, 31, 33, 36–38, 40–42, 53, 65, 70, 73, 79–81, 99, 103, 109, 111–113, 125, 126, 129, 130, 134, 141, 142, 169, 363, 435, 463, 465, 472–474, 481, 487, 509, 510, 587

D
D'Alembert's Principle, 9–10, 20, 30, 491
Damped system
  frequency, 33, 36, 56
  harmonic excitation, 47–54
  oscillator, 30, 31, 42, 51, 53, 54, 57, 63, 96, 103, 109, 113, 118, 491, 492, 523, 559
  period, 34, 37, 42
  steady-state response, 47, 49, 51, 54, 77, 289, 313, 315, 332, 472
  total response, 49
Damping
  coefficient, 29, 31–33, 35–38, 40, 41, 43, 58, 65, 81, 102, 103, 120, 125, 143, 145, 151, 160, 163, 216, 217, 264, 390, 463, 465, 477, 480, 481, 487, 498, 505, 506

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M. Paz, Y. H. Kim, *Structural Dynamics*, https://doi.org/10.1007/978-3-319-94743-3

Damping (*cont.*)
critical, 30–33, 36–38, 40–43, 53, 65, 70, 73, 79–81, 99, 103, 109, 111–113, 125, 126, 129, 130, 134, 141, 142, 169, 363, 435, 463, 465, 471–474, 487, 499, 500, 509, 510, 587
equivalent viscous, 43, 51, 58–60, 78
Degrees-of-freedom, 3–4, 20, 139, 140, 173, 176, 183, 184, 188, 195, 219, 220, 222, 223, 227, 230, 232, 233, 241, 245, 246, 248, 259, 262, 274, 286, 299, 323, 332, 374, 380, 415, 492, 496, 513, 518, 519, 559, 565
Design response spectral curve, 591
Design response spectrum
elastic, 122–126, 128, 131, 137, 142
inelastic, 128–135
Deterministic analysis
Dirac's delta function, 552
direct method, 285
discrete Fourier transform, 459, 461, 553, 571
Direct integration method, 145, 150, 160, 363, 378
Discretization
consistent mass, 222, 262
lumped mass, 262
Distributed coefficient, 264, 498
Distributed property system, 4, 426
Ductility ratio, 129–135, 142, 168, 408, 409
Duhamel's integral
definition, 85–95
numerical evaluation
damped system, 95, 107
undamped system, 95, 107
Dynamic
condensation, 219, 222–227, 234, 239, 241–245, 247, 248, 233–ENF
equilibrium, 9, 10, 20, 95, 143, 145, 150, 159, 178, 270, 276, 278, 342, 363, 376, 385, 390, 393, 439, 443, 491, 510, 523, 561
load factor, 88–91, 109, 430, 433
magnification factor, 50, 55, 72
matrix, 233, 234, 241, 437–442, 444–448
Dynamic matrix for
axial effect, 291, 318, 439–441, 447
flexural-axial effect, 447
flexural effect, 437–439, 445–447
power series expansion, 445–448
torsional effect, 318–320, 330, 441, 442, 447

E
Earthquake
design response spectrum
elastic, 120–126, 128, 131
inelastic, 125, 128–135, 137, 142
effective modal weight, 576, 584
El Centro, CA, (1940), 120, 122, 123, 125, 129, 130, 135, 137, 138, 141, 142
El Centro earthquake, 120, 122, 123, 125, 129, 130, 134, 135, 137, 138, 141, 142
elastoplastic behavior, 128, 134, 135
load effect, 602
San Fernando, CA (1971), 120, 123, 126
tripartite response spectra, 119–122
Eigenproblem, 176, 184, 186, 187, 196, 202, 205, 219, 222, 226, 227, 230, 233–238, 242, 243, 302, 325, 342, 395, 477, 573, 582, 586
Elastoplastic behavior, 128, 132, 134, 135, 141, 142, 156–158, 160, 161, 163–169, 342, 402, 406, 409
Energy
kinetic, 496, 505, 511–514, 516–519, 523
potential, 4, 29, 223, 496–499, 505, 511–514, 516–519, 523
Ensemble, 532, 567
Epicenters, 123
Equations of motion
discrete system, 389, 426, 428, 448, 505
uncoupled, 193–196, 213, 215, 275, 282, 285, 310, 311, 477–480, 483, 559, 562, 568
Equilibrium dynamics, 9, 10, 20, 95, 143, 145, 150, 159, 178, 270, 276, 278, 342, 376, 385, 390, 393, 439, 443, 491, 510, 523, 561
Equivalent lateral force method, 580, 581, 584, 588, 607
Equivalent viscous damping, 43, 51, 58–60, 78
Ergodic process, 532
Euler's relation, 47, 457, 458, 549
Evaluation of
bandwidth (half power), 55–57, 78
logarithmic decrement, 34, 41, 54, 58
ratio, 55, 264, 477–487
resonance, 50, 54–55, 59
uncoupling, 477–480, 482–485, 559, 562, 568

F
Fast Fourier transform (FFT), 461–463, 470, 553, 568
Federal Emergency Management Agency (FEMA), 587, 596
Finite element method (FEM)
displacement function
rectangular element, 380, 381
triangular element, 374, 375
element mass matrix
plate bending, 378–384
triangular, 379
element stiffness matrix
plate bending, 378–384
triangular, 379
plane elasticity problems
plane strain, 372, 376
plane stress, 372, 376
First/fundamental mode, 179, 180, 186, 187, 227, 230, 242, 302, 429, 433, 435, 436, 487, 516, 561
Folding frequency, 459
Forced motion, 217
Force transmitted to the foundation, 69–71, 80, 141
Fourier
analysis, 453–470, 543, 553
coefficients, 420, 456–459, 471, 474, 546, 570
discrete, 458–461, 474, 544–547, 565, 568
exponential form, 457, 458, 542
integral, 542, 543, 571

series, 453–459, 461, 462, 469, 470, 472, 531, 542, 544, 547, 553, 568, 570
transform, 459, 461, 462, 470, 542–545, 547, 548, 553, 559, 560, 565, 568, 571
transform pair, 543, 548, 561
Frame
plane, 291, 304–312, 317, 318, 322, 331, 335, 342, 347, 439
three-dimensional, 281, 317, 335–347
Free body diagram (FBD), 8–10, 20, 30, 46–48, 52, 61, 95, 96, 116, 118, 143, 144, 174, 175, 177, 199, 201, 208, 209, 278, 415, 477, 478, 491, 493, 494, 504, 505, 519, 561, 562
Free vibrations
critically damped, 30–31, 40
overdamped, 31, 40
undamped, 5, 6, 15, 33, 176, 187, 272, 478, 510
underdamped, 32–33, 40
Frequency
analyzer, 548
circular/angular, 12, 543
damped, 37, 41, 42, 44, 104
fundamental, 248, 467, 469, 484, 517, 518, 520
natural, 11–13, 15, 16, 21, 22, 24–26, 29, 31–33, 35, 36, 44, 46, 47, 52, 60, 72, 75, 78, 82, 83, 89, 100, 104, 115, 116, 120, 121, 123, 139, 155, 179, 180, 248, 332, 423, 424, 438, 500, 503, 505, 509–521, 523, 526–528, 555, 573
ratio, 46, 49, 50, 52, 55, 56, 63, 72, 83, 460, 503, 555
response, 56–58, 63, 460, 554–559, 561, 568
Fundamental period, 127, 507, 510, 517, 590, 591, 593, 604

G
Gallagher, R.H., 220
Gaussian distribution, 536
Gauss-Jordan, 221, 229, 233, 235–238, 241, 242, 245, 271
Generalized coordinates, 491, 498, 501, 504, 506, 513, 523–525
Generalized properties
damping, 494, 498, 505–507, 509, 510
force, 494, 497, 498, 505
inertia, 494
mass, 496, 500, 501, 503, 505, 507, 509, 524
stiffness, 494, 499–501, 505, 507, 509, 525
Geometrical stiffness, 266–269, 286, 499, 502, 525
Goodier, J.N., 375, 382
Grid frames, 281, 317–331, 335, 347
Guyan, R.J., 219

H
Harmonic excitations, 45–54, 78, 83, 210, 215, 348
Harmonic motion
damped, 65, 120
undamped, 120, 180
Harmonics/higher harmonics, 12, 80, 113, 120, 143, 180, 207, 209, 386, 426, 431, 437–442, 445, 448, 459–461, 468, 469, 511, 513, 514, 516, 543, 545, 553, 556, 575
Homogeneous equations, 45, 417
Horizontal torsional moments, 599

I
Impulsive load
rectangular, 88–89
triangular, 90–95, 111, 216, 475
Inelastic spectra, 125, 133
Inertial forces, 9, 10, 20, 30, 52, 61, 143, 144, 251, 261, 262, 270, 276–278, 296, 297, 349, 373, 377, 379, 385, 416, 426, 427, 491, 493, 494, 500, 516, 517
Initial conditions, 5, 7, 12, 18, 20, 22, 30, 32, 39, 46, 49, 77, 86, 88, 90, 91, 97, 98, 100, 105, 110, 117, 145, 156, 178, 180, 181, 204, 327, 358, 420, 430
Initial value problem, 12
Integration nonlinear equations
constant acceleration method, 145–147, 150
linear acceleration method, 145, 148–150
International Building Code (IBC), vi, 596, 607
Irregular structures, 594
Isolation, 62, 63, 65, 67, 69, 78, 80

J
Jacobi Method, 184

K
Kinetic energy, 223, 496, 505, 511–514, 516–519, 523
Knopp, K., 446

L
Lateral force, distribution, 596, 598
Linear acceleration method, 139, 145, 148, 150, 151, 158, 285, 315, 333, 342, 389, 391, 397, 401, 402, 409
Loading
general, 85
harmonic, 81, 85, 459
impulsive, 85, 91, 167, 168
periodic, 453–455
rectangular, 91
triangular, 91
Logarithmic decrement, 33–38, 41–43, 58
Lumped
beam, 173, 251, 259, 260, 262, 263, 270, 273, 293, 337
grid, 321
plane frame, 293
three-dimensional frame, 337

M
Magnification factor, 50, 55, 72
Mapped response spectral acceleration, 587, 588, 604
Mass matrix
consistent
beam, 263, 293, 342
grid, 320, 321
plane frame, 297
plane truss, 351
three-dimensional frame, 337, 342
Mathematical models, 3, 20, 103, 199, 222, 277, 478, 487, 518–521, 562

Maxwell's reciprocal theorem, 255
Mean square value, 532–534, 537, 539, 540, 543–550, 554, 555, 557, 563, 568, 570, 571
Mean value, 532, 533, 536, 540, 547, 557, 563, 568, 570
Minimal lateral force procedure, 597
Modal
  damping, 214, 216, 285, 332, 363, 369, 378, 428, 477–487, 574
  displacements, 198, 200, 430, 562, 577, 579
  drift, 577, 580
  force, 194, 196, 198, 285, 357, 428–430, 579
  lateral forces, 575, 577
  matrix, 183, 184, 187, 188, 195, 202, 205, 229, 230, 274, 275, 285, 303, 306, 326, 355, 478, 480, 481, 483, 485, 487, 562, 574, 582
  overturning moment, 578, 580, 585
  seismic coefficient, 577, 584
  seismic response, 573
  shapes for uniform beams, 418, 420, 422–424, 426
  shear force, 216, 574–576, 578, 579, 584
  spring constant, 194
  superposition method, 181, 193, 214, 215, 285, 287, 291, 312, 315, 337, 342, 363, 378, 437, 477, 487, 559, 568
  torsional moment, 575, 578, 580, 585
Mode of vibration, 180, 264, 337, 420
Modified dynamic condensation, 241–244, 248

N
Narrow band process, 541, 550–553
National Building Code (IBC), vi, 590–610
National Earthquake Hazard Reduction Program (NEHRP), 596
Natural frequency
  continuous system, 434
  multi-degree-of-freedom system, 181, 193, 215
  single-degree-of-freedom system, 115, 123, 139, 180, 500, 505, 506, 555
Natural period, 16–18, 21, 89, 92, 100, 103, 117, 125, 129, 142, 150, 151, 154, 161, 163, 179, 184, 197, 386, 395, 473, 521, 573, 574, 577, 582, 583, 586, 587, 593
Newland, D.E., 459
Newmark beta method, 137, 150–155, 165, 389, 401, 402
Newmark, N.M., 121, 124–126, 128, 131, 136, 150–155, 401, 402
Newton's Law of Motion, 7–8, 85, 491, 511, 577
Nodal
  coordinates, 3, 195, 223, 251, 292, 317, 335, 349, 377, 391, 437
  element force, 276–278
Nonlinear response
  incremental equation of motion, 151, 390, 392
  multi-degree-of-freedom system, 215
  single-degree-of-freedom system, 143–144, 156, 390
  unconditionally stable solution, 389, 409
  Wilson-θ method, 342, 389, 391–393, 409
Normal
  distribution, 536, 557, 568
Numerical integration
  piece-wise linear function, 105, 106
Nyquist frequency, 459, 467–469, 545

O
Occupancy importance factor, 597
Orthogonality property
  continuous system, 426, 434
  discrete system, 426, 428
Overdamped system, 31, 40
Overturning moment, 573, 578, 580, 585, 586, 599, 606

P
Participation factor, 200, 202, 204, 205, 213, 232, 506, 573, 577
Particular solutions, 45, 47–49, 97, 98, 110, 116
Paz, M., vii, 233, 241
P-delta effects, 601
Penzien, J., 480, 576
Periodic function, 453, 454, 458, 469, 541–543, 570
Period, natural, 16–18, 21, 89, 92, 100, 103, 117, 125, 129, 142, 150, 151, 154, 161, 163, 179, 184, 197, 386, 395, 473, 521, 574, 577, 582, 583, 586, 587, 593
Phase angle, 15, 50, 51, 70, 71, 79, 82, 181
Potential energy (PE), 4, 223, 496–499, 505, 511–514, 516–519, 523
Power series expansion
  axial and torsional effects, 447
  flexural effects, 445–447
  flexural effects including axial, 447, 448
Principle of conservation of energy, 510, 523
Principle of virtual work, 9, 254, 261, 265, 266, 294, 350, 376, 384, 491–493, 510, 523
Probability density function, 534–538
Pseudovelocity, 119, 120, 125, 134, 135, 142

R
Random
  function, 531, 532, 534, 537, 568
  process, 531, 532, 534, 538, 540, 541, 547, 548, 550–553, 567, 568, 570
  vibration, 214, 531
Rayleigh
  distribution, 537, 538, 558, 568
  improved, 517, 520
  method, 491
  quotient, 186–187
Reduced
  damping matrix, 222
  dynamic matrix, 233, 234, 241
  mass matrix, 222, 226, 230, 234, 240–243
  stiffness matrix, 220, 221, 234, 241, 245, 272, 582
Redundancy/reliability factor, 604–605, 610
Resonance, 47, 50, 54–55, 59, 60, 77, 78, 82, 83, 208
Response modification factor, 595–597, 605
Response spectral acceleration, 587–591, 593–595, 597, 604, 605
Response spectrum
  chart construction, 115

elastic design, 123–126, 131
inelastic design, 131–135
inelastic system, 128–130
tripartite response spectra, 119–122
Response to random excitation
multiple-degree-of-freedom system, 559
N-degree-of-freedom system, 182, 565
single-degree-of-freedom system, 553
two-degree-of-freedom system, 561–565
Restoring force, 4, 59, 128, 141, 143, 147, 156, 157, 163, 166
Root mean square (RMS), 533, 534, 537, 570
Rotary inertia, 416, 434, 443

S
Sample, 459, 467–469, 532, 547, 553, 565, 567, 570
Scaling results, 578, 580–585
Seismic coefficients, 584
Seismic design category, 590, 592, 594, 596, 599–602, 605, 607
Seismic instruments, 71–72
Seismic use group (SUG), 587
Seismic weight, 582, 583, 594–596, 598, 604
Seismograph, 71–73
Shear building
damped, 208–210, 477
definition, 173
forced motion, 217
free vibration, 178, 395
higher harmonics, 180
natural frequencies, 176–181, 184, 188–190, 196, 203
normal modes, 176–181, 184, 188, 189, 246
response to base motion, 199–205, 215
Shear walls, 130, 520–524, 528, 595, 601, 602
Simple undamped oscillator, 5, 12
Single-degree-of-freedom system, 463
damped free vibration, 107, 463, 553, 560
generalize coordinates, 501, 505
generalized, 500
nonlinear response, 143, 160, 378
response spectra, 89, 115, 120, 123–126, 129, 139, 142, 573, 587
response to general dynamic load, 85, 89, 491
Site class definition, 588, 589
Site coefficient, 589, 604
Skeletal structures, 371
Soil modified response spectral
acceleration, 588, 589, 605
Spectral
acceleration, 119, 120, 126, 127, 129, 132, 133, 135, 203, 573–575, 583, 586–591, 593–595, 597, 604, 605
analysis, 543, 546
density analyzer, 548
density function, 547–554, 556–558, 564, 565, 568, 570
displacement, 119, 120, 129, 133–135, 203, 232, 575, 577
function, 544, 553, 556, 565, 570, 571
velocity, 119, 120, 129, 133, 135, 573, 575
Spectrum, 115–120, 123–126, 128, 131–135, 139, 142, 231, 232, 247, 313, 368, 544–547, 550, 553, 568, 573, 591
Spring
hard, 5
linear, 5, 6, 8
soft, 5, 6
Spring constant
in parallel, 6
in series, 6, 7, 13
springs, 6, 7, 13, 20, 29, 37, 38, 42, 43, 69, 83, 102, 103, 149, 151, 154, 163, 174, 175, 178, 194, 210, 463, 465
Square Root Sum of Squares (SRSS) method, 195, 203, 214, 215
Standard deviation, 125, 533, 536, 537, 563, 570
Stationary process, 532, 570
Steady-state response, 47, 49, 51, 66, 72, 207, 208, 211, 217, 218, 274, 288, 289, 313, 315, 332, 432, 454, 460, 467, 470–472, 503
Step-by-step integration
constant acceleration method, 145
linear acceleration method, 139, 145, 158, 285, 315, 333, 389, 391, 397, 409
unconditionally stable, 389, 409
Wilson-θ method, 285, 389, 391–397, 409
Stiffness
coefficients, 36, 52, 144, 145, 151, 174, 176, 218, 252, 254, 255, 257, 258, 260, 264, 266, 268, 291, 292, 294, 295, 319, 349, 390, 404, 437, 446, 447, 499, 502
equation for shear buildings, 173–176
matrix
axial force effect, 268, 269, 291, 292
axial mass effects, 293–297
dynamic, 234, 241
flexural, 292, 320, 335, 496
plane truss, 349–351, 353, 357, 361, 368
space trusses, 361–363
three dimensional frame, 342
torsional, 447
Story drift, 586, 595, 599–601, 606, 607
Story shear force, 584
Support motion, 60–69, 95, 112, 119, 120, 129, 141

T
Taylor's series, 445, 447, 449
Three-dimensional frames, 281, 317, 335–347
Timoshenko, S., 375, 382
Torsional effects, 214, 318–321, 330, 337, 441, 442, 447
Torsional moments, 573, 575, 578, 580, 585, 586, 599
Total design values
base shear force, 583, 584
lateral deflection, 578
lateral seismic force, 579
overturning moment, 578, 586
story drift, 578, 585, 586
story shear force, 578, 586

Total design values (*cont.*)
torsional moment, 578, 586
Transformation matrix, 220–222, 234, 245, 246, 272, 286, 298–300, 313, 322, 324, 332, 338, 340, 341, 351, 353, 362
Transient response, 47, 49
Transmissibility, 62, 63, 65, 66, 70, 71, 78, 80, 81
Triangular plate element, 373–378
Tripartite chart
construction of, 121, 124
elastic design, 131
inelastic design, 131
Trusses
plane, 349–351, 353, 357, 361, 363
space, 361–364, 366–368

U
Undamped free vibration
continuous system, 173, 438
Undamped oscillator, 5, 12, 14, 30, 46, 86–88, 90, 91, 115, 455, 471
Undamped system, 5, 6, 31, 33, 35, 45–47, 77, 85–95, 104, 110, 120, 454, 456, 467, 502, 510
Underdamped system, 32–33, 40
Uniform Building Code (UBC), 126, 127, 587, 599
Uniform Building Code response spectra, 126, 127
Unit impulse relationship, 559
Unit impulse response, 559, 561

V
Variance, 532, 533, 537, 568
Vibration isolation, 62, 63, 67, 69, 78, 80
Vibration, normal modes, 176–181, 184, 193, 215, 420
Virtual work, principle of, 9, 254, 261, 265, 266, 294, 350, 376, 384, 491–493, 510, 523
Viscous damping, 29, 40, 43, 47, 51, 57–60, 78, 79, 143, 390, 484, 487

W
Wakabayashi, M., 484
White noise, 550, 552, 553
Wide-band process, 550
Wiener-Kinchin equations, 549
Wilson-θ method, 342, 389, 391–397, 400, 409